say E_m. Therefore, from Eq. (*i*) we get

$$E_m = \omega N \Phi_m = \omega N B_m A = 2\pi f N B_m A \text{ volt} \qquad ...(ii)$$

where B_m = maximum flux density in Wb/m^2 ; A = area of the coil in m^2

f = frequency of rotation of the coil in rev/second

Substituting this value of E_m in Eq. (*i*), we get $e = E_m \sin\theta = E_m \sin\omega t$ ...(*iii*)

Similarly, the equation of induced alternating current is $i = I_m \sin\omega t$ ...(*iv*)

provided the coil circuit has been closed through a resistive load.

Since $\omega = 2\pi f$, where f is the frequency of rotation of the coil, the above equations of the voltage and current can be writen as

$$e = E_m \sin 2\pi f t = E_m \sin\left(\frac{2\pi}{T}\right)t \text{ and } i = I_m \sin 2\pi f t = I_m \sin\left(\frac{2\pi}{T}\right)t$$

where T = time-period of the alternating voltage or current = $1/f$

It is seen that the induced e.m.f. varies as sine function of the time angle ωt and when e.m.f. is plotted against time, a curve similar to the one shown in Fig. 11.3 is obtained. This curve is known as sine curve and the e.m.f. which varies in this manner is known as *sinusoidal* e.m.f. Such a sine curve can be conveniently drawn, as shown in Fig. 11.4. A vector, equal in length to E_m is drawn. It rotates in the counter-clockwise direction with a velocity of ω radian/second, making one revolution while the generated e.m.f. makes two loops or one cycle. The projection of this vector on Y-axis gives the instantaneous value e of the induced e.m.f. *i.e.* $E_m \sin\omega t$.

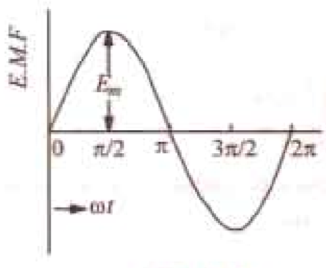

Fig. 11.3

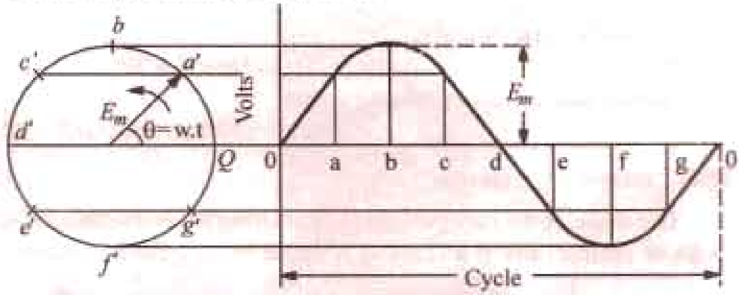

Fig. 11.4

To construct the curve, lay off long X-axis equal angular distance *oa*, *ab*, *bc*, *cd* etc. corresponding to suitable angular displacement of the rotating vector. Now, erect coordinates at the points *a*, *b*, *c* and *d* etc. (Fig. 11.4) and then project the free ends of the vector E_m at the corresponding positions *a'*, *b'*, *c'*, etc to meet these ordinates. Next draw a curve passing through these intersecting points. The curve so obtained is the graphic representation of equation (*iii*) above.

11.3. Alternate Method for the Equations of Alternating Voltages and Currents

In Fig. 11.5 is shown a rectangular coil AC having N turns and rotating in a magnetic field of flux density B Wb./m^2. Let the length of each of its sides A and C be l meters and their peripheral velocity v metre/second. Let angle be measured from the horizontal position *i.e.* from the X-axis. When in horizontal position, the two sides A and C move parallel to the lines of the magnetic flux. Hence, no flux is cut and so no e.m.f. is generated in the coil.

When the coil has turned through angle θ, its velocity can be resolved into two mutually perpendicular components (*i*) $v\cos\theta$ component-parallel to the direction of the magnetic

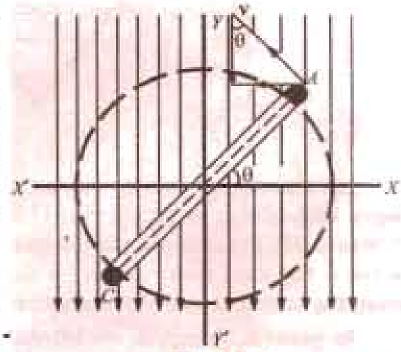

Fig. 11.5

flux and (*ii*) $v \sin \theta$ component-perpendicular to the direction of the magnetic flux. The e.m.f. is generated due entirely to the perpendicular component *i.e.* $v \sin \theta$.

Hence, the e.m.f. generated in one side of the coil which contains N conductors, as seen from Art. 7.7, is given by, $e = N \times Bl\, v \sin \theta$.

Total e.m.f. generated in both sides of the coil is

$$e = 2BNl\, v \sin \theta \text{ volt} \qquad ...(i)$$

Now, e has maximum value of E_m (say) when $\theta = 90°$. Hence, from Eq. (*i*) above, we get, $E_m = 2\, B\, N\, l\, v$ volt. Therefore Eq. (*i*) can be rewritten as $e = E_m \sin \theta$ ...as before

If b = width of the coil in meters ; f = frequency of rotation of coil in Hz, then $v = \pi\, bf$

$$\therefore \quad E_m = 2\, B\, N\, l \times \pi\, b\, f = 2\, p\, i\, f\, N\, B\, A \text{ volts} \qquad ...\text{as before}$$

Example 11.1. *A square coil of 10 cm side and 100 turns is rotated at a uniform speed of 1000 revolutions per minute, about an axis at right angles to a uniform magnetic field of 0.5 Wb/m². Calculate the instantaneous value of the induced electromotive force, when the plane of the coil is (i) at right angles to the field (ii) in the plane of the field.*

(Electromagnetic Theory, A.M.I.E. Sec B, 1992)

Solution. Let the magnetic field lie in the vertical plane and the coil in the horizontal plane. Also, let the angle θ be measured from X-axis.

Maximum value of the induced e.m.f., $E_m = 2\, \pi\, f\, N\, B_m\, A$ volt.

Instantaneous value of the induced e.m.f. $e = E_m \sin \theta$

Now $f = 100/60 = (50/3)$ rps, $N = 100$ $B_m = 0.5$ Wb/m², $A = 10^{-2}$ m²

(*i*) In this case, $\theta = 0°$

$$\therefore \quad e = 0 \; (ii) \text{ Here } \theta = 90°, \quad \therefore \quad e = E_m \sin 90° = E_m$$

Substituting the given values, we get

$$e = 2\pi \times (50/3) \times 100 \times 0.5 \times 10^{-2} = \mathbf{52.3\ V}$$

11.4. Simple Waveforms

The shape of the curve obatined by plotting the instantaneous values of voltage or current as the ordinate against time as a abscissa is called its waveform or wave-shape.

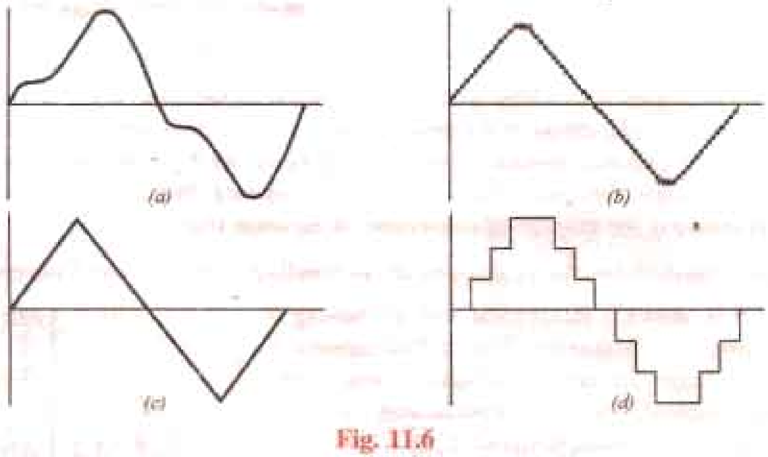

Fig. 11.6

An alternating voltage or current may not always take the form of a systematical or smooth wave such as that shown in Fig. 11.3. Thus, Fig. 11.6 also represents alternating waves. But while it is scarcely possible for the manufacturers to produce sine-wave generators or alternators, yet sine wave is the ideal form sought by the designers and is the accepted standard. The waves deviating from the standard sine wave are termed as distorted waves.

In general, however, *an alternating current or voltage is one the circuit direction of which reverses at regularly recurring intervals.*

11.5. Complex Waveforms

Complex waves are those which depart from the ideal sinusoidal form of Fig. 11.4. All alternating complex waves, which are periodic and have equal positive and negative half cycles can be shown to be made up of a number of pure sine waves, having different frequencies but all these frequencies are integral multiples of that of the lowest alternating wave, called the *fundamental* (or first harmonic). These waves of higher frequencies are called *harmonics*. If the fundamental frequency is 50 Hz, then the frequency of the second *harmonic* is 100 Hz and of the third is 150 Hz and so on. The complex wave may be composed of the fundamental wave (or first harmonic) and any number of other harmonics.

In Fig. 11.7 is shown a complex wave which is made up of a fundamental sine wave of frequency of 50 Hz and third harmoic of frequency of 150 Hz. It is seen that

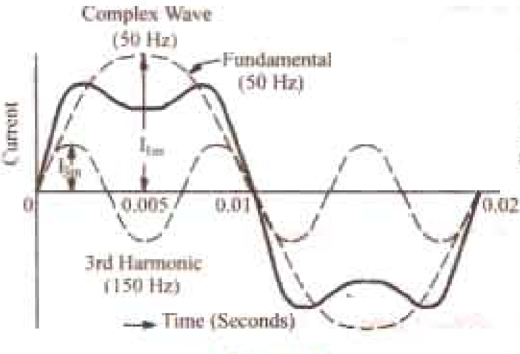

Fig. 11.7

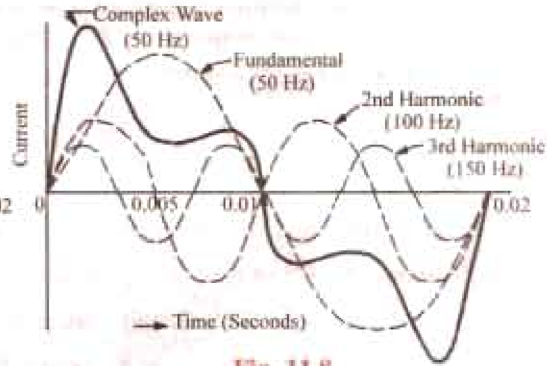

Fig. 11.8

(*i*) the two halves of the complex wave are identical in shape. In other words, there is no distortion. This is always the case when only *odd* harmonic (3rd, 5th, 7th, 9th etc.) are present.

(*ii*) frequency of the complex wave is 50 Hz *i.e.* the same as that of the fundamental sine wave.

In Fig. 11.8 is shown a complex wave which is a combination of fundamental sine wave of frequency 50 Hz and 2nd harmonic of frequency 100 Hz and 3rd harmonic of frequency 150 Hz.

??SL101?TIt is seen that although the frequency of the complex wave even now remains 50 Hz, yet

(*i*) the two halves of the complex wave are not identical. It is always so when *even* harmonics (2nd, 4th, 6th etc.) are present.

(*ii*) there is distortion and greater departure of the wave shape from the purely sinusoidal shape.

Sometimes, a combination of an alternating and direct current flows simultaneously through a circuit. In Fig. 11.9 is shown a complex wave (containing fundamental and third harmonic) combined with a direct current of value I_D. It is seen that the resultant wave remains undistorted in shape but is raised above the axis by an amount I_D. It is worth noting that with reference to the original axis, the two halves of the combined wave are not equal in area.

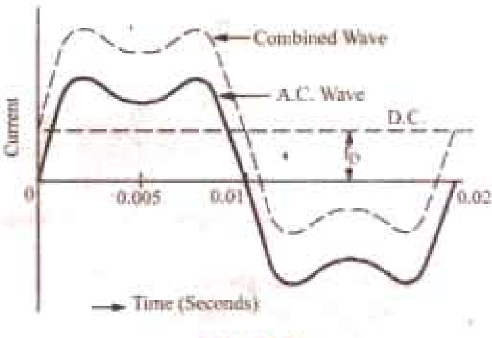

Fig. 11.9

11.6. Cycle

One complete set of positive and negative values of alternating quantity is known as cycle. Hence, each diagram of Fig. 11.6 represents one complete cycle.

A cycle may also be sometimes specified in terms of angular measure. In that case, one complete cycle is said to spread over 360° or 2π radians.

11.7. Time Period

The time taken by an alternating quantity to complete one cycle is called its time period T. For example, a 50-Hz alternating current has a time period of 1/50 second.

11.8. Frequency

The number of cycles/second is called the frequency of the alternating quantity. Its unit is hertz (Hz).

In the simple 2-pole alternator of Fig. 24.1 (*b*), one cycle of alternating current is generated in one revolution of the rotating field. However, if there were 4 poles, then two cycles would have been produced in each revolution. In fact, the frequency of the alternating voltage produced is a function of the speed and the number of poles of the generator. The relation connecting the above three quantities is given as

$f = PN/120$ where N = revolutions in r.p.m. and P = number of poles

For example, an alternator having 20 poles and running at 300 r.p.m. will generate alternating voltage and current whose frequency is $20 \times 300/120 = 50$ hertz (Hz).

It may be noted that the frequency is given by the reciprocal of the time period of the alternating quantity.

$$\therefore \quad f = 1/T \quad \text{or} \quad T = 1/f$$

11.9. Amplitude

The maximum value, positive or negative, of an alternating quantity is known as its amplitude.

11.10. Different Forms of E.M.F. Equation

The standard form of an alternating voltage, as already given in Art. 11.2, is

$$e = E_m \sin \theta = E_m \sin \omega t = E_m \sin 2\pi f t = E_m \sin \frac{2\pi}{T} t$$

By closely looking at the above equations, we find that

(*i*) the maximum value or peak value or amplitude of an *alternating voltage is given by the coefficient of the sine of the time angle.*

(*ii*) *the frequency f is given by the coefficient of time divided by* 2π..

For example, if the equation of an alternating voltage is given by $e = 50 \sin 314t$ then its maximum value of 50 V and its frequency is $f = 314/2\pi = 50$ Hz.

Similarly, if the equation is of the form $e = I_m \sqrt{(R^2 + 4\omega^2 L^2)} \sin 2\omega t$, then its maximum value is $E_m = I_m \sqrt{(R^2 + 4\omega^2 L^2)}$ and the frequency is $2\omega/2\pi$ or ω/π Hz.

Example 11.2. *The maximum values of the alternating voltage and current are 400 V and 20 A respectively in a circuit connected to 50 Hz supply and these quantities are sinusoidal. The instantaneous values of the voltage and current are 283 V and 10 A respectively at t = 0 both increasing positively.*

(i) Write down the expression for voltage and current at time t.

(ii) Determine the power consumed in the circuit. **(Elect. Engg. Pune Univ. 1985)**

Solution. (*i*) In general, the expression for an a.c. voltage is $v = V_m \sin(\omega t + \phi)$ where ϕ is the phase difference with respect to the point where $t = 0$.

Now, $v = 283$ V ; $V_m = 400$ V. Substituting $t = 0$ in the above equation, we get

$283 = 400 (\sin \omega \times 0 + \phi)$ $\therefore$ $\sin \phi = 400/283 = 0.707$; $\therefore$ $\phi = 45°$ or $\pi/4$ radian.

Hence, general expression for voltage is

$v = 400 (\sin 2\pi \times 50 \times t + \pi/4)$

$= 400 \sin (100\pi t + \pi/4)$

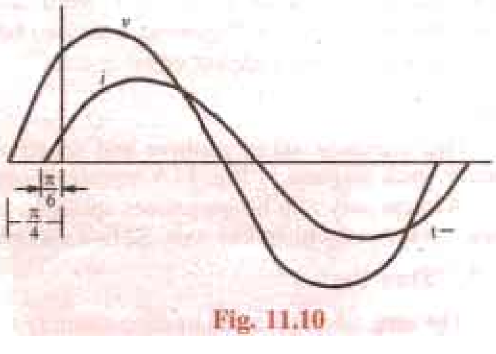

Fig. 11.10

Similarly, at $t = 0$, $10 = 20 \sin(\omega \times 0 + \phi)$ $\therefore$ $\sin\phi = 0.5$ $\therefore$ $\phi = 30°$ or $\pi/6$ radian

Hence, the genral expression for the current is

$i = 20(\sin 100\pi t + 30°) = 20 \sin(100\pi t + \pi/6)$

(*ii*) $P = VI\cos\theta$ where V and I are rms values and θ is the phase difference between the voltage and current.

Now, $V = V_m/\sqrt{2} = 400/\sqrt{2}$; $I = 20/\sqrt{2}$; $\theta = 45° - 30° = 15°$ (See Fig. 11.10)

$\therefore$ $P = (400/\sqrt{2}) \times (20/\sqrt{2}) \times \cos 15° =$ **3864 W**

Example 11.3. *An alternating current of frequency 60 Hz has a maximum value of 120 A. Write down the equation for its instantaneous value. Reckoning time from the instant the current is zero and is becoming positive, find (a) the instantaneous value after 1/360 second and (b) the time taken to reach 96 A for the first time.*

Solution. The instantaneous current equation is

$$i = 120 \sin 2\pi f t = 120 \sin 120\pi t$$

Now when $t = 1/360$ second, then

(*a*) $i = 120 \sin(120 \times \pi \times 1/360)$ ...angle in radians

$= 120 \sin(120 \times 180 \times 1/360)$ ...angle in degree

$= 120 \sin 60° = 103.9$ A

(*b*) $96 = 120 \times \sin 2 \times 180 \times 60 \times t$ ...angle in degree

or $\sin(360 \times 60 \times t) = 96/120 = 0.8$ $\therefore$ $360 \times 60 \times t = \sin^{-1} 0.8 = 53°$ (approx)

$\therefore$ $t = \theta/2\pi f = 53/360 \times 60 =$ **0.00245 second.**

11.11. Phase

By phase of an alternating current is meant the fraction of the time period of that alternating current which has elaspsed since the current last passed through the zero position of reference. For example, the phase of current at point A is $T/4$ second, where T is time period or expressed in terms of angle, it is $\pi/2$ radians (Fig. 11.11). Similarly, the phase of the rotating coil at the instant shown in Fig. 11.1 is ωt which is, therefore, called its phase angle.

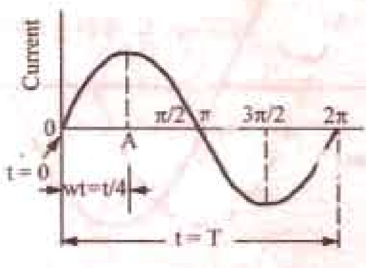

Fig. 11.11

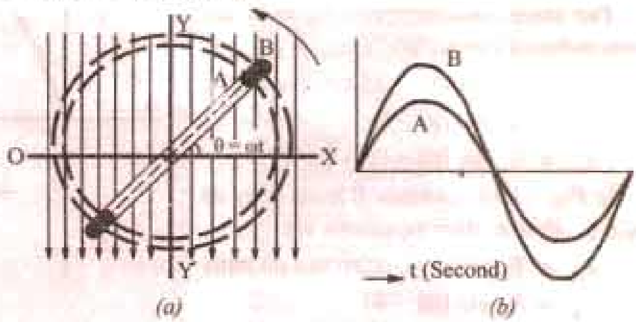

Fig. 11.12

In electrical engineering, we are, however, more concerned with relative phases or phase differences between different alternating quantities, rather than with their absolute phases. Consider two single-turn coils of different sizes [Fig. 11.12 (*a*)] arranged radially in the same plane and rotating with the same angular velocity in a common magnetic field of uniform intensity. The e.m.fs. induced in both coils will be of the same frequency and of sinusoidal shape, although the values of instantaneous e.m.fs. induced would be different. However, the two alternating e.m.fs. would reach their maximum and zero values at the same time as shown in Fig. 11.12 (*b*). Such alternating voltages (or currents) are said to be in phase with each other. The two voltages will have the equations

$$e_1 = E_{m1} \sin \omega t \quad \text{and} \quad e_2 = E_{m2} \sin \omega t$$

11.12. Phase Difference

Now, consider three similar single-turn coils displaced from each other by angles α and β and rotating in a uniform magnetic field with the same angular velocity [Fig. 11.13 (*a*)].

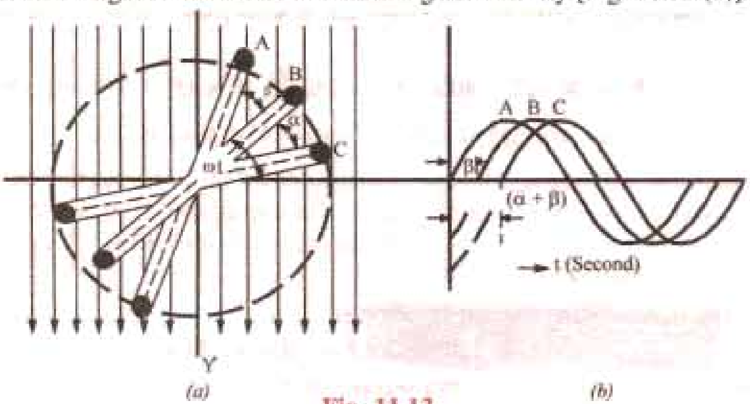

Fig. 11.13

In this case, the value of induced e.m.fs. in the three coils are the same, but there is one important difference. The e.m.fs. in these coils do not reach their maximum or zero values simultaneously but one after another. The three sinusoidal waves are shown in Fig. 11.13 (*b*). It is seen that curves *B* and *C* are displaced from curve *A* and angles β and (α + β) respectively. Hence, it means that phase difference between *A* and *B* is β and between *B* and *C* is α but between *A* and *C* is (α + β). The statement, however, does not give indication as to which e.m.f. reaches its maximum value first. This deficiency is supplied by using the terms '*lag*' or '*lead*'.

A leading alternating quantity is one which reaches its maximum (or zero) value earlier as compared to the other quantity.

Similarly, a lagging alternating quantity is one which reaches its maximum or zero value later than the other quantity. For example, in Fig. 11.13 (*b*), *B* lags behind *A* by β and *C* lags behind *A* by (α + β) because they reach their maximum values later.

The three equations for the instantaneous induced e.m.fs. are (Fig. 11.14)

$$e_A = E_m \sin \omega t \text{ ...reference quantity}$$
$$e_B = E_m \sin (\omega t - \beta)$$
$$e_C = E_m \sin [\omega t - (\alpha + \beta)]$$

In Fig. 11.14, quantity *B* leads *A* by an angle φ. Hence, their equations are

$$e_A = E_m \sin \omega t \text{ ...reference quantity}$$
$$e_B = E_m \sin (\omega t - \phi)$$

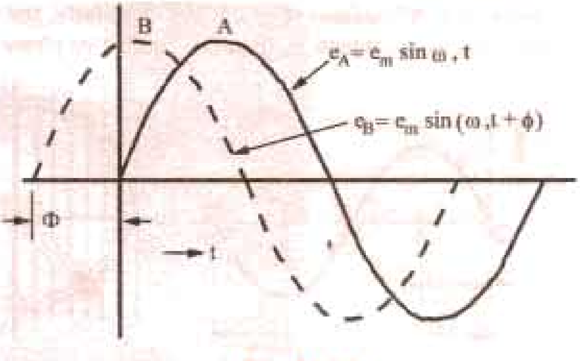

Fig. 11.14

A plus (+) sign when used in connection with phase difference denotes 'lead' whereas a minus (–) sign denotes 'lag'.

11.13. Root-Mean-Square (R.M.S.) Value

The r.m.s. value of an alternating current *is given by that steady (d.c.) current which when flowing through a given circuit for a given time produces the same* **heat** *as produced by the alternating current when flowing through the same circuit for the same time.*

It is also known as the *effective* or *virtual* value of the alternating current, the former term being used more extensively. For computing the r.m.s. value of symmetrical sinusoidal alternating currents, either mid-ordinate method or analytical method may be used, although for symmetrical but non-sinusoidal waves, the mid-ordinate method would be found more convenient.

A simple experimental arrangement for measuring the equivalent d.c. value of a sinusoidal current is shown in Fig. 11.15. The two circuits have identical resistances but one is connected to battery and the other to a sinusoidal generator. Wattmeters are used to measure heat power in each circuit. The voltage applied to each circuit is so adjusted that heat power production in each circuit is the same. In that case, the direct current will equal $I_m/\sqrt{2}$ which is called r.m.s. value of the sinusoidal current.

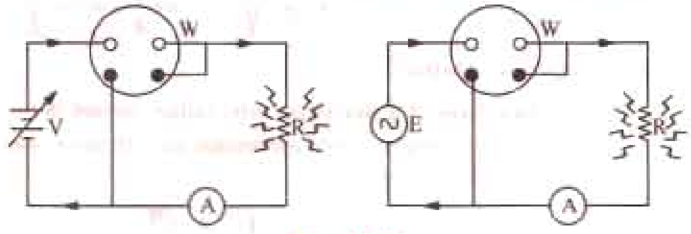

Fig. 11.15

11.14. Mid-ordinate Method

In Fig. 11.16 are shown the positive half cycles for both symmetrical sinusoidal and non-sinusoidal alternating currents. Divide time base 't' into n equal intervals of time each of duration t/n seconds. Let the average values of instantaneous currents during these intervals be respectively i_1, i_2, i_3 .... i_n (*i.e.* mid-ordinates in Fig. 11.16). Suppose that this alternating current is passed through a circuit of resistance R ohms. Then,

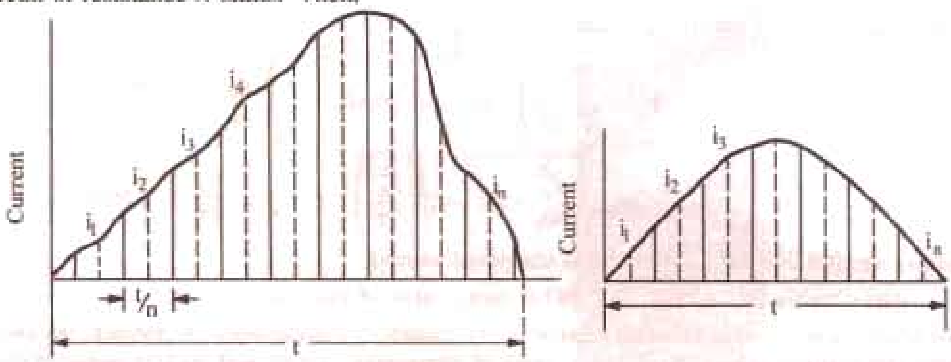

Fig. 11.16

Heat produced in Ist interval $= 0.24 \times 10^{-3}\, i_1^2\, Rt/n$ kcal ($\because 1/J = 1/4200 = 0.24 \times 10^{-3}$)

Heat produce in 2nd interval $= 0.24 \times 10^{-3}\, i_2^2\, Rt/n$ kcal

: : : : :

: : : : :

: : : : :

Heat produced in nth interval $= 0.24 \times 10^{-3}\, i_n^2\, Rt/n$ kcal

Total heat produced in t seconds is $= 0.24 \times 10^{-3}\, Rt \left(\dfrac{i_1^2 + i_2^2 + ... + i_n^2}{n}\right)$ kcal

Now, suppose that a direct current of value I produces the same heat through the same resistance during the same time t. Heat produced by it is $= 0.24 \times 10^{-3}\, I^2Rt$ kcal. By definition, the two amounts of heat produced should be equal.

$$\therefore \quad 0.24 \times 10^{-3} I^2 Rt = 0.24 \times 10^{-3}\, Rt \left(\frac{i_1^2 + i_2^2 + ... + i_n^2}{n}\right)$$

$$\therefore \quad I^2 = \frac{i_1^2 + i_2^2 + ... + i_n^2}{n} \quad \therefore \quad I = \sqrt{\left(\frac{i_1^2 + i_2^2 + ... + i_n^2}{n}\right)}$$

= square root of the mean of the squares of the instantaneous currents

Similarly, the r.m.s. value of alternating voltage is given by the expression

$$V = \sqrt{\left(\frac{v_1^2 + v_2^2 + ... + v_n^2}{n}\right)}$$

11.15. Analytical Method

The standard form of a sinusoidal alternating current is $i = I_m \sin \omega t = I_m \sin \theta$.

The mean of the squares of the instantaneous values of current over one complete cycle is (even the value over half a cycle will do).

$$= \int_0^{2\pi} \frac{i^2 \, d\theta}{(2\pi - 0)}$$

The square root of this value is $= \sqrt{\left(\int_0^{2\pi} \frac{i^2 \, d\theta}{2\pi}\right)}$

Hence, the r.m.s. value of the alternating current is

$$I = \sqrt{\left(\int_0^{2\pi} \frac{i^2 d\theta}{2\pi}\right)} = \sqrt{\left(\frac{I_m^2}{2\pi}\int_0^{2\pi} \sin^2 \theta \, d\theta\right)} \qquad \text{(put } i = I_m \sin \theta\text{)}$$

Now, $\cos 2\theta = 1 - 2 \sin^2 \theta \quad \therefore \quad \sin^2 \theta = \frac{1 - \cos 2\theta}{2}$

$$\therefore \qquad I = \sqrt{\left(\frac{I_m^2}{4\pi}\int_0^{2\pi} (1 - \cos 2\theta) \, d\theta\right)} = \sqrt{\left(\frac{I_m^2}{4\pi}\left| \theta - \frac{\sin 2\theta}{2} \right|_0^{2\pi}\right)}$$

$$= \sqrt{\left(\frac{I_m^2}{4\pi} \times 2\pi\right)} = \sqrt{\left(\frac{I_m^2}{2\pi}\right)} \quad \therefore \quad \mathrm{I} = \frac{I_m}{\sqrt{2}} = 0.707 \, I_m$$

Hence, we find that for a symmetrical sinusoidal current

r.m.s. value of current = 0.707 × max. value of current

The r.m.s. value of an alternating current is of considerable importance in practice, because the ammeters and voltmeters record the r.m.s. value of alternating current and voltage respectively. In electrical engineering work, *unless indicated otherwise, the values of the given current and voltage are always the r.m.s. values.*

It should be noted that the average heating effect produced during one cycle is

$$= I^2 R = (I_m/\sqrt{2})^2 \, R = \frac{1}{2} I_m^2 \, R$$

11.16. R.M.S. Value of a Complex Wave

In their case also, either the mid-ordinate method (when equation of the wave is not known) or analytical method (when equation of the wave is known) may be used. Suppose a current having the equation $i = 12 \sin \omega t + 6 \sin (3\omega t - \pi/6) + 4 \sin (5\omega t + \pi/3)$ flows through a resistor of R ohm. Then, in the time period T second of the wave, the effect due to each component is as follows :

Fundamental .... $(12/\sqrt{2})^2 \; RT$ watt

3rd harmonic .... $(6/\sqrt{2})^2 \; RT$ watt

5th harmonic .... $(4/\sqrt{2})^2 \; RT$ watt

$\therefore \qquad$ Total heating effect $= RT\,[(12/\sqrt{2})^2 + (6/\sqrt{2})^2 + (4/\sqrt{2})^2]$

If I is the r.m.s. value of the complex wave, then equivalent heating effect is I^2RT

$$\therefore \qquad I^2RT = RT\,[(12/\sqrt{2})^2 + (6/\sqrt{2})^2 + (4/\sqrt{2})^2]$$

$$\therefore \qquad I = \sqrt{[(12/\sqrt{2})^2 + (6/\sqrt{2})^2 + (4/\sqrt{2})^2]} = 9.74 \text{ A}$$

Had there been a direct current of (say) 5 amperes flowing in the circuit also*, then the r.m.s. value would have been

$$= \sqrt{(12 + \sqrt{2})^2 + (6/\sqrt{2})^2 + (4/\sqrt{2})^2 + 5^2} = 10.93 \text{ A}$$

Hence, for complex waves the rule is as follows : *The r.m.s. value of a complex current wave is equal to the square root of the sum of the squares of the r.m.s. values of its individual components.*

11.17. Average Value

The average value I_a of an alternating current is expressed *by that steady current which transfers across any circuit the same* **charge as** *is transferred by that alternating current during the same time.*

In the case of a symmetrical alternating current (*i.e.* one whose two half-cycles are exactly similar, whether sinusoidal or non-sinusoidal), the average value over a complete cycle is zero. Hence, in their case, the average value is obtained by adding or integrating the instantaneous values of current over one half-cycle only. *But in the case of an unsymmetrical alternating current (like half-wave rectified current) the average value must always be taken over the whole cycle.*

(i) Mid-ordinate Method

With reference to Fig. 11.16, $I_{av} = \dfrac{i_1 + i_2 + ... + i_n}{n}$

This method may be used both for sinusoidal and non-sinusoidal waves, although it is specially convenient for the latter.

(ii) Analytical Method

The standard equation of an alternating current is, $i = I_m \sin\theta$

$$I_{av} = \int_0^{\pi} \frac{i\,d\theta}{(\pi - 0)} = \frac{I_m}{\pi}\int_0^{\pi} \sin\theta\, d\theta \qquad \text{(putting value of } i\text{)}$$

$$= \frac{I_m}{\pi}\left|-\cos\theta\right|_0^{\pi} = \frac{I_m}{\pi}\left|+1-(-1)\right| = \frac{2I_m}{\pi} = \frac{I_m}{\pi/2} = \frac{\text{twice the maximum current}}{\pi}$$

$\therefore \quad I_{av} = I_m/\tfrac{1}{2}\pi = 0.637\, I_m \quad \therefore$ **average value of current = 0.637 × maximum value**

Note. R.M.S. value is always greater than average value except in the case of a rectangular wave when both are equal.

11.18. Form Factor

It is defined as the ratio, $K_f = \dfrac{\text{r.m.s. value}}{\text{average value}} = \dfrac{0.707\, I_m}{0.637\, I_m} = 1.1$. (for sinusoidal alternating currents only)

In the case of sinusoidal alternating voltage also, $K_f = \dfrac{0.707\, E_m}{0.637\, E_m} = 1.11$

As is clear, the knowledge of form factor will enable the r.m.s. value to be found from the arithmetic mean value and *vice-versa.*

11.19. Crest or Peak or Amplitude Factor

It is defined as the ratio $K_a = \dfrac{\text{maximum value}}{\text{r.m.s. value}} = \dfrac{I_m}{I_m/\sqrt{2}} = \sqrt{2} = 1.414$ (for sinusoidal a.c. only)

For sinusoidal alternating voltage also, $K_a = \dfrac{E_m}{E_m/\sqrt{2}} = 1.414$

---

* The equation of the complex wave, in that case, would be,
$i = 5 + 12 \sin\omega t + 6 \sin(3\omega t - \pi/6) + 4 \sin(5\omega t + \pi/3)$

Knowledge of this factor is of importance in dielectric insulation testing, because the dielectric stress to which the insulation is subjected, is proportional to the maximum or peak value of the applied voltage. The knowledge is also necessary when measuring iron losses, because the iron loss depends on the value of maximum flux.

Example 11.4. *An alternating current varying sinusoidally with a frequency of 50 Hz has an RMS value of 20 A. Write down the equation for the instantaneous value and find this value (a) 0.0025 second (b) 0.0125 second after passing through a positive maximum value. At what time, measured from a positive maximum value, will the instantaneous current be 14.14 A ?*

(Elect. Science-I Allahabad Univ. 1992)

Solution. $I_m = 20\sqrt{2} = 28.2$ A. $\omega = 2\pi \times 50 = 100\,\pi$ rad/s.

The equation of the sinusoidal current wave with reference to point O (Fig. 11.17) as zero time point is $\quad i = 28.2 \sin 100\,\pi t$ **ampere**

Since time values are given from point *A* where voltage has positive and maximum value, the equation may itself be referred to point *A*. In the case, the equation becomes :

$$i = 28.2 \cos 100\,\pi t$$

(*i*) **When $t = 0.0025$ second**

$i = 28.2 \cos 100\pi \times 0.0025$ ...angle in radian

$= 28.2 \cos 100 \times 180 \times 0.0025$ ...angle in degrees

$= 28.2 \cos 45° = 20$ A ...point *B*

(*ii*) **When $t = 0.0125$ second**

$i = 28.2 \cos 100 \times 180 \times 0.0125$

$= 28.2 \cos 225° = 28.2 \times (-1/\sqrt{2})$

$= -20$ A ...point *C*

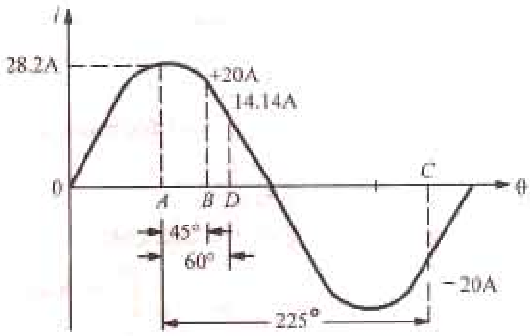

Fig. 11.17

(*iii*) **Here $i = 14.14$ A**

$\therefore \quad 14.14 = 28.2 \cos 100 \times 180\, t \quad \therefore \quad \cos 100 \times 180\, t = \dfrac{1}{2}$

or $\quad 100 \times 180\, t = \cos^{-1}(0.5) = 60°$, $t =$ **1/300 second** ...point *D*

Example 11.5. *An alternating current of frequency 50 Hz has a maximum value of 100 A. Calculate (a) its value 1/600 second after the instant the current is zero and its value decreasing thereafter/wards (b) how many seconds after the instant the current is zero (increasing thereafter wards) will the current attain the value of 86.6 A ?* **(Elect. Technology. Allahabad Univ. 1991)**

Solution. The equation of the alternating current (assumed sinusoidal) with respect to the origin *O* (Fig. 11.18) is

$$i = 100 \sin 2\pi \times 50t = 100 \sin 100\,\pi t$$

(*a*) It should be noted that, in this case, time is being measured from point *A* (where current is zero and decreasing thereafter) and not from point *O*.

If the above equation is to be utilized, then, this time must be referred to point *O*. For this purpose, half time-period *i.e.* 1/100 second has to be added to 1/600 second. The given time as referred to point *O* becomes

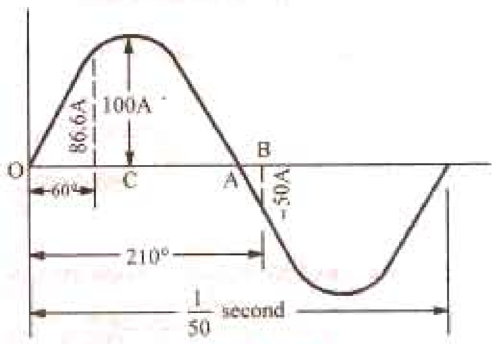

Fig. 11.18

$$= \frac{1}{100} + \frac{1}{600} = \frac{7}{600} \text{ second}$$

$\therefore \quad i = 100 \sin 100 \times 180 \times 7/600 = 100 \sin 210°$

$$= 100 \times -1/2 = \mathbf{-50\ A} \qquad \text{...point } B$$

(*ii*) In this case, the reference point is *O*.

$\therefore \quad 86.6 = 100 \sin 100 \times 180\, t \quad$ or $\quad \sin 18{,}000\, t = 0.866$

or $\quad 18{,}000\, t = \sin^{-1}(0.866) = 60° \quad \therefore \quad t = 60/18{,}000 =$ **1/300 second**

Example 11.6. *Calculate the r.m.s. value, the form factor and peak factor of a periodic votage having the following values for equal time intervals changing suddenly from one value to the next : 0, 5, 10, 20, 50, 60, 50, 20, 10, 5, 0, –5, –10 V etc. What would be the r.m.s value of sine wave having the same peak value ?*

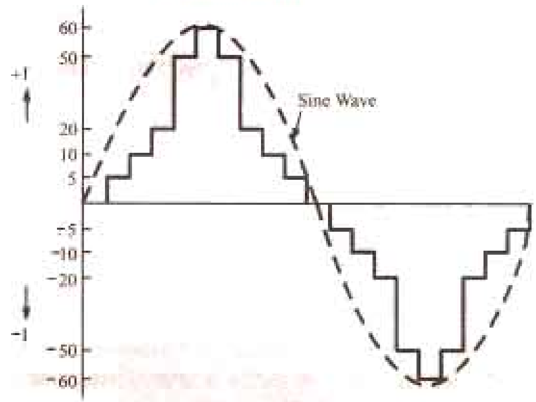

Fig. 11.19

Solution. The waveform of the alternating voltage is shown in Fig. 11.19. Obviously, it is not sinusoidal but it is symmetrical. Hence, though r.m.s value may be full one cycle, the average value has necessarily to be considered for half-cycle only, otherwise the symmetrical negative and positive half-cycles will cancel each other out.

$$\text{Mean value of } v^2 = \frac{0^2 + 5^2 + 10^2 + 20^2 + 50^2 + 60^2 + 50^2 + 20^2 + 10^2 + 5^2}{10} = \mathbf{965\ V}$$

$$\therefore \quad \text{r.m.s. value} = \sqrt{965} = \mathbf{31\ V\ (approx.)}$$

$$\text{Average value (half-cycle)} = \frac{0 + 5 + 10 + 20 + 50 + 60 + 50 + 20 + 10 + 5}{10} = \mathbf{23\ V}$$

$$\text{Form factor} = \frac{\text{r.m.s. value}}{\text{average value}} = \frac{31}{23} = \mathbf{1.35.} \quad \text{Peak factor} = 60/31 = \mathbf{2\ (approx.)}$$

R.M.S. value of a sine wave of the same peak value = $0.707 \times 60 =$ **42.2 V.**

Alternative Solution

If '*t*' be the regular time interval, then area of the half-cycle is

$= (5t + 10t + 20t + 50t)\, 2 + 60t = 230t$, Base $= 10t \quad \therefore$ Mean value $= 230t/10t = 23$ V.

Area when ordinates are squared $= (25t + 100t + 400t + 2500t)\, 2 + 3600t = 9650t$, Base $= 10t$

$\therefore \quad$ Mean height of the squared curve $= 9650t/10\, t = 965$

$\therefore \quad$ r.m.s. value $= \sqrt{965} = 31$ V

Further solution is as before.

Example 11.7. *Calculate the reading which will be given by a hot-wire voltmeter if it is connected across the terminals of a generator whose voltage waveform is represented by*

$$v = 200 \sin \omega t + 100 \sin 3\omega t + 50 \sin 5\omega t$$

Solution. Since hot-wire voltmeter reads only r.m.s value, we will have to find the r.m.s. value of the given voltage. Considering one complete cycle,

$$\text{R.M.S. value} \qquad V = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} v^2\, d\theta} \quad \text{where } \theta = \omega t$$

$$\text{or} \qquad V^2 = \frac{2}{2\pi}\int_0^{2\pi} (200 \sin\theta + 100 \sin 3\theta + 50 \sin 5\theta)^2 \, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (200^2 \sin^2\theta + 100^2 \sin^2 3\theta + 50^2 \sin^2 5\theta$$

$$+ 2 \times 200.100 \sin\theta . \sin 3\theta + 2 \times 100.50. \sin 3\theta . \sin 5\theta$$

$$+ 2 \times 50.200 . \sin 5\theta . \sin\theta)\, d\theta$$

$$= \frac{1}{2\pi}\left(\frac{200^2}{2} + \frac{100^2}{2} + \frac{50^2}{2}\right)2\pi = 26{,}250$$

$$\therefore \qquad V = \sqrt{26{,}250} = \mathbf{162\ V}$$

Alternative Solution

The r.m.s. value of individual components are $(200/\sqrt{2})$, $(100/\sqrt{2})$ and $(50/\sqrt{2})$. Hence, as stated in Art. 11.16,

$$V = \sqrt{V_1^2 + V_2^2 + V_3^2} = \sqrt{(200/\sqrt{2})^2 + (100/\sqrt{2})^2 + (50/\sqrt{2})^2} = \mathbf{162\ V}$$

11.20. R.M.S. Value of H.W. Rectified Alternating Current

Half-wave (H.W.) rectified alternating current is one whose one half-cycle has been suppressed *i.e.* one which flows for half the time during one cycle. It is shown in Fig. 11.20 where suppressed half-cycle is shown dotted.

As said earlier, for finding r.m.s. value of such an alternating current, summation would be carried over the period for which current *actually* flows *i.e.* from 0 to π, though it would be averaged for the whole cycle *i.e.* from 0 to 2π.

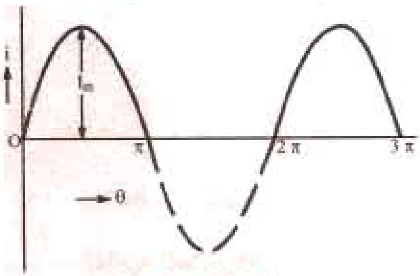

Fig. 11.20

∴ R.M.S. current

$$I = \sqrt{\left(\int_0^{\pi} \frac{i^2 d\theta}{2\pi}\right)} = \sqrt{\left(\frac{I_m^2}{2\pi}\int_0^{\pi} \sin^2\theta \, d\theta\right)}$$

$$= \sqrt{\frac{I_m^2}{4\pi}\int_0^{\pi}(1 - \cos 2\theta)\, d\theta}$$

$$= \sqrt{\left(\frac{I_m^2}{4\pi}\left|\theta - \frac{\sin 2\theta}{2}\right|_0^{\pi}\right)} = \sqrt{\left(\frac{I_m^2}{4\pi} \times \pi\right)} = \sqrt{\left(\frac{I_m^2}{4}\right)} \quad \therefore \quad I = \frac{I_m}{2} = \mathbf{0.5 I_m}$$

11.21. Average Value of H.W. Rectified Alternating Current

For the same reasons as given in Art. 11.20, integration would be carried over from 0 − π

$$\therefore \qquad I_{av} = \int_0^{\pi} \frac{i d\theta}{2\pi} = \frac{I_m}{2\pi}\int_0^{\pi} \sin\theta \, d\theta \qquad (\because \quad i = I_m \sin\theta)$$

$$= \frac{I_m}{2\pi}\left|-\cos\theta\right|_0^{\pi} = \frac{I_m}{2\pi} \times 2 = \frac{I_m}{\pi}$$

11.22. Form Factor of H.W. Rectified Alternating Current

$$\text{Form factor} = \frac{\text{r.m.s. value}}{\text{average value}} = \frac{I_m/2}{I_m/\pi} = \frac{\pi}{2} = 1.57$$

Example 11.8. *An alternating voltage e = 200 sin 314t is applied to a device which offers an ohmic resistance of 20 Ω to the flow of current in one direction, while preventing the flow of current in opposite direction. Calculate RMS value, average value and form factor for the current over one cycle.* **(Elect. Engg. Nagpur Univ. 1992)**

Solution. Comparing the given voltage equation with the standard form of alternating voltage equation, we find that $V_m = 200$ V, $R = 20\ \Omega$, $I_m = 200/20 = 10$ A. For such a half-wave rectified current, RMS value = $I_m/2 = 10/2 =$ **5A.**

Average current = $I_m/\pi = 10/\pi =$ **3.18 A** ; Form factor 5/3.18 = **1.57**

Example 11.9. *Compute the average and effective values of the square voltage wave shown in Fig. 11.21.*

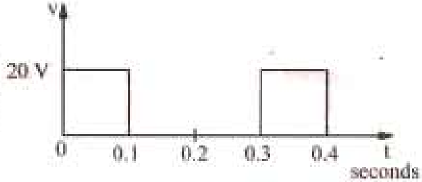

Fig. 11.21

Solution. As seen, for $0 < t < 0.1$ *i.e.* for the time interval 0 to 0.1 second, $v = 20$ V. Similarly, for $0.1 < t < 0.3$, $v = 0$. Also time-period of the voltage wave is 0.3 second.

$$\therefore \quad V_{av} = \frac{1}{T}\int_0^T v\ dt = \frac{1}{0.3}\int_0^{0.1} 20\ dt$$

$$= \frac{1}{0.3}(20 \times 0.1) = \mathbf{6.67\ V}$$

$$V^2 = \frac{1}{T}\int_0^T v^2 dt = \frac{1}{0.3}\int_0^{0.1} 20^2\ dt = \frac{1}{0.3}(400 \times 0.1) = 133.3;\ V = \mathbf{11.5\ V}$$

Example 11.10. *Calculate the RMS value of the function shown in Fig. 11.22 if it is given that for $0 < t < 0.1$, $y = 10(1 - e^{-100t})$ and $0.1 < t < 0.2$, $y = 10\,e^{-50(t-0.1)}$*

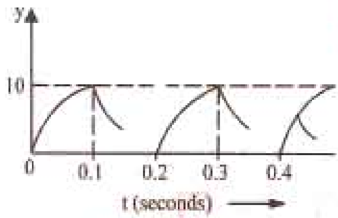

Fig. 11.22

Solution.

$$Y^2 = \frac{1}{0.2}\left\{\int_0^{0.1} y^2 dt + \int_{0.1}^{0.2} y^2 dt\right\}$$

$$= \frac{1}{0.2}\left\{\int_0^{0.1} 10^2\,(1 - e^{-100t})^2 dt + \int_{0.1}^{0.2} (10e^{-50(t-0.1)})^2 dt\right\}$$

$$= \frac{1}{0.2}\left\{\int_0^{0.1} 100\,(1 + e^{-200t} - 2e^{-100t})dt + \int_{0.1}^{0.2} 100\,e^{-100(t-0.1)}dt\right\}$$

$$= 500\left\{\left[t - 0.005e^{-200t} + 0.02\,e^{-100t}\right]_0^{0.1} + \left[-0.0\,e^{-100(t-0.1)}\right]_{0.1}^{0.2}\right\}$$

$$= 500\left\{\left[\left(0.1 - 0.005e^{-20} + 0.02e^{-10}\right) - \left(0 - 0.005 + 0.02\right)\right] + \left[\left(-0.01e^{-10}\right) - \left(-0.01\right)\right]\right\}$$

$$= 500 \times 0.095 = 47.5 \quad \therefore \quad Y = \sqrt{47.5} = \mathbf{6.9}$$

Example 11.11. *The half cycle of an alternating signal is as follows : It increases uniformaly from zero at $0°$ to F_m at $\alpha°$, remains constant form $\alpha°$ $(180 - \alpha)°$, decreases uniformaly from F_m at $(180 - \alpha)°$ to zero at $180°$. Calculate the average and effective values of the signal.*

(Elect. Science-I, Allahabad Univ. 1992)

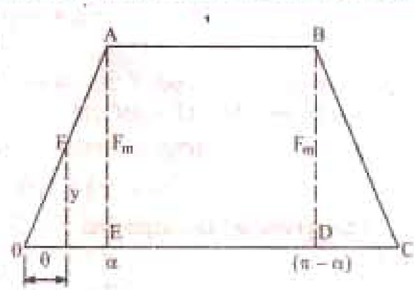

Fig. 11.23

Solution. For finding the average value, we would find the total area of the trapezium and divide it by π (Fig. 11.23).

Area = $2 \times \Delta\, OAE$ + rectangle $ABDE = 2 \times (1/2) \times F_m\alpha + (\pi - 2\alpha)\,F_m = (\pi - \alpha)\,F_m$

$\therefore$ average value $= (\pi - \alpha)\,F_m/\pi$

RMS Value From similar triangles, we get $\dfrac{y}{\theta} = \dfrac{F_m}{\alpha}$ or $y^2 \dfrac{Fm^2}{\alpha^2}\theta^2$

This gives the equation of the signal over the two triangles *OAE* and *DBC*. The signal remains constant over the angle α to $(\pi - \alpha)$ *i.e.* over an angular distance of $(\pi - \alpha) - \alpha = (\pi - 2\alpha)$

Sum of the squares = $\dfrac{2F_m^2}{\alpha^2}\int_\theta^\alpha \theta^2\, d\theta + F_m^2(\pi - 2\alpha)\; F_m^2(\pi - 4\alpha/3).$

The mean value of the squares is = $\dfrac{1}{\pi}F_m^2\left(\pi - \dfrac{4\alpha}{3}\right) = F_m^2\left(1 - \dfrac{4\alpha}{3\pi}\right)$

$$\text{r.m.s. value} = F_m\sqrt{\left(1 - \frac{4\,\alpha}{3\pi}\right)}$$

Example 11.12. *Find the average and r.m.s values of the a.c. voltage whose waveform is given in Fig. 11.24 (a)*

Solution. It is seen [(Fig. 11.24 (a)] that the time period of the waveform is 5s. For finding the average value of the waveform, we will calculate the net area of the waveform over one period and then find its average value for one cycle.

$A_1 = 20 \times 1 = 20\ V - s,\ A_2 = -5 \times 2 = -10\ \text{V} - \text{s}$

Net area over the full cycle $= A_1 + A_2 = 20 - 10 = 10\ \text{V} - \text{s}.$

Average value $= 10\ V\text{–s}/5\text{s} = 2\ \text{V}.$

Fig. 11.24 (*b*) shows a graph of $v^2\,(t)$. Since the negative voltage is also squared, it becomes positive.

Average value of the area $= 400\ V^2 \times 1\ \text{s} + 25\ V^2 \times 2\ \text{s} = 450\ V^2 - \text{s}$. The average value of the sum of the square $= 450\ V^2/\text{–s}/5\text{s} = 90V^2$ rms value $= \sqrt{90\,V^2} = 9.49$ V.

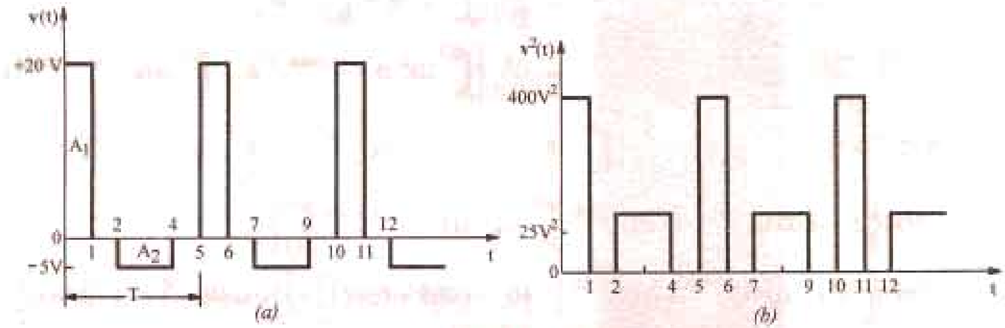

Fig. 11.24

Example 11.13. *What is the significance of the r.m.s and average values of a wave ? Determine the r.m.s. and average value of the waveform shown in Fig. 11.25*

(Elect. Technology, Indore Univ., 1984)

Solution. The slope of the curve *AB* is $BC/AC = 20/T$. Next, consider the function *y* at any time *t*. It is seen that $DE/AE = BC/AC = 10/T$

or $(y - 10)/t = 10/T$

or $y = 10 + (10/T)t$

This gives us the equation for the function for one cycle.

$$Y_{av} = \frac{1}{T}\int_0^T y\, dt = \frac{1}{T}\int_0^T\left(10 + \frac{10}{T}t\right)dt$$

$$= \frac{1}{T}\int_0^T\left[10.dt + \frac{10}{T}.t.\,dt\right] = \frac{1}{T}\left|10t + \frac{5t^2}{T}\right|_0^T = 15$$

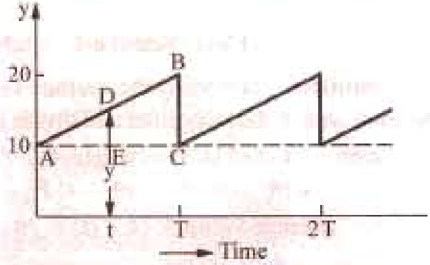

Fig. 11.25

$$\text{Mean square value} = \frac{1}{T}\int_0^T y^2 dt = \int_0^T \left(10+\frac{10}{T}t\right)^2 dt$$

$$= \frac{1}{T}\int_0^T \left(100+\frac{100}{T^2}t^2+\frac{200}{T}t\right)dt = \frac{1}{T}\left|100t+\frac{100t^3}{3T^2}+\frac{100t^2}{T}\right|_0^T = \frac{700}{3}$$

or RMS value $= 10\sqrt{7/3}$ $=$ **15.2**

Example 11.14. *For the trapezoidal current wave-form of Fig. 11.26, determine the effective value.*

(Elect. Technology, Vikram Univ. Ujjain 1985, Similar Example, Nagpur Univ. 1999)

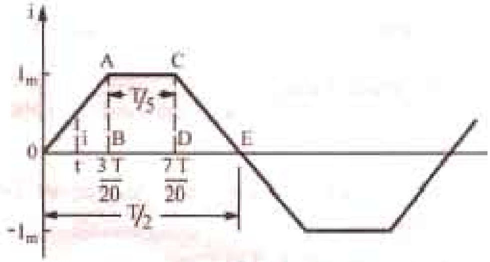

Fig. 11.26

Solution. For $0 < t < 3T/20$, equation of the current can be found from the relation

$$\frac{i}{t} = \frac{I_m}{3T/20} \quad \text{or } i = \frac{20I_m}{3T}.t$$

When $3T/20 < t < 7T/20$, equation of the current is given by $i = I_m$. Keeping in mind the fact that ΔOAB is identical with ΔCDE,

$$\text{RMS value of current} = \sqrt{\frac{1}{T/2}\left[2\int_0^{3T/20} i^2 dt + \int_{3T/20}^{7T/20} I_m^2 dt\right]}$$

$$= \sqrt{\frac{2}{T}\left[2\left(\frac{20I_m}{3T}\right)^2\int_0^{3T/20} t^2 dt + I_m^2\int_{3T/20}^{7T/20} dt\right]} = \frac{3}{5}I_m^2$$

$$\therefore \qquad I = \sqrt{(3/5)}.I_m = \mathbf{0.775\ I_m}$$

Incidentally, the average value is given by

$$I_{av} = \frac{2}{T}\left\{2\int_0^{3T/20} i dt + \int_{3T/20}^{7T/20} I_m dt\right\} = \frac{2}{T}\left\{2\int_0^{3T/20}\left(\frac{20I_m}{3T}\right)t dt + I_m\int_{3T/20}^{7T/20} dt\right\}$$

$$= \frac{2}{T}\left\{2\left(\frac{20I_m}{3T}\right)\left|\frac{t^2}{2}\right|_0^{3T/20} + I_m \left| t \right|_{3T/20}^{7T/20}\right\} = \frac{7}{10}.I_m$$

Example 11.15. *A sinusoidal alternating voltage of 110 V is applied across a moving-coil ammeter, a hot-wire ammeter and a half-wave rectifier, all connected in series. The rectifier offers a resistance of 25 Ω in one direction and infinite resistance in opposite direction. Calculate (i) the readings on the ammeters (ii) the form factor and peak factor of the current wave.*

(Elect. Engg.-I Nagpur Univ. 1992)

Solution. For solving this question, it should be noted that

(*a*) Moving-coil ammeter, due to the inertia of its moving system, registers the average current for the *whole* cycle.

(*b*) The reading of hot-wire ammeter is proportional to the average heating effect over the *whole* cycle. It should further be noted that in a.c. circuits, the given voltage and current values, unless indicated otherwise, always refer to r.m.s values.

$$E_m = 110/0.707 = 155.5 \text{ V (approx.)} ;\ I_m/2 = 155.5/25 = 6.22 \text{ A}$$

Average value of current for positive half cycle $= 0.637 \times 6.22 = 3.96$ A

Value of current in the negative half cycle is zero. But, as said earlier, due to inertia of the coil, M.C. ammeter reads the average value for the *whole* cycle.

(*i*) M.C. ammeter reading $= 3.96/2 =$ **1.98 A**

Let R be the resistance of hot-wire ammeter. Average heating effect over the positive half cycle is $\frac{1}{2} I_m^2 . R$ watts. But as there is no generation of heat in the negative half cycle, the average heating effect over the whole cycle is $\frac{1}{4} I_m^2 R$ watt.

Let I be the d.c. current which produces the same heating effect, then

$$I^2 R = \frac{1}{4} I_m^2 R \qquad \therefore \qquad I = I_m/2 = 6.22/2 = 3.11\ A.$$

Hence, hot-wire ammeter will read **3.11 A**

(*ii*) Form factor = $\frac{\text{r.m.s value}}{\text{average value}} = \frac{3.11}{1.98} =$ **1.57** ; Peak factor = $\frac{\text{max. value}}{\text{r.m.s. value}} = \frac{6.22}{3.11} =$ **2**

Example 11.16. *Find the form-factor of the wave form given in fig.*

[Nagpur University November 1991, Similar example, Sambalpur University 1985]

Solution.

Form-factor = $\frac{\text{RMS value}}{\text{Average value}}$

Average value of the current

$= 1/4 \times \int_0^4 (50/4) \times t \times dt = 25$ amp

Let RMS value of the current be I amp

$$I^2 \times 4 = \int_0^4 (12.5 \times t)^2 . dt$$

$$= \left[\frac{12.5 \times 12.5 \times t^2}{3}\right]_0^4 = (1/3) \times (12.5 \times 12.5 \times 4 \times 4 \times 4)$$

Thus $I = \frac{50}{\sqrt{3}}$ 28.87 amp. Hence, form factor $\frac{28.87}{2.5} = 1.1548$

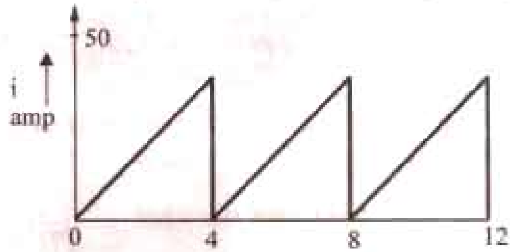

Fig. 11.27

Example 11.17. *A half-wave rectifier which prevents current flowing in one direction is connected in series with an a.c. ammeter and a permanent-magnet moving-coil ammeter. The supply is sinusoidal. The reading on the a.c. ammeter is 10 A. Find the reading given by the other ammeter. What should be the readings on the ammeters, if the other half-wave were rectified insteand of being cut off ?*

Solution. It should be noted that an a.c. ammeter reads r.m.s. value whereas the d.c. ammeter reads the average value of the rectified current.

As shown in Art. 11.20 from H.W. rectified alternating current, $I = I_m/2$ and $I_{av} = I_m/\pi$

As a.c. ammeter reads 10 A, hence r.m.s. value of the current is 10 A.

$\therefore \quad 10 = I_m/2 \quad$ or $\quad I_m = 20$ A

$\therefore \quad I_{av} = 20/\pi =$ **6.365 A** – reading of d.c. ammeter.

The full-wave rectified current wave is shown in Fig. 11.28. In this case mean value of i^2 over a complete cycle is given as

$$= 2\int_0^{\pi} \frac{i^2\, d\theta}{2\pi - 0} = \frac{1}{\pi}\int_0^{\pi} I_m^2\ 2\sin^2\theta\, d\theta$$

$$= \frac{I_m^2}{2\pi}\int_0^{\pi}(1-\cos 2\theta)d\theta = \frac{I_m^2}{2\pi}\left|\theta - \frac{\sin 2\theta}{2}\right|_0^{\pi} = \frac{I_m^2}{2}$$

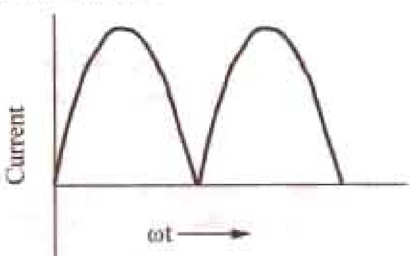

Fig: 11.28

$\therefore \quad I = I_m/\sqrt{2} = 20/\sqrt{2} = 14.14$ A $\therefore$ a.c. ammeter will read **14.14 A**

Now, mean value of i over a complete cycle

$$= \frac{2\int_0^{\pi} I_m \sin\theta\, d\theta}{2\pi} \;=\; \frac{I_m}{\pi}\int_0^{\pi}\sin\theta d\theta = \frac{I_m}{\pi}\left|-\cos\theta\right|_0^{\pi} = \frac{2I_m}{\pi} = \frac{2\times 20}{\pi} = \mathbf{12.73\ A}$$

This value, as might have been expected, is twice the value obtained in the previous case.

$\therefore$ d.c. ammeter will read **12.73 A.**

Example 11.18. *A full-wave rectified sinusodial voltage is clipped at $1/\sqrt{2}$ of its maximum value. Calculate the average and RMS values of such a voltage.*

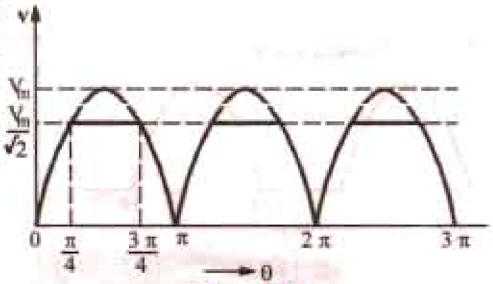

Fig. 11.29

Solution. As seen from Fig. 11.29, the rectified voltage has a period of π and is represented by the following equations during the different intervals.

$$0 < \theta < \pi/4\ ;\ v = V_m \sin\theta$$

$$\pi/4 < \theta < 3\pi/4\ ;\ v = V_m/\sqrt{2} = 0.707\, V_m$$

$$3\pi/4 < \theta < \pi\ ;\ v = V_m \sin\theta$$

$$\therefore\quad Y_{av} = \frac{1}{\pi}\left\{\int_0^{\pi/4} v\,d\theta + \int_{\pi/4}^{3\pi/4} v\,d\theta + \int_{3\pi/4}^{\pi} v\,d\theta\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi/4} V_m \sin\theta\, d\theta + \int_{\pi/4}^{3\pi/4} 0.707\, V_m d\theta + \int_{3\pi/4}^{\pi} V_m \sin\theta\, d\theta\right\}$$

$$= \frac{V_m}{\pi}\left\{\left|-\cos\theta\right|_0^{\pi/4} + 0.707\left|\theta\right|_{\pi/4}^{3\pi/4} + \left|-\cos\theta\right|_{3\pi/4}^{\pi}\right\} = \frac{V_m}{\pi}(0.293 + 1.111 + 0.293) = \mathbf{0.54\ V_m}$$

$$V^2 = \frac{1}{\pi}\left\{\int_0^{\pi/4} V_m^2 \sin^2\theta\, d\theta + \int_{\pi/4}^{3\pi/4} (0.707\, V_m)^2 d\,\theta + \int_{3\pi/4}^{\pi} V_m^2 \sin^2\theta\, d\theta\right\} = 0.341\, V_m^2$$

$$\therefore\qquad V = \mathbf{0.584\ V_m}$$

Example 11.19. *A delayed full-wave rectified sinusoidal current has an average value equal to half its maximum value. Find the delay angle θ.* **(Basic Circuit Analysis, Nagpur 1992)**

Solution. The current waveform is shown in Fig. 11.30.

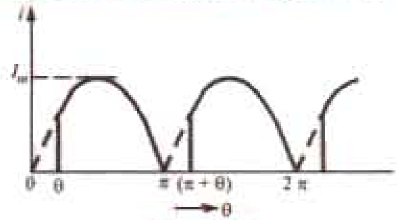

Fig. 11.30

$$I_{av} = \frac{1}{\pi}\int_0^{\pi} I_m \sin\theta\, d\theta = \frac{I_m}{\pi}\ (-\cos\pi + \cos\theta)$$

Now, $I_{av} = I_m/2$

$$\therefore\quad \frac{I_m}{\pi}(-\cos\pi + \cos\theta) = \frac{I_m}{2}$$

$$\therefore\quad \cos\theta = 0.57,\ \theta = \cos^{-1}(0.57) = \mathbf{55.25^\circ}$$

Example 11.20. *The waveform of an output current is as shown in Fig. 11.31. It consists of a portion of the positive half cycle of a sine wave between the angle θ and 180°. Determine the effective value for θ = 30°.*

(Elect. Technology, Vikram Univ. 1984)

Solution. The equation of the given delayed half-wave rectified sine wave is $i = I_m \sin\omega t = I_m \sin\theta$. The effective value is given by

$$I = \sqrt{\frac{1}{2\pi}\int_{\pi/6}^{\pi} i^2 d\theta} \ \text{ or } I^2 = \frac{1}{2\pi}\int_{\pi/6}^{\pi} I_m^2 \sin^2\theta d\theta$$

$$= \frac{I_m^2}{4\pi}\int_{\pi/6}^{\pi}(1-\cos 2\theta)\, d\theta = \frac{I_m^2}{4\pi}\left|\left(\theta - \frac{\sin 2\theta}{2}\right)\right|_{\pi/6}^{\pi}$$

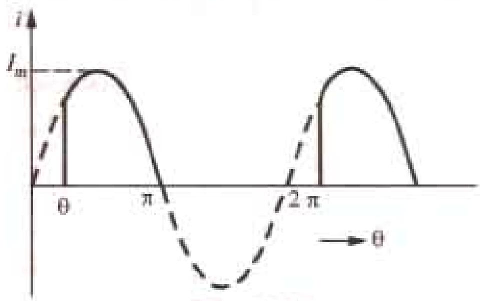

Fig. 11.31

$$= 0.242\, I_m^2$$

or $\quad I = \sqrt{0.242 I_m^2} = \mathbf{0.492\ I_m}$

Example 11.21. *Calculate the "form factor" and "peak factor" of the sine wave shown in Fig. 11.32.*

(Elect. Technology-I, G walior Univ. 1988)

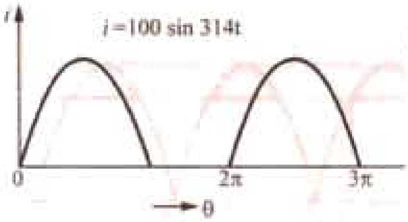

Fig. 11.32

Solution. For $0 < \theta < \pi$, $i = 100 \sin\theta$ and for $\pi < \theta < 2\pi$, $i = 0$. The period is 2π.

$$\therefore \quad I_{av} = \frac{1}{2\pi}\left\{\int_0^{\pi} i\,d\theta + \int_{\pi}^{2\pi} 0\,d\theta\right\}$$

$$= \frac{1}{2\pi}\left\{100\int_0^{\pi}\sin\theta\, d\theta\right\} = 31.8\text{ A}$$

$$I^2 = \frac{1}{2\pi}\int_0^{\pi} i^2 d\theta = \frac{100^2}{2\pi}\int_0^{\pi}\sin^2\theta\, d\theta = \frac{100^2}{4} = 2500\,;\; I = 50\text{A}$$

$\therefore$ form factor = 50/31.8 = **1.57** ; peak factor = 100/50 = 2

Example 11.22. *Find the average and effective values of voltage of sinusodial waveform shown in Fig. 11.33.*

(Elect. Science-I Allahabad Univ. 1991)

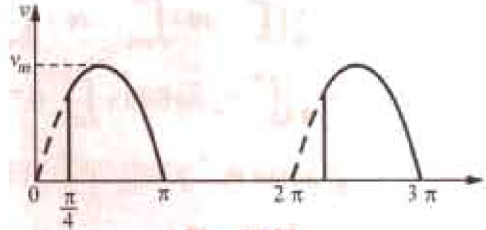

Fig. 11.33

Solution. Although, the given waveform would be integrated from π/4 to π, it would be averaged over the whole cycle beause it is unsymmetrical. The equation of the given sinusoidal waveform is $v = 100 \sin\theta$.

$$\therefore \quad V_{av} = \frac{1}{2\pi}\int_{\pi/4}^{\pi} 100\sin\theta\, d\theta = \frac{100}{2\pi}\left| -\cos\theta \right|_{\pi/4}^{\pi} = 27.2\text{ V}$$

$$V^2 = \frac{1}{2\pi}\int_{\pi/4}^{\pi} 100^2\sin^2\theta\, d\theta = \frac{100^2}{4\pi}(1-\cos 2\theta)\, d\theta = \frac{100^2}{4\pi}\left|\theta - \frac{\sin 2\theta}{2}\right|_{\pi/4}^{\pi} = \frac{100^2}{4\pi}\left(\pi - \frac{\pi}{4} + \frac{1}{2}\right)$$

$$\therefore \quad V = 47.7\text{ V}$$

Example 11.23. *Find the r.m.s. and average values of the saw tooth waveform shown in Fig. 11.34 (a).*

Solution. The required values can be found by using either graphical method or analytical method.

Graphical Method

The average value can be found by averaging the function from $t = 0$ to $t = 1$ in parts as given below :

$$\text{Average value of } (f) = \frac{1}{T}\int_0^T f(t)\, dt = \frac{1}{T} \times \text{(net area over one cycle)}$$

Now, area of a right-angled triangle = (1/2) × (base) × (altitude).

Hence, area of the triangle during $t = 0$ to $t = 0.5$ second is

$$A_1 = \frac{1}{2}\times(\Delta t)\times(-2) = \frac{1}{2}\times\frac{1}{2}\times -2 = -\frac{1}{2}$$

Similarly, area of the triangle from $t = 0.5$ to $t = 1$ second is

$$A_2 = \frac{1}{2}\times(\Delta t)\times(+2) = \frac{1}{2}\times\frac{1}{2}\times 2 = \frac{1}{2}$$

Net area from $t = 0$ to $t = 1.0$ second is $A_1 + A_2 = -\frac{1}{2} + \frac{1}{2} = 0$

Hence, average value of $f(t)$ over one cycle is zero.

For finding the r.m.s. value, we will first square the ordinates of the given function and draw a new plot for $f^2(t)$ as shown in Fig. 11.34 (*b*). It would be seen that the squared ordinates from a parabola.

Area under parabolic curve = $\frac{1}{3}\times$ base $\times$ altitude. The area under the curve from $t = 0$ to $t = 0.5$ second is : $A_1 = \frac{1}{3}(\Delta t)\times 2^2 = \frac{1}{3}\times\frac{1}{2}\times 4 = \frac{2}{3}$

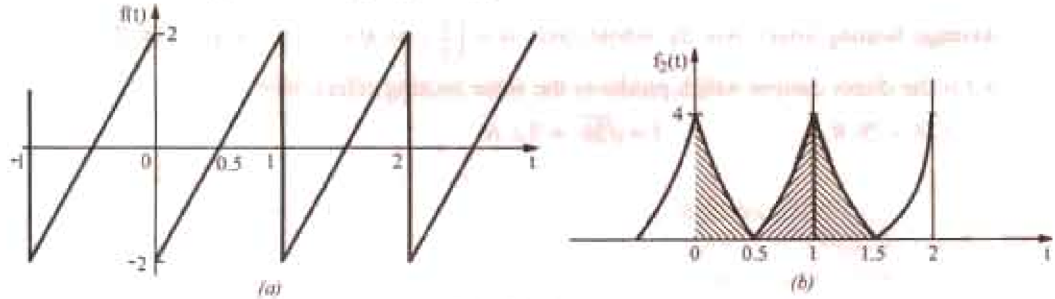

Fig. 11.34

Similarly, for $t = 0.5$ to $t = 1.0$ second $A_2 = \frac{1}{3}(\Delta t)\times 4 = \frac{1}{3}\times\frac{1}{2}\times 4 = \frac{2}{3}$

Total area $A_1 + A_2 = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$, r.m.s. value = $\sqrt{\frac{1}{T}\int_0^T f^2(t)\,dt} = \sqrt{\text{average of } f^2(t)}$

$\therefore$ r.m.s. value $= \sqrt{4/3} =$ **1.15**

Analytical Method

The equation of the straight line from $t = 0$ to $t = 1$ in Fig. 11.34 (*a*) is

$$f(t) = 4t - 2;\ f^2(t) = 16t^2 - 16t + 4$$

$$\text{Average value } \frac{1}{T}\int_0^T (4t-2)\,dt = \frac{1}{T}\left|\frac{4t^2}{2} - 2t\right|_0^T = 0$$

$$\text{r.m.s value} = \sqrt{\frac{1}{T}\int_0^T (16t^2 - 16t + 4)dt} = \sqrt{\frac{1}{T}\left|\frac{16t^3}{3} - \frac{16t^2}{2} + 4t\right|_0^T} = \mathbf{1.15}$$

Example 11.24. *A circuit offers a resistance of 20 Ω in one direction and 100 Ω in the reverse direction. A sinusoidal voltage of maximum value 200 V is applied to the above circuit in series with*

(a) a moving-iron ammeter *(b) a moving-coil ammeter*

(c) a moving-coil instrument with a full-wave rectifier *(d) a moving-coil ammeter.*

Calculate the reading of each instrument.

Solution. (*a*) The deflecting torque of an *M/I* instrument is proportional to (current)^2. Hence, its reading will be proportional to the average value of i^2 over the whole cycle. Therefore, the reading of such an instrument :

$$= \sqrt{\left[\frac{1}{2\pi}\left(\int_0^{\pi} 10^2\sin^2\theta d\theta + \int_{\pi}^{2\pi} 2^2\sin^2\theta d\theta\right)\right]}$$

$$= \sqrt{\left[\frac{1}{2\pi}\left(\frac{100}{2}\left|\theta - \frac{\sin 2\theta}{2}\right|_0^{\pi} + \frac{4}{2}\left|\theta - \frac{\sin 2\theta}{2}\right|_{\pi}^{2p}\right)\right]} = \sqrt{26} = \mathbf{5.1\ A}$$

(*b*) An M/C ammeter reads the average current over the whole cycle.

Average current over positive half-cycle is = 10 × 0.637 = 6.37 A

Average current over positive half-cycle is = − 2 × 0.637 = − 1.27 A

$\therefore$ Average value over the whole cycle is = (6.37 − 1.27)/2 = **2.55 A** (*c*) In this case, due to the full-wave rectifier, the current passing through the operating coil of the instrument would flow in the positive direction during both the positive and negative half cycles.

$\therefore$ reading = (6.37 + 1.27)/2 = **3.82 A**

(*d*) Average heating effect over the positive half-cycle is $= \frac{1}{2} I_{m_1}^2 R$

Average heating effect over the negative half-cycle is $= \frac{1}{2} I_{m_2}^2 R$

where $I_{m1} = 200/20 = 10$ A; $I_{m2} = 200/100 = 2$ A

Average heating effect over the whole cycle is $= \left(\frac{1}{2}\times 10^2 R + \frac{1}{2}\times 2^2 \times R\right)/2 = 26\,R$

If I is the direct current which produces the same heating effect, then

$I^2 R = 26\,R \qquad \therefore\ I = \sqrt{26} = $ **5.1 A**

Example 11.25. *A moving coil ammeter, a hot-wire ammeter and a resistance of 100 Ω are connected in series with a rectifying device across a sinusoidal alternating supply of 200 V. If the device has a resistance of 100 Ω to the current in one direction and 5.00 Ω to current in opposite direction, calculate the readings of the two ammeters.*

(Elect. Theory and Meas. Madras University, 1985)

Solution. R.M.S. current in one direction is = 200/(100 + 100) = 1 A

Average current in the first *i.e.* positive half cycle is = 1/1.11 = 0.9 A

Similarly, r.m.s. value in the negative half-cycle is = − 200/(100 + 500) = − 1/3 A

Average value = (−1/3)/1.11 = −0.3 A

Average value over the whole cycle is = (0.9 − 0.3)/2 = 0.3 A

Hence, M/C ammeter reads **0.3 A**

Average heating effect during the +ve half cycle $= I_{rms}^2 \times R = I^2 \times R = R$

Similarly, average heating effect during the −ve half-cycle is $= (-1/3)^2 \times R = R/9$

Here, R is the resistance of the hot-wire ammeter.

Average heating effect over the whole cycle is $= \frac{1}{2}\left(R + \frac{R}{9}\right) = \frac{5R}{9}$

If I is the direct current which produces the same heating effect, than

$I^2 R = 5R/9 \qquad \therefore\ I = \sqrt{5}/3 = 0.745$ A

Hence, hot-wire ammeter indicates **0.745 A**

Example 11.26. *A resultant current wave is made up of two components : a 5A d.c. component and a 50-Hz a.c. component, which is of sinusoidal waveform and which has a maximum value of 5A.*

(*i*) *Draw a sketch of the resultant wave.*

(*ii*) *Write an analytical expression for the current wave, reckoning t = 0 at a point where the a.c. component is at zero value and where di/dt is positive.*

(*iii*) What is the average value of the resultant current over a cycle ?

(*iv*) What is the effective or r.m.s. value of the resultant current ? **[Similar Problem: Bombay Univ. 1996]**

Solution. (*i*) The two current components and resultant current wave have been shown in Fig. 11.35.

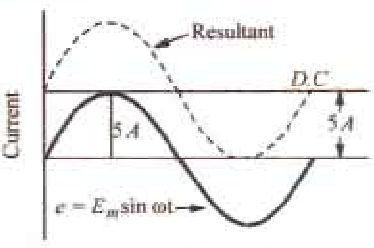

Fig. 11.35

(*ii*) Obviously, the instantaneous value of the resultant current is given by $i = (5 + 5 \sin \omega t) = (5 + 5 \sin \theta)$

(*iii*) Over one complete cycle, the average value of the alternating current is zero. Hence, the average value of the resultant current is equal to the value of d.c. component *i.e.* **5A**

(*iv*) Mean value of i^2 over complete cycle is

$$= \frac{1}{2\pi}\int_0^{2\pi} i^2\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} (5 + 5\sin\theta)^2\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(25 + 50\sin\theta + 25\sin^2\theta)\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[25 + 50\sin\theta + 25\left(\frac{1-\cos 2\theta}{2}\right)\right]d\theta = \frac{1}{2\pi}\int_0^{2\pi}(37.5+50\sin\theta - 12.5\cos 2\theta)\,d\theta$$

$$= \frac{1}{2\pi}\left| 3.7.5\cos\theta - 50\,\theta\frac{12.5}{2}\sin 2\theta\right|_0^{2\pi} = \frac{75\pi}{2\pi} = 37.5\text{ A} \;\therefore\; \text{R.M.S. value } I = \sqrt{37.5} = \mathbf{6.12\ A}$$

Note. In general, let the combined current be given by $i = A + B\sqrt{2}\sin\omega t = A + B\sqrt{2}\sin\theta$ where A represents the value of direct current and B the r.m.s. value of alternating current.

The r.m.s. value of combined current is given by

$$I_{rms} = \sqrt{\left(\frac{1}{2\pi}\int_0^{2\pi} i^2\,d\theta\right)} \text{ or } I^2_{r.m.s.} = \frac{1}{2\pi}\int_0^{2\pi} i^2\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}(A + B\sqrt{2}\sin\theta)^2\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(A^2 + 2B^2\sin^2\theta + 2\sqrt{2}\,AB\sin\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}(A^2 + B^2 - B^2\cos 2\theta + 2\sqrt{2}\,AB\sin)\,d\theta$$

$$= \frac{1}{2\pi}\left[A^2\theta + B^2\theta - \frac{B^2\sin 2\theta}{2} - 2\sqrt{2}AB\cos\right]_0^{2\pi} = \frac{1}{2\pi}\left[2\pi A^2 + 2\pi B^2 - 2\sqrt{2}\,2AB + 2\sqrt{2}AB\right]$$

$$= A^2 + B^2 \quad \therefore \quad I_{rms} = \sqrt{(A^2 + B^2)}$$

The above example could be easily soved by putting $A = 5$ and $B = 5/\sqrt{2}$ (because $B_{max} = 5$)

$$\therefore \quad I_{rms} = \sqrt{5^2 + (5/\sqrt{2})^2} = \mathbf{6.12\ A}$$

Example 11.27. *Determine the r.m.s. value of a semi-circular current wave which has a maximum value of a.*

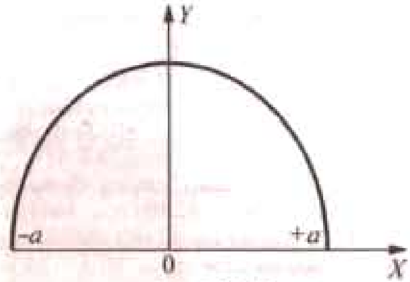

Fig. 11.36

Solution. The equation of a semi-circular wave (shown in Fig. 11.36) is

$$x^2 + y^2 = a^2 \quad \text{or} \quad y^2 = a^2 - x^2$$

$$\therefore \quad I_{rms} = \sqrt{\frac{1}{2a}\int_{-a}^{+a} y^2 dx} \quad \text{or} \quad I^2_{rms} = \frac{1}{2a}\int_{-a}^{+a}(a^2 - x^2)\,dx$$

$$= \frac{1}{2a}\int_{-a}^{+a}(a^2\,dx - x^2\,dx) = \frac{1}{2a}\left|a^2x - \frac{x^3}{3}\right|_{-a}^{+a} = \frac{1}{2a}\left(a^3 - \frac{a^3}{3} + a^3 - \frac{a^3}{3}\right) = \frac{2a^2}{3}$$

$$\therefore \quad I_{rms} = \sqrt{2a^2/3} = \mathbf{0.816\ a}$$

Example 11.28. *Calculate the r.m.s. and average value of the voltage wave shown in Fig. 11.37.*

Solution. In such cases, it is difficult to develop a single equation. Hence, it is usal to consider two equations, one applicable from 0 to 1 and an other form 1 to 2 millisecond.

For t lying between 0 and 1 ms, $v_1 = 4$, For t lying between 1 and 2 ms, $v_2 = -4t + 4$

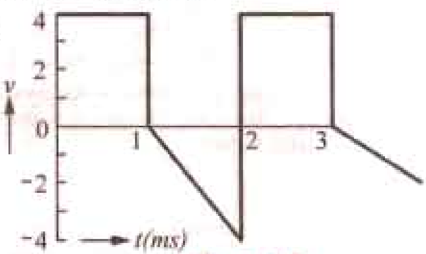

Fig. 11.37

$$\therefore \quad v_{rms} = \sqrt{\frac{1}{2}\left(\int_0^1 v_1^2\,dt + \int_1^2 v_2^2\,dt\right)}$$

$$V^2_{rms} = \frac{1}{2}\left[\int_0^1 4^2\,dt + \int_1^2(-4t+4)^2\,dt\right]$$

$$= \frac{1}{2}\left[\left|16t\right|_0^1 + \left|\frac{16t^3}{3}\right|_1^2 + \left|16t\right|_1^2 - \left|\frac{32t^2}{2}\right|_1^2\right]$$

$$= \frac{1}{2}\left[16+\frac{16\times8}{3}-\frac{16}{3}+16\times2-16\times1-\frac{32\times4}{2}+\frac{32\times1}{2}\right]=\frac{32}{3} \quad \therefore \; V_{rms}=\sqrt{32/3}=\mathbf{3.265\ volt}$$

$$V_{av} = \frac{1}{2}\left[\int_0^1 v_1 dt + \int_1^2 v_2\, dt\right] = \frac{1}{2}\left[\int_0^1 4dt + \int_1^2 (-4t+4)\, dt\right] = \frac{1}{2}\left[\left|4t\right|_0^1 + \left|-\frac{4t^2}{2}+4t\right|_1^2\right] = \mathbf{1\ volt}$$

Totorial Problems No. 11.1

1. Calculate the maximum value of the e.m.f. generated in a coil which is rotating at 50 rev/s in a uniform magnetic field of 0.8 Wb/m^3. The coil is wound on a square former having sides 5 cm in length and is wound with 300 turns. **[188.5 V]**

2. (*a*) What is the peak value of a sinusoidal alternating current of 4.78 r.m.s. amperes ?
(*b*) What is the r.m.s. value of a rectangular voltage wave with an amplitude of 9.87 V ?
(*c*) What is the average value of a sinusoidal alternating current of 31 A maxmium value ?
(*d*) An alternating current has s periodic time of 0.03 second. What is its frequency ?
(*e*) An alternating current is represented by i = 70.7 sin 520 *t*. Determine (*i*) the frequency (*ii*) the current 0.0015 second after passing through zero, increasing positively.
[6.76 A ; 9.87 V ; 19.75 A ; 33.3 Hz ; 82.8 Hz ; 49.7 A]

3. A sinusoidal alternating voltage has an r.m.s. value of 200 V and a frequency of 50 Hz. It crosses the zero axis in a positive direction when $t = 0$. Determine (*i*) the time when voltage first reaches the instantaneous value of 200 V and (*ii*) the time when voltage after passing through its maximum positive value reaches the value of 141.4 V.
[(*i*) (0.0025 second (*ii*) 1/300 second)]

4. Find the form factor and peak factor of the triangular wave shown in Fig. 11.38 **[1.155; 1.732]**

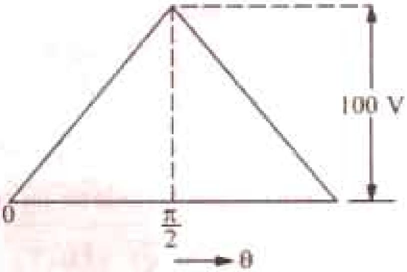

Fig. 11.38

5. An alternating voltage of 200 sin 471 *t* is applied to a h.w. rectifier which is in series with a resistance of 40 Ω. If the resistance of the rectifier is infinite in one direction and zero in the other, find the r.m.s. value of the current drawn from the supply source. **[2.5 A]**

6. A sinusoidally varying alternating current has an average value of 127.4 A. When its value is zero, then its rate change is 62,800 A/s. Find an analytical expression for the sine wave. **[][*i* = 200 sin 100 π*t*]**

7. A resistor carries two alternating currents haveing the same frequency and phase and having the same value of maximum current *i.e.* 10 A. One is sinusoidal and the other is rectangular in waveform. Find the r.m.s. value of the resultant current. **[12.24 A]**

8. A copper-oxide rectifier and a non-inductive resistance of 20 Ω are connected in series across a sinusoidal a.c. supply of 230 V (r.m.s.). The resistance of the rectifier is 2.5 Ω in forward direction and 3,000 Ω in the reverse direction. Calculate the r.m.s. and average values of the current.
[r.m.s. value = 5.1 A, average value = 3.22 A]

9. Find the average and effective values for the waveshape shown in Fig. 11.39 if the curves are parts of a sine wave. **[27.2 V, 47.7V]** (*Elect. Technology, Indore Univ. 1979*)

10. Find the effective value of the resultant current in a wire which carries simultaneously a direct current of 10 A and a sinusoidal alternating current with a peak value of 15 A.
[14.58A] (*Elect. Technology, Vikram Univ. Ujjain 1978*)

11. Determine the r.m.s. value of the voltage defined by $e = 5 + 5 \sin (314\, t + \pi/6)$
[6.12 V] (*Elect. Technology, Indore Univ. July 1979*)

12. Find the r.m.s. value of the resultant current in a wire which carries simultaneously a direct current of 10 A and a sinusoidal alternating current with a peak value of 10 A. **[12.25 A]** (*Elect. Technology-I; Delhi Univ. 1981*)

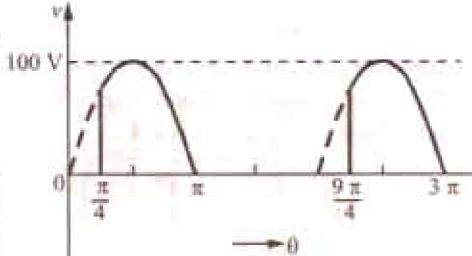

Fig. 11.39

13. An alternating voltage given by e = 150 sin 100π*t* is applied to a circuit which offers a resistance of 50 ohms to the current in one direction and completely prevents the flow of current in the opposite direction. Find the r.m.s.

and average vlaues of this current and its form factor.

[1.5 A, 0.95 A, 1.57] (*Elect. Technology, Indore Univ. Nov. 1978*)

14. Find the relative heating effects of three current waves of equal maximum value, one rectangular, the second semi-circular and the third sinusoidal in waveform [1: 2/2, 1/2] (*Sheffield Univ. U.K.*)

15. Calculate the average and root mean-square value, the form factor and peak factor of a periodic current wave have the following values for equal time intervals over half-cycle, changing suddenly from one value of the next. [0, 40, 60, 80, 100, 80, 60, 40, 0] (*A.M.I.E. June 1992*)

16. A sinusoidal alternating voltage of amplitude 100 V is applied across a circuit containing a rectifying device which entirely prevents current flowing in one direction and offers a resistance of 10 ohm to the flow of current in the other direction. A hot wire ammeter is used for measuring the current. Find the reading of instrument. (*Elect. Technology. Punjab Univ. May 1988*)

11.23. Representation of Alternating Quantities

It has already been pointed out that an attempt is made to obtain alternating voltages and currents having sine waveform. In any case, a.c. computations are based on the assumption of sinusodial voltages and currents. It is, however, cumbersome to continuously handle the instantaneous values in the form of equations of waves like $e = E_m \sin \omega t$ etc. A conventional method is to employ vector method of representing these sine waves. These vectors may then be manipulated instead of the sine functions to achieve the desired result. *In fact, vectors are a shorthand for the representation of alternating voltages and currents and their use greatly simplifies the problems in a.c. work.*

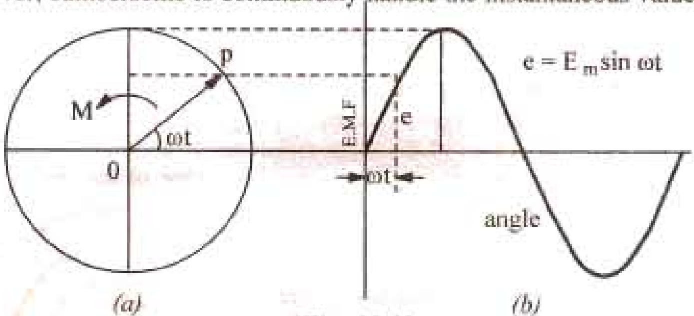

Fig. 11.40

A vector is a physical quantity which has magnitude as well as direction. Such vector quantities are completely known when particulars of their magnitude, direction and the sense in which they act, are given. They are graphically represented by straingth lines called vectors. The length of the line represents the magnitude of the alternating quantity, the inclination of the line with respect to some axis of reference gives the direction of that quantity and an arrow-head placed at one end indicates the direction in which that quantity acts.

The alternating voltages and currents are represented by such vectors rotating counter-clockwise with the same frequency as that of the alternating quantity. In Fig. 11.40 (*a*), *OP* is such a vector which represents the maximum value of the alternating current and its angle with *X* axis gives its phase. Let the alternating current be represented by the equation $e = E_m \sin \omega t$. It will be seen that the projection of *OP* and *Y*-axis at any instant gives the instantaneous value of that alternating current.

$$\therefore \quad OM = OP \sin \omega t \text{ or } e = OP \sin \omega t = E_m \sin \omega t$$

It should be noted that a line like *OP* can be made to represent an alternating voltage of current if it satisfies the following conditions :

(*i*) Its length should be equal to the peak or maximum value of the sinusoidal alternating current to a suitable scale. (*ii*) It should be in the horizontal position at the same instant as the alternating quantity is zero and increasing. (*iii*) Its angular velocity should be such that it completes one revolution in the same time as taken by the alternating quantity to complete one cycle.

11.24. Vector Diagram using R.M.S. Values

Instead of using maximum values as above, it is very common practice to draw vector diagrams using r.m.s. values of alternating quantities. But it should be understood that in that case, the projection of the rotating vector on the *Y*-axis does not give the instantaneous value of that alternating quantity.

11.25. Vector Diagrams of Sine Waves of Same Frequency

Two or more sine waves of the same frequency can be shown on the same vector diagram because the various vectors representing different waves all rotate counter-clockwise at the same frequency and maintain a fixed position relative to each other. This is illustrated in Fig. 11.41 where a voltage e and current i of the same frequency are shown. The current wave is supposed to pass upward through zero at the instant when $t = 0$ while at the same time the voltage wave has already advanced an angle α from its zero value. Hence, their equations can be written as

$$i = I_m \sin \omega t$$

and $$e = E_m \sin (\omega t + \alpha)$$

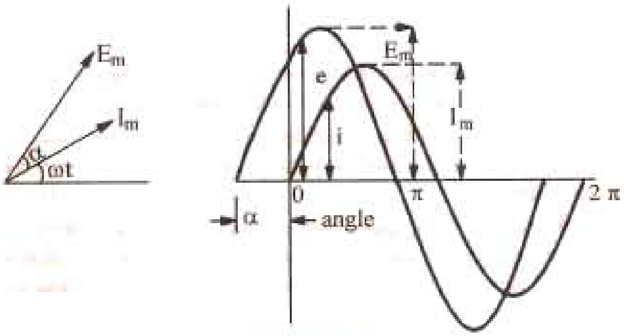

Fig. 11.41

Sine wave of different frequencies cannot be represented on the same vector diagram in a still picture because due to difference in speed of different vectors, the phase angles between them will be continuously changing.

11.26. Addition of Two Alternating Quantities

In Fig. 11.42 (*a*) are shown two rotating vectors representing the maximum values of two sinusoidal voltage waves represented by $e_1 = E_{m1} \sin \omega t$ and $e_2 = E_{m2} \sin (\omega t - \phi)$. It is seen that the sum of the two sine waves of the same frequency is another sine wave of the same frequency but of a different maximum value and phase. The value of the instantanous resultant voltage e_r, at any instant is obtained by algebraically adding the projections of the two vectors on the Y-axis. If these projections are e_1 and e_2, then, $e_r = e_1 + e_2$ at that time. The resultant curve is drawn in this way by adding the ordinates. It is found that the resultant wave is a sine wave of the same frequency as the component waves but lagging behind E_{m1} by an angle α. The vector diagram of Fig. 11.42 (*a*) can be very easily drawn. Lay off E_{m2} lagging $\phi°$ behind E_{m1} and then complete the parallelogram to get E_r.

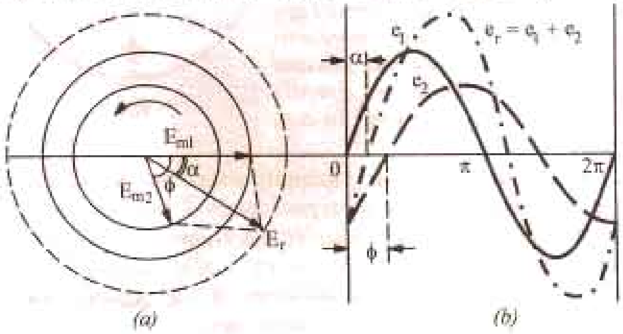

Fig. 11.42

Example 11.29. *Add the following currents as waves and as vectors.*

$$i_1 = 7 \sin \omega t \text{ and } i_2 = 10 \sin (\omega t + \pi/3)$$

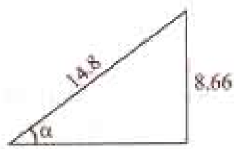

Fig. 11.43

Solution. As Waves

$$i_r = i_1 + i_2 = 7 \sin \omega t + 10 \sin (\omega t + 60°)$$
$$= 7 \sin \omega t + 10 \sin \omega t \cos 60° + 10 \cos \omega t \sin 60°$$
$$= 12 \sin \omega t + 8.66 \cos \omega t$$

Dividing both sides by $\sqrt{(12^2 + 8.66^2)} = 14.8$, we get

$$i_r = 14.8 \left(\frac{12}{14.8} \sin \omega t + \frac{8.66}{14.8} \cos \omega t \right)$$
$$= 14.8 (\cos \alpha \sin \omega t + \sin \alpha \cos \omega t)$$

where $\cos \alpha = 12/14.8$ and $\alpha = 8.66/14.8$ —as shown in Fig. 11.43

$\therefore$ $i_r = 14.8 \sin (\omega t + \alpha)$

where $\tan \alpha = 8.66.12$ or $\alpha = \tan^{-1}(8.66/12) = 35.8°$

$\therefore$ $i_r = 14.8 \sin (\omega t + 35.8°)$

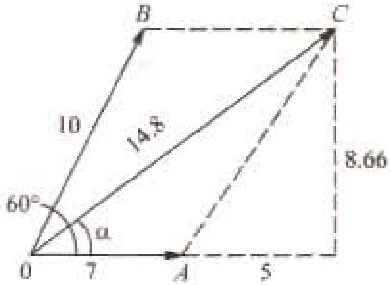

Fig. 11.44

As Vectors

Vector diagram is shown in Fig. 11.44. Resolving the vectors into their horizontal and vertical components, we have

$$X\text{ – component} = 7 + 10\cos 60° = 12$$

$$Y\text{ – component} = 0 + 10\sin 60° = 8.66$$

$$\text{Resultant} = \sqrt{(12^2 + 8.66^2)} = 14.8 \text{ A}$$

and $\alpha = \tan^{-1}(8.66/12) = 35.8°$

Hence, the resultant equation can be written as

$$i_r = 14.8 \sin(\omega t + 35.8°)$$

11.27. Addition and Subtractrion of Vectors

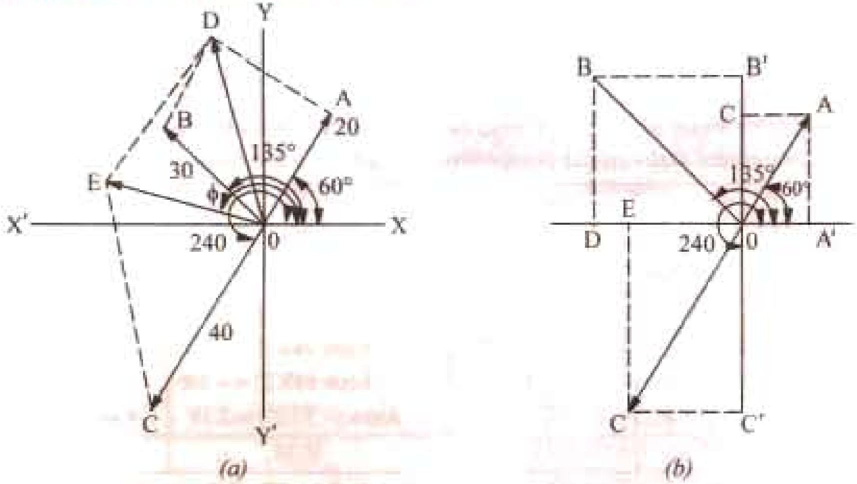

Fig. 11.45

(*i*) **Addition.** In a.c. circuit problems we may be concerned with a number of alternating voltages or currents of the same frequency but of different phases and it may be required to obtain the resultant voltage or current. As explained earlier (Art. 11.23) if the quantities are sinusoidal, they may be represented by a number of rotating vectors having a common axis of rotation and displaced from one another by fixed angles which are equal to the phase differences between the respective alternating quantities. The instantaneous value of the resultant voltage is given by the algebraic sum of the projections of the different vectors on Y-axis. The maximum value (or r.m.s. value if the vectors represent that value) is obtained by compounding the several vectors by using the parallelogram and polygon laws of vector addition.

However, another easier method is to resolve the various vectors into their X-and Y-components and then to add them up us as shown in Example 11.30 and 31.

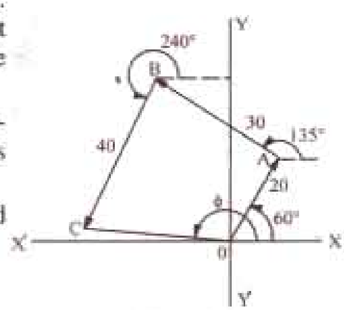

Fig. 11.46

Suppose we are given the following three alernating e.m.fs. and it is required to find the equation of the resultant e.m.f.

$$e_1 = 20\sin(\omega t + \pi/3)$$

$$e_2 = 30\sin(\omega t + 3\pi/4)$$

$$e_3 = 40\sin(\omega t + 4\pi/3)$$

Then the vector diagram can be drawn as explained before and solved in any of the following three ways :

(*i*) By compounding according to parallelogram law as in Fig. 11.45 (*a*) (*ii*) By resolving the various vectors into their *X*-and *Y*-components as in Fig. 11.45 (*b*).

(*iii*) By laying off various vectors end-on end at their proper phase angles and then measuring the closing vector as shown in Fig. 11.46.

Knowing the magnitude of the resultant vector and its inclination ϕ with *X* axis, the equation of the resultant e.m.f. can be written as $e = E_m \sin(\omega t + \phi)$.

Example 11.30. *Represent the following quantities by vectors :*

5 sin (2π ft – 1) ; 3 cos (2π ft + 1) ; 2 sin (2π ft + 2.5) and 4 sin (2π ft – 1)

Add the vectors and express the result in the form : A sin (2π ft ± Φ)

Solution. It should be noted that all quantities have the same frequency *f*, hence they can be represented vertically on the same vector diagram and added as outlined in Art. 11.27. But before doing this, it would be helpful to express all the quantities as sine functions. Therefore, the second expression 3 cos (2π *ft* + 1) can be written as

$$3\sin\left(2\pi ft + 1 + \frac{\pi}{2}\right) = 3\sin(2\pi ft + 1 + 1.57) = 3\sin(2\pi ft + 2.57)$$

The maxmimum value of each quantity, its phase with respect to the quantity of reference *i.e.* *X* sin 2π*ft*, its horizontal and vertical components are given in the table below :

Quantity	*Max. value*	*Phase*		*Horizontal component*	*Vertical component*
		radians	*angles*		
(*i*) 5 sin (2π*ft* – 1)	5	–1	–57.3°	5 × cos (–57.3°) = 2.7	5 sin (–57.3°) = – 4.21
(*ii*) 3 sin (2π*ft* + 2.57)	3	+2.57	147.2°	3 × cos 147.2° = – 2.52	3 sin 147.2° = 1.63
(*iii*) 2 sin (2π*ft* + 2.5)	2	+ 2.5	143.2°	2 cos 143.2° = – 1.6	2 sin 143.2° = 1.2
(*iv*) 4 sin (2π*ft* – 1)	4	– 1	– 57.3°	4 cos (– 57.3°) = 2.16	4 sin (–57.3°) = – 3.07
Total				0.74	– 4.75

The vector diagram is shown in Fig. 11.47 in which *OA*, *OB*, *OC* and *OD* represent quantities (*i*), (*ii*), (*iii*) and (*iv*) given in the table.

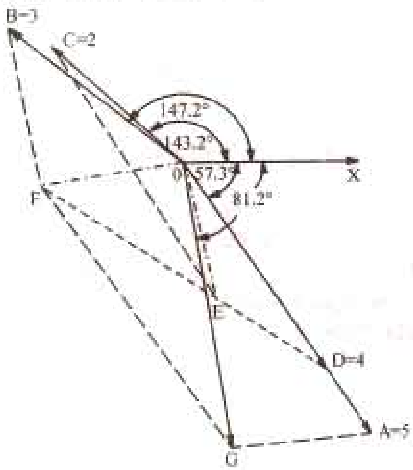

Fig. 11.47

Fig. 11.48

Their resultant is given by *OG* and the net horizontal and vertical components are shown in Fig. 11.48. Resultant $= \sqrt{[0.74^2 + (-4.75)^2]} = 4.81$ and $\tan\theta = -4.75 / 0.74$

$\therefore \quad \theta = \tan^{-1}(-4.75/0.74) = -81.2° = -1.43$ radians

The equation of the resultant quantity is **4.81 sin (2π*ft* – 1.43)**

Example 11.31. *Three voltages represented by*

$$e_1 = 20 \sin \omega t;\ e_2 = 30 \sin (\omega t - \pi/4) \text{ and } e_3 = 40 \cos (\omega t + \pi/6)$$

act together in a circuit. Find an expression for the resultant voltage. Represent them by appropriate vectors. **(Electro-technics Madras Univ. 1981)** *(Elec. Circuit Nagpur Unvi. 1991)*

Solution. First, let us draw the three vectors representing the maximum values of the given alternating voltages.

$e_1 = 20$ sing ωt – here phase angle with X-axis is zero, hence the vector will be drawn parallel to the X-axis

$e_2 = 30 \sin (\omega t - \pi/4)$ – its vector will be below OX by 45°

$e_3 = 40 \cos (\omega t + \pi/6) = 40 \sin (90° + \omega t + \pi/6)$*

$= 40 \sin (\omega t + 120°)$ – its vector will be at 120° with OX in CCW direction.

These vectors are shown in Fig. 11.49 (*a*). Resolving them into X-and Y-components, we get

X - component = 20 + 30 cos 45° - 40 cos 60°= 21.2 V

Y - component = 40 sin 60° – 30 sin 45° = 13.4 V

As seen from Fig. 11.49 (*b*), the maximum value of the resultant voltage is

$$OD = \sqrt{21.2^2 + 13.4^2} = 25.1 \text{ V}$$

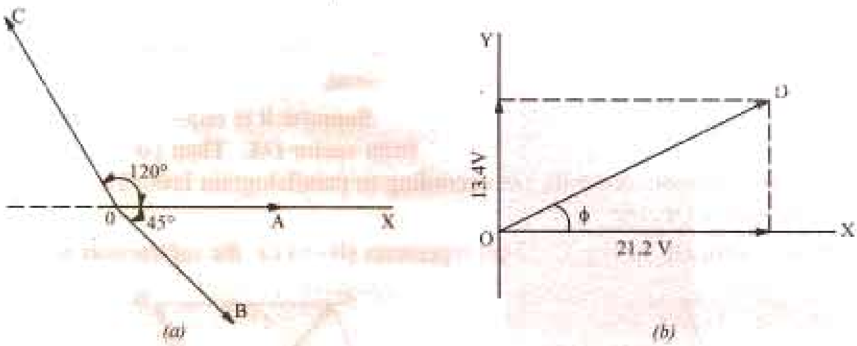

Fig. 11.49

The phase angle of the resultant voltage is given by $\tan \phi = \dfrac{\text{Y-compenent}}{\text{X-component}} = \dfrac{13.4}{27.2} = \mathbf{0.632}$

$\therefore \quad \phi = \tan^{-1} 0.632 = 32.3° = $ **0.564 radian**

The equaiton of the resultant voltage wave is $e = 25.1 \sin (\omega t + 32.3°)$ or $e = 25.1 \sin (\omega t + 0.564)$

Example 11.32. *Four circuits A, B, C and D are connected is series across a 240 V, 50-Hz supply. The voltages across three of the circuits and their phase angles relative to the current through them are, V_A, 80 V at 50° leading, V_B, 120 V at 65° lagging · V_C, 135 V at 80° leading. If the supply voltage leads the current by 15°, find from a vector digram drawn to scale the voltage V_D across the circuit D and its phase angle.*

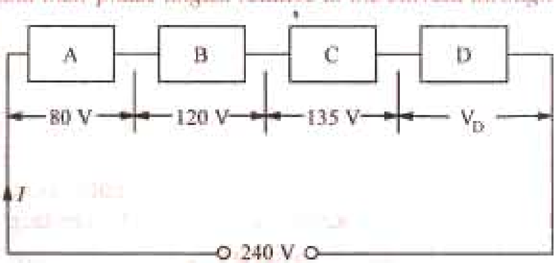

Fig. 11.50

Solution. The circuit is shown in Fig. 11.50.

(*a*) The vector diagram is shown in Fig. 11.50. The current vector OM is drawn horizontally and is taken as reference vector. Taking a scale of 1 cm = 20 V, vector OA is drawn 4 cm in length and leading OM by an angle of 50°. Vector

* $\cos \theta = \sin (90° + \theta)$

OB represents 120 V and is drawn lagging behind OM by 65°. Their vector sum, as found by Parallelogram Law of Vectors, is given by vector OG.

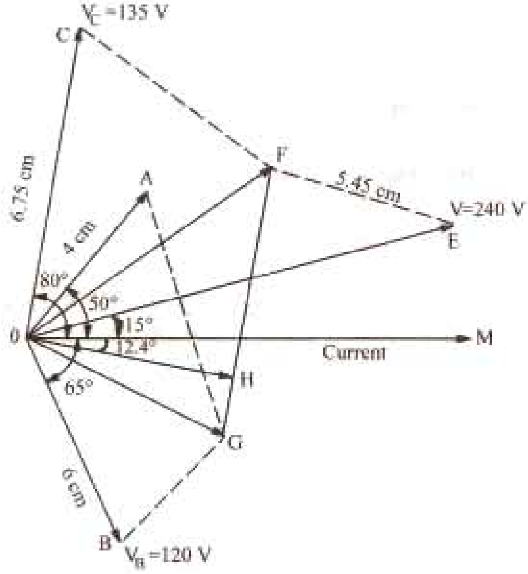

Fig. 11.51

Next, vector OC is drawn ahead of OM by 80° representing 135 V. Vector OF represents the vector sum of OG and OC. Vector OE represesnets the applied votlage of 240 V and is drawn 15° ahead of current vector OM. The vector difference of OE and OF gives the required voltage V_D. It is equal to FE. It measures 5.45 cm which means that it represents 20 × 5.45 = 109 V. This vector is transferred to position OH by drawing OH parallel to FE. It is seen that OH lags behind the current vector OM by 12.4°.

Hence, V_D' = **109** volts lagging behind the current by **12.4°**.

(*ii*) Subtraction of Vectors

If difference of two vectors is required, then one of the vectors is reversed and this reversed vector is then compounded with the other vector as usual.

Suppose it is required to subtract vector OB from vector OA. Then OB is reversed as shown in Fig. 11.52 (*a*) and compounded with OA according to parallelogram law. The vector difference (**A-B**) is given by vector OC.

Similarly, the vector OC in Fig. 12.52 (*b*) represents (**B–A**) *i.e.* the subtraction of **OA** form **OB**.

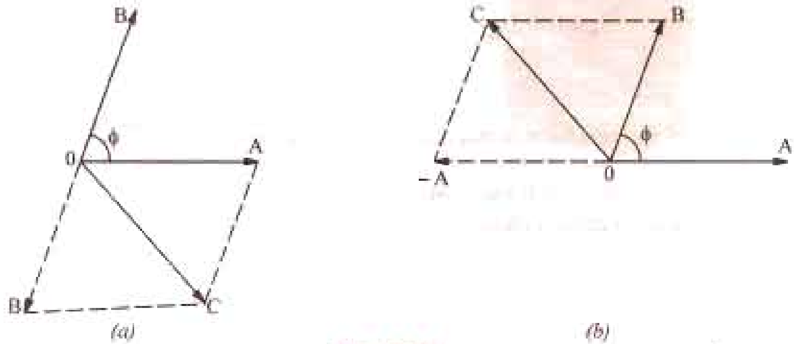

Fig 11.52

Example 11.33. *Two currents i_1 and i_2 are given by the expressions*

$i_1 = 10 \sin(314t + \pi/4)$ amperes and $i_2 = 8 \sin(313\,t - \pi/3)$ amperes

Find (a) $i_1 + i_2$ and (b) $i_1 - i_2$. Express the answer in the form $i = I_m \sin(314\,t \pm \phi)$

Solution. (*a*) The current vectors representing maximum values of the two currents are shown in Fig. 11.53 (*a*). Resolving the currents into their X-and Y-components, we get

X- component = 10 cos 45° + 8 cos 60° = $10/\sqrt{2} + 8/2 = 11.07$ A

Y- component = 10 sin 45° − 8 sin 60 ° = 0.14 A

$\therefore \quad I_m = \sqrt{11.07^2 + 0.14^2} = 11.08$ A

$\tan\phi = (0.14/11.07) = 0.01265 \quad \therefore \quad \phi = 44'$

Hence, the equation for the resultant current is i = 11.08 sin (314 t + 44′) amperes

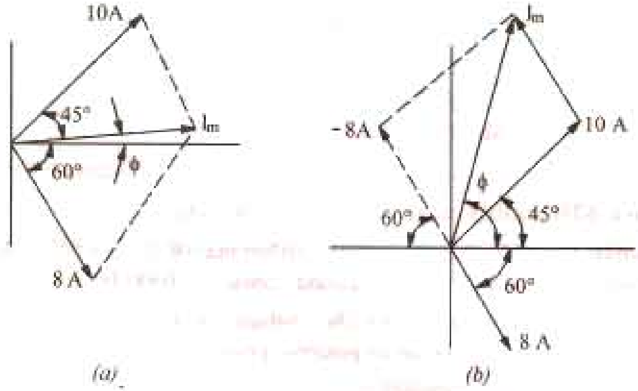

Fig. 11.53

(b) X – component = 10 cos 45° – 8 cos 60° = 3.07 A

Y – component = 10 sin 45° + 8 sin 60° = 14 A

$$\therefore \quad I_m = \sqrt{3.07^2 + 14^2} = 14.33 \text{ A} \qquad \text{...Fig. 11.53 } (b)$$

$$\phi = \tan^{-1}(14/3.07) = 77° \; 38'$$

Hence, the equation of the resultant current is

$$i = 14.33 \sin (314 + 77° \; 38') \text{ amperes}$$

Example 11.34. *The maximum values of the alternating voltage and current are 400 V and 20 A respectively in a circuit connected to a 50 Hz supply. The instantaneous values of voltage and current are 283 V and 10 A respectively at time t = 0, both increasing positively.*

(i) Write down the expression for voltage and current at time t.

(ii) Determine the power consumed in the circuit.

Take the voltage and current to be sinusoidal **[Nagpur University, Nov. 1998]**

Solution. $V_m = 400$, $I_m = 20$, $\omega = 314$ rad./sec

(i) Let the expressions be as follows :

$v(t) = V_m \text{ Sin } (\omega t + \theta_1) = 400 \text{ Sin } (314\, t + \theta_1)$

$I(t) = I_m \text{ Sin } (\omega t + \theta_2) = 20 \text{ Sin } (314\, t + \theta_2)$

where θ_1 and θ_2 indicate the concerned phase-shifts with respect to some reference. Substituting the given instantaneous values at $t = 0$,

$\theta_1 = 45°$ and $\theta_2 = 30°$

The required expressions are :

$V(t) = 400 \text{ Sin } (314\, t + 45°)$

$i(t) = 20 \text{ Sin } (314\, t + 30°)$

Thus, the voltage leads the current by 15°.

V = RMS voltage = 400/1.414 t = 283 V

I = RMS voltage = 20/1.414 = 14.14 A

Power-factor, cos ϕ = cos 15° = 0.966 lagging, since current lags behind the voltage.

(ii) Power = V I cos ϕ = 3865 watts

Additional Hint : Draw these two wave forms

Example 11.35. *Voltage and Current for a circuit with two elements in series are expersed as follows :*

$$v(t) = 170 \sin(6280\,t + \pi/3) \text{ Volts}$$
$$i(t) = 8.5 \sin(6280\,t + \pi/2) \text{ Amps}$$

(i) Plot the two waveforms. (ii) Determine the frequency in Hz. (iii) Determine the power factor stating its nature. (iv) What are the values of the elements ?

[Nagpur University, April 1996]

Solution. (*ii*) ω = 6280 radiation/sec, $f = \omega/2\pi$ = 1000 Hz

(*i*) Two sinusoidal waveforms with a phase-difference of 30° (= $\pi/2 - \pi/3$) are to be drawn. Each waveform completes a cycle in 1 milli-second, since f = 1000 Hz.

The waveform for current leads that for the voltage by 30°. At ωt = 0, the current is at its positive peak, while the votlage will be at its positive peak for $\omega t = \pi/6$ = 30°. Peak value are 170 volts and 8.5 amp.

(*iii*) RMS value of votlage = $170/\sqrt{2} = 120$ volts

RMS value of current = $8.5/\sqrt{2} = 6$ amp.

Impedance = V/I = 120/6 = 20 ohms

Power factor = Cos 30°, Leading = 0.866, leading.

Since the current leads the voltage, the two elements must be R and C.

$R = Z \cos\phi = 20 \times 0.866 = 17.32$ ohms

$X_c = Z \sin\phi = 20 \times 0.50 = 10$ ohms

$$C = 1/(\omega X_c) = \frac{1000 \times 1000}{6280 \times 10} = 15.92 \text{ mF}$$

Example 11.36. *Three sinusoidally alternating currents of rms values 5,7.5, and 10 A are having same frequency of 50 Hz, with phase angles of 30°, – 60° and 45°.*

(i) Find their average values. (ii) Write equations for their instantaneous values. (iii) Draw waveforms and phasor diagrams taking first current as the reference. (iv) Find their instantaneous values at 100 mSec from the original reference. **[Nagpur University, Nov. 1996]**

Solution. (*i*) Average value of and alternating quantity in case of sinusoidal nature of variation = (RMS Value)/1.11

Average value of first current = 5/1.11 = 4.50 A

Average value of second current = 7.5/1.11 = 6.76 A

Average value of third current = 10/1.11 = 9.00 A

(*ii*) Instantaneous Values : $\omega = 2\pi \times 50 = 314$ rad/sec

$i_1(t) = 5\sqrt{2}\ \text{Sin}\,(314\,t + 30°)$

$i_2(t) = 7.5\sqrt{2}\ \text{Sin}\,(314\,t - 60°)$

$i_3(t) = 10\sqrt{2}\ \text{Sin}\,(314\,t + 45°)$

(*iii*) First current is to be taken as a reference, now. From hte expressions, second current lags behind the first current by 90°. Third current leads the first current by 15°. Waveforms with this description are drawn in Fig. 11.54 (*a*) and the phasor diagrams, in Fig. 11.54 (*b*).

(*iv*) A 50 Hz a.c. quantity completes a cycle in 20 m sec. In 100 m sec, it completes five cycles. Original reference is the starting point required for this purpose. Hence, at 100 m sec from the reference.

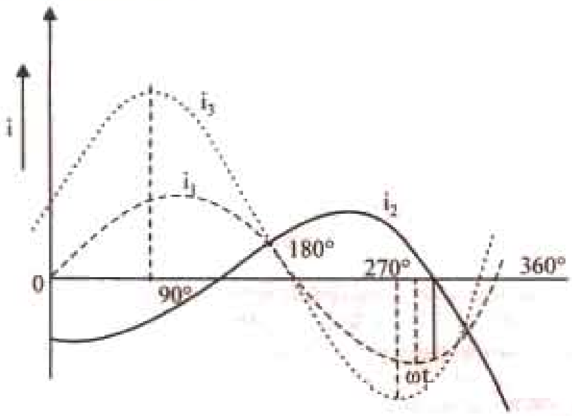

Fig. 11.54 (a)

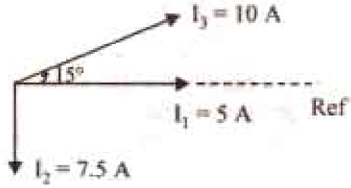

Fig. 11.54 (b)

(v) instantaneous value of $i_1(t) = 5\sqrt{2}\,\text{Sin } 30° = 3.53$ A

instantaneous value of $i_2(t) = 7.5\sqrt{2}\,\text{Sin }(-60°) = -9.816$ A

instantaneous value of $i_3(t) = 10\sqrt{2}\,\text{Sin }(45°) = 10$ A

Example 11.37. *Determine the form factor and peak factor for the unshaded waveform,in Fig. 11.55.* **[Bombay University, 2000]**

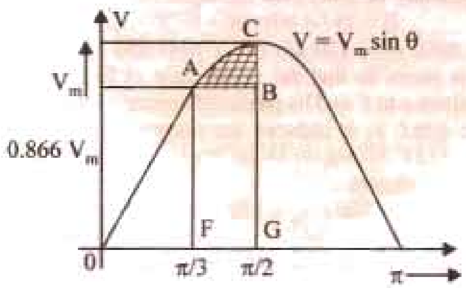

Fig. 11.55

Solution. $v(\theta) = V_m \sin\theta$, except for the region between

$\theta = 60°$ to $\theta = 90°$, wherein $v = 0.866\, V_m$.

Area under the curve will be worked out first, for calculating the average value.

Area OAF = $V_m \int_0 \sin\theta d\theta = 0.5\, V_m$ Area

FABG = $0.866\, V_m\,(\pi/2 - \pi/3) = 0.4532\, V_m$

Area GCD = V_m. Total area under the curve = $V_m\,(1 + 0.432 + 0.5)$

Average value, $V_{av} = (1.9532\, V_m)/3.14$

$= 0.622\, V_m$

For evaluating rms value, the square of the function is to be taken, its mean value calculated and square-root of the mean value found out.

Area under the squared function :

For Portion OF : $V_m^2 \int_0^{\pi/2} \sin^2\theta d\theta = V_m^2/2 \int_0^{\pi/2} (1 - \cos 2\theta)\, d\theta = 0.307\, V_m^2$

For Portion FG : $(0.866\, V_m)^2 \times \pi \times (1/2 - 1/3) = 0.3925\, V_m^2$

For Portion GD : $V_m^2 \int_0^{\pi/2} \sin^2\theta\, d\theta = 0.785\, V_m^2$

Total area = $V_m^2\,[0.307 + 0.3925 + 0.785] = 1.4845\, V_m^2$

Let R.M.S. Value be V_e

$V_e^2\,\pi = 1.4845\, V_m^2$, or $V_e = 0.688\, V_m$

Form factor = RMS Value/Average Value = 0.688/0.622 = 1.106

Peak factor = Peak Value/RMS Value = 1.0/0.688 = 1.4535

Tutorial Problems No. 11.2

1. The values of the instantaneous currents in the branches of a parallel circuit are as follows :
$i_1 = 5 \sin 346\, t$; $i_2 = 10 \sin (346t + \pi/4)$; $i_3 = 7.5 \sin (346\, t + \pi/2)$; $i_4 = 8 \sin (346\, t - \pi/3)$
Express the resultant line current in the same form as the original expression and determine the r.m.s. value and the frequency of this current. **[12.5 A; 55 Hz]**
2. Four coils are connected in series. Each has induced in it a sinusoidal e.m.f. of 100 V, 50 Hz and there is a phase difference of 14 electrical degrees between one coil and the next. What is the total e.m.f. generated in the circuit ? **[384 V]**
3. The instantaneous voltage across each of the four coils connected in series is given by
$v_1 = 100 \sin 471\, t$; $v_2 = 250 \cos 471\, t$; $v_3 = 150 \sin (471\, t + \pi/6)$; $v_4 = 200 \sin (471\, t - \pi/4)$
Determine the total p.d. exprssed in similar form to those given. What will be the resultant p.d. If v_2 is reversed in sign ? **[$v = 414 \sin (471\, t + 26.5°)$; $v = 486 \sin (471\, t - 40°)$]**
4. An alternating voltage of $v = 100 \sin 376.8\, t$ is applied to a circuit consisting of a coil having a resistance of 6Ω and an inductance of 21.22 mH.
(*a*) Express the current flowing in the circuit in the form $i = Im \sin (376.8\, t \pm \phi)$
(*b*) If a moving-iron voltmeter, a wattmeter and a frequency meter are connected in the circuit, what would be the respective readings on the instruments ?
[$i = 10 \sin (376.8\, t - 53.1°)$; 70.7 V; 300 W; 60 Hz]
5. Three circuits *A*, *B* and *C* are connected in series across a 200-V supply. The voltage across circuit *A* is 50 V lagging the supply voltge by 45° and the voltage across circuit *C* is 100 V leading the supply voltage by 30°. Determine graphically or by calculation, the voltage across circuit *B* and its phase displacement from the supply voltage. **[79.4 V ; 10° 38′ lagging]**
6. Three alternating currents are given by
$i_1 = 141 \sin (\omega t + \pi/4)$ $i_2 = 30 \sin (\omega t + \pi/2)$ $i_3 = 20 \sin (\omega t - \pi/6)$
and are fed into a common conductor. Find graphically or otherwise the equation of the resultant current and its r.m.s. value. **[$i = 167.4 \sin (\omega t + 0.797)$, $I_{rms} = 118.4$ A]**
7. Four e.m.fs $e_1 = 100 \sin \omega t$, $e_2 = 80 \sin (\omega t - \pi/6)$, $e_3 = 120 \sin (\omega t + \pi/4)$ and $e_4 = 100 \sin (\omega t - 2\pi/3)$ are induced in four coils connected in series so that the vector sum of four e.m.fs. is obtained. Find graphically or by calcultion the resultant e.m.f. and its phase difference with (*a*) e_1 and (*b*) e_2.If the connections to the coil in which the e.m.f. e_2 is induced are reversed, find the new resultant e.m.f **[208 sin (ωt – 0.202) (*a*) 11°34′ lag (*b*) 18′26′ lead; 76 sin (ωt + 0.528)]**
8. Draw to scale a vector diagram showing the follwing voltages :
$v_1 = 100 \sin 500\, t$; $v_2 = 200 \sin (500\, t + \pi/3)$; $v_3 = -50 \cos 500\, t$; $v_4 = 150 \sin (500\, t - \pi/4)$
Obtain graphically or otherwise, their vector sum and express this in the form $V_m \sin (500\, t \pm \phi)$, using v_1 as the reference vector. Give the r.m.s. value and frequency of the resultant voltage.
[360.5 sin (500 *t* + 0.056); 217 V; 79.6 Hz]

11.28. A.C. Through Resistance, Inductance and Capacitance

We will now consider the phase angle introduced between an alternating voltage and current when the circuit contains resistance only, inductance only and capcitance only. *In each case, we will assume that we are given the alternating voltage of equation $e = E_m \sin \omega t$* and will proceed to find the equation and the phase of the alternating current produced in each case.

11.29. A.C. Through Pure Ohmic Resistance Alone

The circuit is shown in Fig. 11.56. Let the applied voltage be given by the equation.

$$v = V_m \sin \theta = V_m \sin \omega t \qquad ...(i)$$

Let R = ohmic resistnace ; i = instantaneous current

Obviously, the applied voltage has to supply ohmic voltage drop only. Hence for equilibrium

$$v = iR;$$

Putting the value of 'v' from above, we get $V_m \sin \omega t = iR$; $i = \dfrac{V_m}{R} \sin \omega t$ ...(*ii*)

Current 'i' is maximum when $\sin \omega t$ is unity $\therefore$ $I_m = V_m/R$ Hence, equation (*ii*) becomes, $i = I_m \sin \omega t$ ...(*iii*)

Comparing (*i*) and (*ii*), we find that the alternating voltage and current are in phase with each other as shown in Fig. 11.57. It is also shown vectorially by vectors V_R and I in Fig. 11.54.

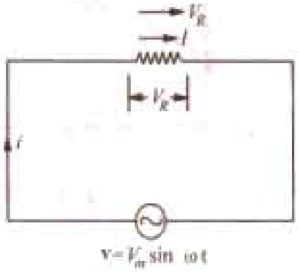

Fig. 11.56

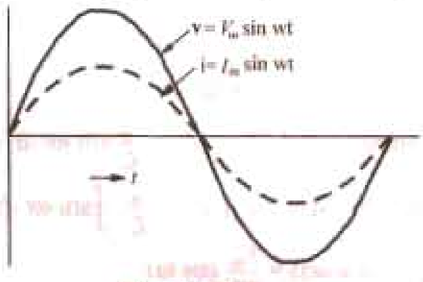

Fig. 11.57

Power. Instantaneous power, $p = vi = V_m I_m \sin^2 \omega t$ ...(Fig. 11.58)

$$= \frac{V_m I_m}{2}(1 - \cos 2\omega t) = \frac{V_m I_m}{2} - \frac{V_m I_m}{2}\cos 2\,\omega t$$

Power consists of a constant part $\frac{V_m I_m}{2}$ and a fluctuating part $\frac{V_m I_m}{2}$ cos 2 ωt of frequency double that of voltage and current waves. For a complete cycle, the average value of $\frac{V_m I_m}{2}$ cos $2\omega t$ is zero.

Hence, power for the whole cycle is

$$P = \frac{V_m I_m}{2} = \frac{V_m}{\sqrt{2}} \times \frac{I_m}{\sqrt{2}}$$

or $\quad P = V \times I$ watt

where $\quad V$ = r.m.s. value of applied voltage.

$\quad\quad\quad I$ = r.m.s. value of the current.

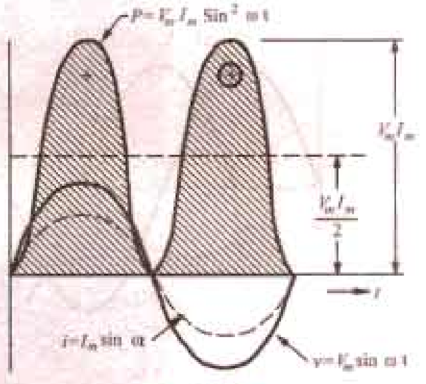

Fig. 11.58

It is seen from Fig. 11.58 that no part of the power cycle becomes negative at any time. In other words, in a purely resistive circuit, power is never zero. This is so because the instantaneous values of votlage and current are always either both positive or negative and hence the product is always positive.

Example 11.38. *A 60-Hz voltage of 115 V (r.m.s.) is impressed on a 100 ohm resistnace :*

(i) Write the time equations for the voltage and the resulting current. Let the zero point of the voltage wave be at t = 0 (ii) Show the voltage and current on a time diagram. (iii) Show the voltage and current on a phasor diagram.

[Elect Techonology. Hyderabad Univ. 1992, Similar Example, U.P. Technical Univ. 2001]

Solution. (*i*) $V_{max} = \sqrt{2}\,V = \sqrt{2} \times 115 = 163$ V

$I_{max} = V_{max}/R = 163/100 = 1.63$ A; $\phi = 0$; $\omega = 2\pi f = 2\pi \times 60 = 377$ rad/s

The required equations are : $v(t) = 1.63 \sin 377\,t$ and $i(t) = 1.63 \sin 377\,t$

(*ii*) and (*iii*) These are similar to those shown in Fig. 11.56 and 11.57

11.30. A.C. Through Pure Inductance Alone

Whenever an alternating voltage is applied to a purely inductive coil*, a back e.m.f. is produced due to the self-inductance of the coil. The back e.m.f., at every step, opposes the rise or fall of

* By purely inductive coil is meant one that has no ohmic resistance and hence no I^2R loss. Pure inductance is actually no attainable, though it is very nearly approached by a coil wound with such thick wire that its resistance is negligible. If it has some actual resistance, then it is represented by a separate equivalent resistance joined in series with it.

current through the coil. As there is no ohmic voltage drop, the applied voltage has to overcome this self-induced e.m.f. only. So at every step

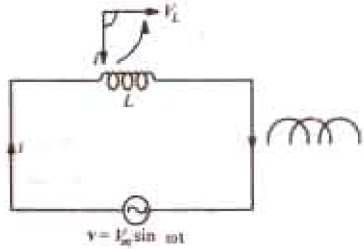

Fig. 11.59

$$v = L\frac{di}{dt}$$

Now $\quad v = V_m \sin \omega t$

$$\therefore \quad V_m \sin \omega t = L\frac{di}{dt} \quad \therefore \; di = \frac{V_m}{L} \sin \omega t \, dt$$

Integrating both sides, we get $i = \frac{V_m}{L} \int \sin \omega t \, dt$

$$= \frac{V_m}{\omega L}(-\cos \omega t) = \frac{V_m}{\omega L} \cos \omega t \qquad \text{...(constant of integration = 0)}$$

$$\therefore \quad = \frac{V_m}{\omega L} \sin\left(\omega t - \frac{\pi}{2}\right) = \frac{V_m}{X_L} \sin(\omega t - \pi/2) \qquad \text{...(ii)}$$

Max. value of i is $I_m = \frac{V_m}{\omega L}$ when $\sin\left(\omega t - \frac{\pi}{2}\right)$ is unity.

Hence, the equation of the current becomes $i = I_m \sin(\omega t - \pi/2)$.

So, we find that if applied voltage is represented by $v = V_m \sin \omega t$, then current flowing in a *purely* inductive circuit is given by $i = I_m \sin\left(\omega t - \frac{\pi}{2}\right)$

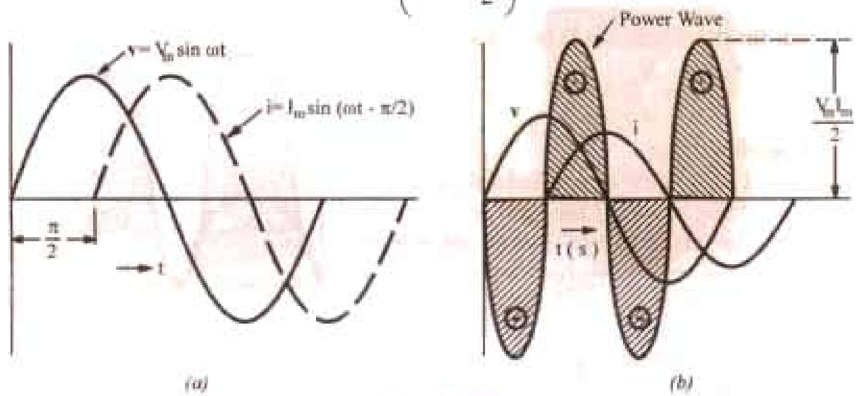

Fig. 11.60

Clearly, the current lags behind the applied voltage by a quarter cycle (Fig. 11.60) or the phase difference between the two is $\pi/2$ with voltage leading. Vectors are shown in Fig. 11.59 where voltage has been taken along the reference axis. We have seen that $I_m = V_m/\omega L = V_m/X_L$. Here '$\omega L$' plays the part of 'resistance'. It is called the (inductive) *reactance* X_L of the coil and is given in ohms if L is in henry and ω is in radian/second.

Now, $X_L = \omega L = 2\pi f L$ ohm. It is seen that X_L depends directly on frequency of the voltage. Higher the value of f, greater the reactance offered and *vice-versa*.

Power

Instantaneous power $= v_i = V_m I_m \sin \omega t \sin\left(\omega t - \frac{\pi}{2}\right) = -V_m I_m \sin \omega t . \cos \omega t$ * $= -\frac{V_m I_m}{2} \sin 2\omega t$

Power for whole cycle is $P = -\frac{V_m I_m}{2} \int_0^{2\pi} \sin 2\omega t \, dt = 0$

* Or $p = \frac{1}{2} E_m I_m [\cos 90° - \cos(2\omega t - 90°)]$. The constant component $= \frac{1}{2} E_m I_m \cos 90° = 0$. The pulsating component is $-\frac{1}{2} E_m I_m \cos(2\omega t - 90°)$ whose average value over one complete cycle is zero.

It is also clear from Fig. 11.60 (*b*) that the average demand of power from the supply for a complete cycle is zero. Here again it is seen that power wave is a sine wave of frequency double that of the voltage and current waves. The maximum value of the instantaneous power is $V_m I_m/2$.

Example 11.39. *Through a coil of inductance 1 henry, a current of the wave-form shown in Fig. 11.61 (a) is flowing. Sketch the wave form of the voltage across the inductance and calculate the r.m.s. value of the voltage.* **(Elect. Technology, Indore Univ. 1985)**

Solution. The instantaneous current i (t) is given by

(*i*) $0 < t < 1$ second, here slope of the curve is $1/1 = 1$.

$\therefore \quad i = 1 \times t = t$ ampere

(*ii*) $1 < t < 3$ second, here slope is $\dfrac{1-(-1)}{2} = 1$

$\therefore \quad t = 1 - (1)(t-1) = 1 - (t-1) = (2-t)$ ampere

(*iii*) $3 < t < 4$ second, here slope is $\dfrac{1-0}{1} = -1$

(*b*) $i = -1 - (-1)(t-3) = (t-4)$ ampere

The corresponding voltage are (*i*) $v_1 = Ldi/dt = 1 \times 1 = 1$ V (*ii*) $v_2 = Ldi/dt = 1 \times \dfrac{d}{dt}(2-t) = -1$ V

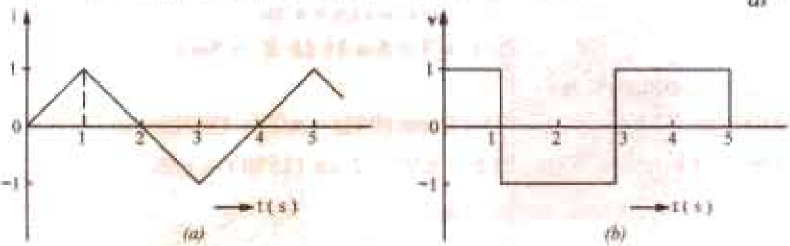

Fig. 11.61

(*iii*) $v_3 = Ldi/dt = 1 \times \dfrac{d}{dt}(t-4) = 1$ V

The voltage waveform is sketched in Fig. 11.61. Obviously, the r.m.s. value of the symmetrical square voltage waveform is **1 V.**

Example 11.40. *A 60-Hz voltage of 230-V effective value is impressed on an inductance of 0.265 H.*

(i) Write the time equation for the voltage and the resulting current. Let the zero axis of the voltage wave be at t = 0. (ii) Show the voltage and current on a phasor diagram. (iii) Find the maximum energy stored in the inductance. **(Elect. Engineering, Bhagalpur Univ. 1985)**

Solution. $V_{max} = \sqrt{2}\, V = \sqrt{2} \times 230\, V, f. = 60\, Hz.$

$\omega = 2\pi f = 2\pi \times 60 = 377$ rad/s, $X_L = \omega L = 377 \times 0.265 = 100\ \Omega$

(*i*) The time equation for voltage is $v\,(t) = 230\sqrt{2} \sin 377\, t$;

$I_{max} = V_{max}/X_L = 230\sqrt{2}/100 = 2.3\sqrt{2}\ \phi = 90°$ (lag)

$\therefore$ Current equation is $i\,(t) = 2.3\sqrt{2} \sin(377\,t - \pi/2)$ or $= 2.3\sqrt{2} \cos 377\, t$.

(*ii*) It is shown in Fig. 11.56. (*iii*) $E_{max} = \dfrac{1}{2} L I_{max}^2 = \dfrac{1}{2} \times 0.265 \times (2.3\sqrt{2})^2 =$ **1.4 J**

11.31. Complex Voltage Applied to Pure Insductance

In Art. 11.30, the applied voltage was a pure sine wave (*i.e.* without harmonics) given by

$$v = V_m \sin \omega t.$$

The current was given by $i = I_m \sin(\omega t - \pi/2)$

Now, it is applied voltage has a complex form and is (say) given by *

$$v = V_{1m} \sin \omega t + V_{3m} \sin 3\omega t + V_{5m} \sin 5\omega t$$

then the reactances offered to the fundamental voltage wave and the harmonics would be different.

For the fundamental wave, $X_1 = \omega L$. Form 3rd harmonic ; $X_3 = 3\omega L$. For 5th harmonic ; $x_5 = 5\omega L$.

Hence, the current would be given by the equation.

$$i = \frac{V_{1m}}{\omega L} \sin\left(\omega t - \frac{\pi}{2}\right) + \frac{V_{3m}}{3\omega L} \sin\left(3\omega t - \frac{\pi}{2}\right) + \frac{V_{5m}}{5\omega L} \sin\left(5\omega t - \frac{\pi}{2}\right)$$

Obviously, the harmonics in the current wave are much smaller than in the voltage wave. For example, the 5th harmonic of the current wave is of only 1/5th of the harmonic in the voltage wave. It means that the self-inductance of a coil has the effect of 'smoothening' current waveform when the voltage waveform is complex *i.e.* contains harmonics.

Example 11.41. *The voltage applied to a purely inductive coil of self-inductance 15.9 mH is given by the equation, $v = 100 \sin 314\, t + 75 \sin 942\, t + 50 \sin 1570\, t$. Find the equation of the resulting current wave.*

Solution. Here $\omega = 314$ rad/s $\quad \therefore\ X_1 = \omega L = (15.9 \times 10^{-3}) \times 314 = 5\ \Omega$

$$X_3 = 3\omega L = 3 \times 5 = 15\ \Omega;\ X_5 = 5\omega L = 5 \times 5 = 25\ \Omega$$

Hence, the current equation is

$$i = (100/5) \sin(314\, t - \pi/2) + (75/15) \sin(942t - \pi/2) + (50/25) \sin(1570t - \pi/2)$$

or $i = \mathbf{20 \sin(314 - \pi/2) + 5 \sin(942t - \pi/2) + 2 \sin(1570\, t - \pi/2)}$

11.32. A.C. Through Pure Capacitance Alone

When an alternating voltage is applied to the plates of a capacitor, the capacitor is charged first in one direction and then in the opposite direction. When reference to Fig. 11.62, let

v = p.d. developed between plates at any instant

q = Charge on plates at that instant.

The $\quad q = Cv \quad$ ...where C is the capacitance

$\quad = C V_m \sin \omega t \quad$ ...putting the value of v.

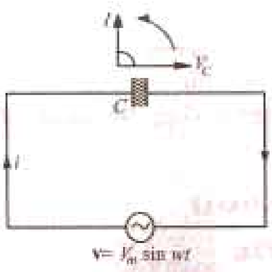

Fig. 11.62

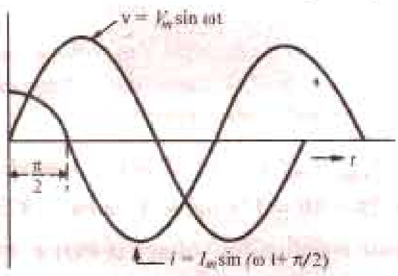

Fig. 11.63

Now, current i is given by the rate of flow of charge.

$$\therefore\quad i = \frac{dq}{dt} = \frac{d}{dt}(Cv_m \sin \omega t) = \omega C V_m \cos \omega t \text{ or } i = \frac{V_m}{1/\omega C} \cos \omega t = \frac{V_m}{1/\omega C} \sin\left(\omega t + \frac{\pi}{2}\right)$$

Obviously, $I_m = \frac{V_m}{1/\omega C} = \frac{V_m}{X_C} \quad \therefore\ i = I_m \sin\left(\omega t + \frac{\pi}{2}\right)$

* It is assumed that the harmonics have no individual phase differences.

The denominator $X_C = 1/\omega C$ is known as capacitive *reactance* and is in ohms if C is in farad and ω in radian/second. It is seen that if the applied voltage is given by $v = V_m \sin \omega t$, then the current is given by $i = I_m \sin(\omega t + \pi/2)$.

Hence, we find that the current in a pure capacitor leads its voltage by a quarter cycle as shown in Fig. 11.63 or phase diffence between its voltage and current is $\pi/2$ with the current leading. Vector representation is given in Fig. 11.63. Note that V_c is taken along the reference axis.

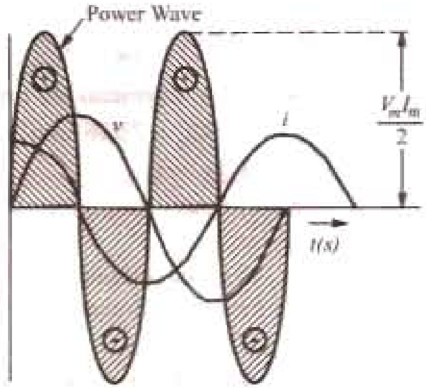

Fig. 11.64

Power. Instantaneous power

$$p = vi = V_m \sin \omega t . I_m \sin(\omega t + 90°)$$
$$= V_m I_m \sin \omega t \cos \omega t\,^* = \frac{1}{2} V_m I_m \sin 2\omega t$$

Power for the whole cycle

$$= \frac{1}{2} V_m I_m \int_0^{2\pi} \sin 2\omega t \, dt = 0$$

This fact is graphically illustrated in Fig. 11.64. We find that in a purely capacitive circuit **, the average demand of power from supply is zero (as in a purely inductive circuit). Again, it is seen that power wave is a sine wave of frequency double that of the voltage and current waves. The maximum value of the instantaneous power is $V_m I_m/2$.

Example 11.42. *A 50-Hz voltage of 230 volts effective value is impressed on a capacitance of 26.5 μF. (a) Write the time equations for the voltage and the resulting current. Let the zero axis of the voltage wave be at t = 0. (b) Show the voltage and current on a time diagram.*

(c) Show the voltage and current on a phasor diagram. (d) Find the maximum energy stored in the capacitance. Find the relative heating effects of two current waves of equal peak value, the one sinusoidal and the other rectangular in waveform. **(Elect. Technology, Allahabad Univ. 1991)**

Solution.

$$V_{max} = 230\sqrt{2} = 235\,V$$
$$\omega = 2\pi \times 50 = 314 \text{ rad/s}; X_C = 1/\omega C = 10^6/314 \times 26.5 = 120\ \Omega$$
$$I_{max} = V_{max}/X_C = 325/120 = 2.71 \text{ A}, \phi = 90° \text{ (lead)}$$

(*a*) $v(t) = 325 \sin 314\,t$; $i(t) = 2.71 \sin(314t + \pi/2) = 2.71 \cos 314\,t$

(*b*) and (*c*) These are shown in Fig. 11.59.

(*d*) $E_{max} = \frac{1}{2} CV^2_{max} = \frac{1}{2}(26.5 \times 10^{-6}) \times 325^2 =$ **1.4 J**

(*e*) Let I_m be the peak value of both waves.

For sinusoidal wave : $H \propto I^2 R \propto (I_m/\sqrt{2})^2 R \propto I_m^2 R/2$. For rectangular wave : $H \propto I_m^2 R$ - Art. 12.15.

$$\frac{H \text{ rectangular}}{H \text{ sinusoidal}} = \frac{I_m^2 R}{I_m^2 R/2} = \mathbf{2}$$

Example 11.43. *A 50-μF capacitor is connected across a 230-V, 50-Hz supply. Calculate (a) the rectance offered by the capacitor (b) the maximum current and (c) the r.m.s. value of the current drawn by the capacitor.*

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* Or power $p = \frac{1}{2} E_m I_m [\cos 90° - \cos(2\omega t - 90°)]$. The constant component is again zero. The pulsating component averaged over one complete cycle is zero.

** By pure capacitor is meant one that has neither resistance nor dielectric loss. If there is loss in a capacitor, then it may be represented by loss in (*a*) high resistance joined in parallel with the pure capacitor or (*b*) by a comparatively low resistance joined in series with the pure capacitor. But out of the two alternatives usually, (*a*) is chosen (Art. 13.8).

Solution. (a) $X_C = \frac{1}{\omega C} = \frac{1}{2\pi f_c} = \frac{1}{2\pi \times 50 \times 50 \times 10^{-6}} = 63.6\ \Omega$

(c) Since 230 V represents the r.m.s. value,

$\therefore\ I_{r.m.s.} = 230/X_c = 230/63.6 = 3.62$ A (b) $I_m = I_{r.m.s.} \times \sqrt{2} = 3.62 \times \sqrt{2} = 5.11$ A

Example 11.44. *The voltage applied across 3-branched circuit of Fig. 11.65 is given by v = 100 sin (5000t + π/4). Calculate the branch currents and total current.*

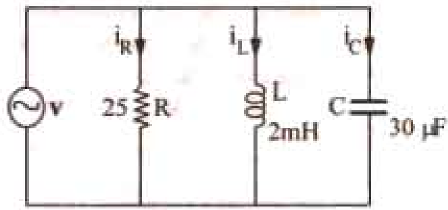


Fig. 11.65

Solution. The total instantaneous current is the vector sum of the three branch currents.

$i_t = i_R + i_L + i_C$

Now $i_R = v/R = 100 \sin(5000\,t + \pi/4)/25$

$= 4 \sin(5000\,t + \pi/4)$

$$i_L = \frac{1}{L}\int v\,dt = \frac{10^3}{2}\int 100 \sin\left(5000t + \frac{\pi}{4}\right)dt$$

$$= \frac{10^3 \times 100}{2}\left[\frac{-\cos(5000\,t + \pi/4)}{5000}\right] = -10\cos(5000\,t + \pi/4)$$

$$i_C = C\frac{dv}{dt} = C.\frac{d}{dt}[100 \sin(5000\,t + \pi/4)]$$

$$= 30 \times 10^{-6} \times 100 \times 5000 \times \cos(5000\,t + \pi/4) = 15\cos(5000\,t + \pi/4)$$

$$i_t = 4\sin(5000\,t + \pi/4) - 10\cos(5000\,t + \pi/4) + 15\cos(5000t + \pi/4)$$

$$= 4\sin(5000\,t + \pi/4) + 5\cos(5000\,t + \pi/4)$$

OBJECTIVE TESTS – 11

1. An a.c. current given by $i = 14.14 \sin(\omega t + \pi/6)$ has an r.m.s value of — amperes.
   (a) 10
   (b) 14.14
   (c) 1.96
   (d) 7.07
   and a phase of — degrees.
   (e) 180
   (f) 30
   (g) – 30
   (h) 210
2. If $e_1 = A \sin \omega t$ and $e_2 = B \sin(\omega t - \phi)$, then
   (a) e_1 lags e_2 by θ
   (b) e_2 lags e_1 by θ
   (c) e_2 leads e_2 by θ
   (d) e_1 is in phase with e_2
3. From the two voltage equations $e_A = E_m \sin 100\pi t$ and $e_B = E_m \sin(100\pi t + \pi/6)$, it is obvious that
   (a) A leads B by 30°
   (b) B achieves its maximum value 1/600 second before A does.
   (c) B lags behind A
   (d) A achieves its zero value 1/600 second before B.
4. The r.m.s. value of a half-wave rectified current is 10A, its value for full-wave rectification would be — amperes.
   (a) 20
   (b) 14.14
   (c) 20/π
   (d) 40/π
5. A resultant current is made of two components : a 10 A d.c. component and a sinusoidal component of maximum value 14.14 A. The average value of the resultant current is — amperes.
   (a) 0
   (b) 24.14
   (c) 10
   (d) 4.14
   and r.m.s. value is — amperes.
   (e) 10
   (f) 14.14
   (g) 24.14
   (h) 100

6. The r.m.s. value of sinusoidal a.c. current is equal to its value at an angle of — degree
   (*a*) 60 (*b*) 45 (*c*) 30 (*d*) 90
7. Two sinusoidal currents are given by the equations : $i_1 = 10 \sin(\omega t + \pi/3)$ and $i_2 = 15 \sin(\omega t - \pi/4)$. The phase difference between them is — degrees.
   (*a*) 105 (*b*) 75 (*c*) 15 (*d*) 60
8. As sine wave has a frequency of 50 Hz. Its angular frequency is — radian/second.
   (*a*) $50/\pi$ (*b*) $50/2\pi$ (*c*) 50π (*d*) 100π
9. An a.c. current is given by $i = 100 \sin 100$. It will achieve a value of 50 A after — second.
   (*a*) 1/600 (*b*) 1/300 (*c*) 1/1800 (*d*) 1/900
10. The reactance offered by a capacitor to alternating current of frequency 50 Hz is 10Ω. If frequency is increased to 100 Hz reactance becomes—ohm.
   (*a*) 20 (*b*) 5
   (*c*) 2.5 (*d*) 40
11. A complex current wave is given by $i = 5 + 5 \sin 100\pi t$ ampere. Its average value is — ampere.
   (*a*) 10 (*b*) 0
   (*c*) $\sqrt{50}$ (*d*) 5

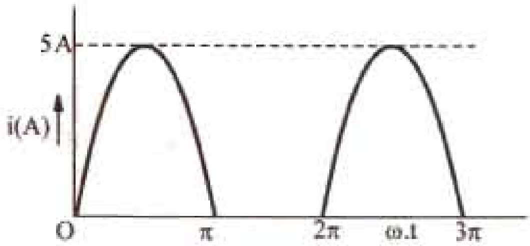


Fig. 11.66

12. The current through a resistor has a waveform as shown in Fig. 11.66. The reading shown by a moving coil ammeter will be— ampere.
   (*a*) $5/\sqrt{2}$ (*b*) $2.5/\sqrt{2}$
   (*c*) $5/\pi$ (*d*) 5

(Principles of Elect. Engg. Delhi Univ. July 1984)

12 COMPLEX NUMBERS

12.1. Mathematical Representation of Vectors

There are various forms or methods of representing vector quantities, all of which enable those operations which are carried out graphically in a phasor diagram, to be performed analytically. The various methods are :

(*i*) *Symbolic Notation.* According to this method, a vector quantity is expressed algebraically in terms of its rectangular components. Hence, this form of representation is also known as Rectangular or Cartesian form of notation or representation.

(*ii*) *Trigonometrical Form* (*iii*) *Exponential Form* (*iv*) *Polar Form.*

12.2. Symbolic Notation

A vector can be specified in terms of its *X*-component and *Y*-component. For example, the vector OE_1 (Fig. 12.1) may be completely described by stating that its horizontal component is a_1 and vertical component is b_1. But instead of stating this verbally, we may express symbolically

$$\mathbf{E}_1 = a_1 + jb_1$$

where symbol *j*, known as an operator, indicates that component b_1 is perpendicular to component a_1 and that the two terms are *not* to be treated like terms in any algebraic expression. The vector written in this way is said to be written in '*complex form*'. In Mathematics, a_1 is known as real component and b_1 as imaginary component but in electrical engineering, these are known as *in phase* (or active) and *quadrature* (or reactive) components respectively.

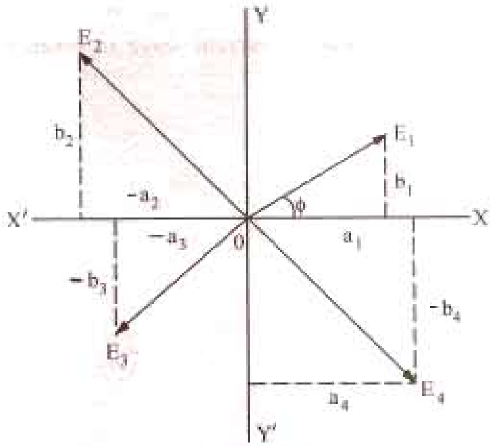


Fig. 12.1

The other vectors OE_2, OE_3 and OE_4 can similarly, be expressed in this form.

$$\mathbf{E}_2 = -a_2 + jb_2\ ;\ \mathbf{E}_3 = -a_3 - jb_3\ ;\ \mathbf{E}_4 = +a_4 - jb_4$$

It should be noted that in this book, a vector quantity would be represented by letters in heavy type and its numerical or scalar value by the same letter in ordinary type.* Other method adopted for indicating a vector quantity is to put an arrow about the letter such as $\vec{E}$.

The numerical value of vector E_1 is $\sqrt{a_1^2 + b_1^2}$. Its angle with *X*-axis is given by $\phi = \tan^{-1}(b_1/a_1)$.

12.3. Significance of Operator *j*

The letter *j* used in the above expression is a symbol of an operation. Just as symbols $\times$, $+$ $\sqrt{\ }$, $\int$

* The magnitude of a vector is sometimes called 'modulus' and is represented by $|E|$ or E.

etc. are used with numbers for indicating certain operations to be performed on those numbers, similarly, symbol j is used to indicate the counter-clockwise rotation of a vector through 90°. It is assigned a value of $\sqrt{(-1)}$ *. The double operation of j on a vector rotates it counter-clockwise through 180° and hence reverses its sense because

$$jj = j^2 = \sqrt{(-1)^2} = -1$$

When operator j is operated on vector **E**, we get the new vector j**E** which is displaced by 90° in counter-clockwise direction from **E** (Fig. 12.2). Further application of j will give $j^2\mathbf{E} = -\mathbf{E}$ as shown.

If the operator j is applied to the vector $j^2\mathbf{E}$, the result is $j^3\mathbf{E} = -j\mathbf{E}$. The vector $j^3\mathbf{E}$ is 270° counter-clockwise from the reference axis and is directly opposite to $j\mathbf{E}$. If the vector $j^3\mathbf{E}$ is, in turn, operated on by j, the result will be

$$j^4\mathbf{E} = \left[\sqrt{(-1)}\right]^4 \mathbf{E} = \mathbf{E}$$

Hence, it is seen that successive applications of the operator j to the vector **E** produce successive 90° steps of rotation of the vector in the counter-clockwise direction without in any way affecting the magnitude of the vector.

It will also be seen from Fig. 12.2 that the application of $-j$ to **E** yields $-j\mathbf{E}$ which is a vector of identical magnitude but rotated 90° *clockwise* from **E**.

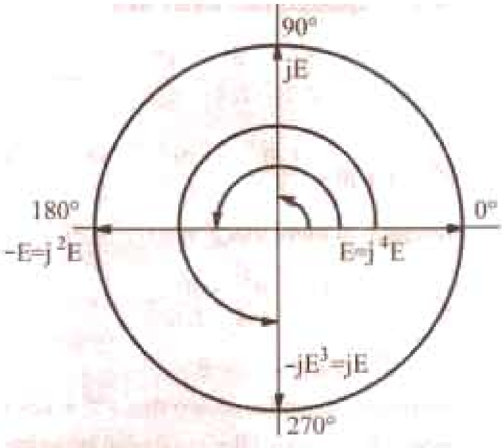


Fig. 12.2

Summarising the above, we have

$$\begin{aligned}
j &= 90^\circ \text{ ccw rotation} = \sqrt{(-1)}\\
j^2 &= 180^\circ \text{ ccw rotation} = [\sqrt{(-1)}]^2 = -1;\\
j^3 &= 270^\circ \text{ ccw rotation} = [\sqrt{(-1)}]^3 = -\sqrt{(-1)} = -j\\
j^4 &= 360^\circ \text{ ccw rotation} = [\sqrt{(-1)}]^4 = +1;\\
j^5 &= 450^\circ \text{ ccw rotation} = [\sqrt{(-1)}]^5 = -\sqrt{(-1)} = j
\end{aligned}$$

It should also be noted that $\dfrac{1}{j} = \dfrac{j}{j^2} = \dfrac{j}{-1} = -j$

12.4. Conjugate Complex Numbers

Two complex numbers are sid to be conjugate if they differ only in the algebraic sign of their quadrature components. Accordingly, the numbers $(a + jb)$ and $(a - jb)$ are conjugate. The sum of two conjugate numbers gives in-phase (or active) component and their difference gives quadrature (or reactive) component.

12.5. Trigonometrical Form of Vector

From Fig. 12.3, it is seen that *X-component of* **E** is $E \cos\theta$ and Y-component is $E \sin\theta$. Hence, we can represent the vector **E** in the form : $\mathbf{E} = E(\cos\theta + j\sin\theta)$

---

* In Mathematics, $\sqrt{(-1)}$ is denoted by i but in electrical engineering j is adopted because letter i is reserved for representing current. This helps to avoid confusion.

This is equivalent to the rectangular form $\mathbf{E} = a + jb$ because $a = E \cos\theta$ and $b = E \sin\theta$. In general, $\mathbf{E} = E(\cos\theta \pm j\sin\theta)$.

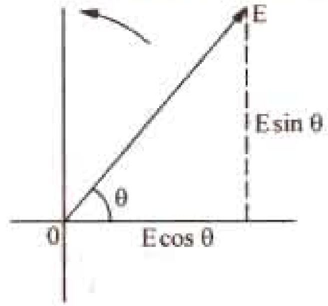


Fig. 12.3

12.6. Exponential Form of Vector

It can be proved that $e^{\pm j\theta} = (\cos\theta \pm j\sin\theta)$

This equation is known as Euler's equation after the famous mathematician of 18th century : Leonard Euler.

This equation follows directly from an inspection of Maclaurin* series expansions of sin θ, cos θ and $e^{j\theta}$.

When expanded into series form :

$$\cos\theta = 1 - \frac{\theta^2}{\angle 2} + \frac{\theta^4}{\angle 4} - \frac{\theta^6}{\angle 6} + \ldots \text{ and } \sin\theta = \theta - \frac{\theta^3}{\angle 3} + \frac{\theta^5}{\angle 5} - \frac{\theta^7}{\angle 7} + \ldots$$

$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{\angle 2} + \frac{(j\theta)^3}{\angle 3} + \frac{(j\theta)^4}{\angle 4} + \frac{(j\theta)^5}{\angle 5} + \frac{(j\theta)^6}{\angle 6} + \ldots$$

Keeping in mind that $j^2 = -1, j^3 = -j, j^4 = 1, j^5 = -j, j^6 = -1$, we get

$$e^{j\theta} = \left(1 - \frac{\theta^2}{\angle 2} + \frac{\theta^4}{\angle 4} - \frac{\theta^6}{\angle 6} + \ldots\right) + j\left(\theta - \frac{\theta^3}{\angle 3} + \frac{\theta^5}{\angle 5} - \frac{\theta^7}{\angle 7} + \ldots\right)$$

$\therefore \quad e^{j\theta} = \cos\theta + j\sin\theta$

Similarly, it can be shown that $e^{-j\theta} = \cos\theta - j\sin\theta$

Hence $\mathbf{E} = E(\cos j\theta \pm \sin\theta)$ can be written as $\mathbf{E} = Ee^{\pm j\theta}$. This is known as exponential form of representing vector quantities. It represents a vector of numerical value E and having phase angle of $\pm\theta$ with the reference axis.

12.7. Polar Form of Vector Representation

The expression $E(\cos\theta + j\sin\theta)$ is written in the simplified form of $E\angle\theta$. In this expression, E represents the magnitude of the vector and θ its inclination (in ccw direction) with the X-axis. For angles in clockwise direction the expression becomes $E\angle -\theta$. In general, the expression is written as $E\angle\pm\theta$. *It may be pointed out here that $E\angle\pm\theta$ is simply a short-hand or symbolic style of writing $Ee^{\pm j\theta}$.* Also, the form is purely conventional and does not possess the mathematical elegance of the various other forms of vector representation given above.

Summarizing, we have the following alternate ways of representing vector quantities

(*i*) Rectangular form (or complex form) $\mathbf{E} = a + jb$

(*ii*) Trigonometrical form $\mathbf{E} = E(\cos\theta \pm j\sin\theta)$

(*iii*) Exponential form $\mathbf{E} = Ee^{\pm j\theta}$

(*iv*) Polar form (conventional) $\mathbf{E} = E\angle\pm\theta$.

Example 12.1. *Write the equivalent exponential and polar forms of vector 3 + j4. How will you illustrate the vector means of diagram ?*

---

* Functions like cos θ, sin θ and $e^{j\theta}$ etc. can be expanded into series form with the help of Macalurin's Theorem. The theorem states : $f(\theta) = f(0) + \frac{f'(0)\theta}{1} + \frac{f''(0)\theta^2}{\angle 2} + \frac{f'''(0)\theta^3}{\angle 3} + \ldots$ where $f(\theta)$ is function of θ which is to be expanded, $f(0)$ is the value of the function when $\theta = 0$, $f'(0)$ is the value of first derivative of $f(\theta)$ when $\theta = 0$, $f''(0)$ is the value of second derivative of function $f(\theta)$ when $\theta = 0$ etc.

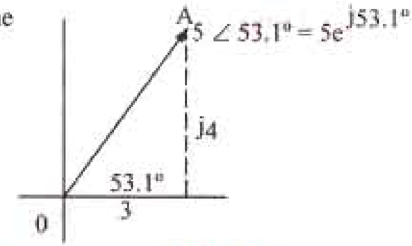


Fig. 12.4

Solution. With reference to Fig. 12.4., magnitude of the vector is $= \sqrt{3^2 + 4^2} = 5$. $\tan \theta = 4/3$.

$\therefore \quad \theta = \tan^{-1}(4/3) = 53.1°$

$\therefore$ Exponential form $= 5\, e^{j53.1°}$

The angle may also be expressed in radians.

Polar form $= 5 \angle 53.1°$.

Example 12.2. *A vector is represented by $20\, e^{-j2\pi/3}$. Write the various equivalent forms of the vector and illustrate by means of a vector diagram, the magnitude and position of the above vector.*

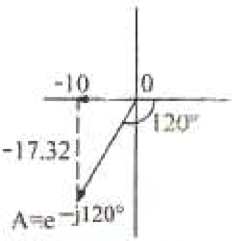


Fig. 12.5

Solution. The vector is drawn in a direction making an angle of $2\pi/3 = 120°$ in the clockwise direction (Fig. 12.5). The clockwise direction is taken because the angle is negative.

(*i*) **Rectangular Form** $a = 20 \cos(-120°) = -10$;

$b = 20 \sin(-120°) = -17.32$

$\therefore$ Expression is $= (-10 - j17.32)$

(*ii*) **Polar Form** is $20 \angle -120°$

12.8. Addition and Subtraction of Vector Quantities

Rectangular form is best suited for addition and subtraction of vector quantities. Suppose we are given two vector quantities $\mathbf{E}_1 = a_1 + jb_1$ and $\mathbf{E}_2 = a_2 + jb_2$ and it is required to find their sum and difference.

Addition. $\quad \mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = a_1 + jb_1 + a_2 + jb_2 = (a_1 + a_2) + j(b_1 + b_2)$

The magnitude of resultant vector $\mathbf{E}$ is $\sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}$

The position of $\mathbf{E}$ with respect to X-axis is $\theta = \tan^{-1}\left(\dfrac{b_1 + b_2}{a_1 + a_2}\right)$

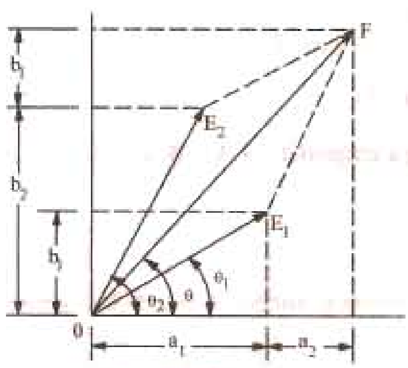


Fig. 12.6

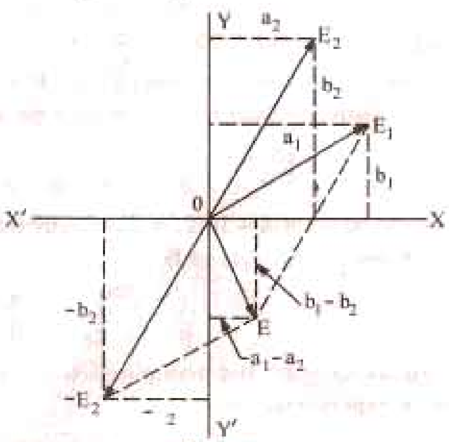


Fig. 12.7

A graphic representation of the addition process is shown in Fig. 12.6

Subtraction. $\quad \mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2 = (a_1 + jb_1) - (a_2 + jb_2) = (a_1 - a_2) + j(b_1 - b_2)$

Magnitude of $\quad \mathbf{E} = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$

Its position with respect to x-axis is given by the angle $\theta = \tan^{-1}\left(\dfrac{b_1 - b_2}{a_1 - a_2}\right)$.

The graphic representation of the process of subtraction is shown in Fig. 12.7.

12.9. Multiplication and Division of Vector Quantities

Multiplication and division of vectors becomes very simple and easy *if they are represented in the polar or exponential form.* As will be shown below, the rectangular form of representation is not well-suited for this process.

(i) **Multiplication** – *Rectangular form*

Let the two vectors be given by $\mathbf{A} = a_1 + jb_1$ and $\mathbf{B} = a_2 + jb_2$

$$\therefore \quad \mathbf{A} \times \mathbf{B} = \mathbf{C} = (a_1 + jb_1)(a_2 + jb_2) = a_1a_2 + j^2b_1b_2 + j(a_1b_2 + b_1a_2)$$

$$= (a_1a_2 - b_1b_2) + j(a_1b_2 + b_1a_2) \qquad (\because j^2 = -1)$$

The magnitude of $\mathbf{C} = \sqrt{[(a_1a_2 - b_1b_2)^2 + (a_1b_2 + b_1a_2)^2]}$

In angle with respect to X-axis is given by $\theta = \tan^{-1}\left(\dfrac{a_1b_2 + b_1a_2}{a_1a_2 - b_1b_2}\right)$

(ii) Division – *Rectangular Form :* $\dfrac{\mathbf{A}}{\mathbf{B}} = \dfrac{a_1 + jb_1}{a_2 + jb_2} = \dfrac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)}$

Both the numerator and denominator have been multiplied by the conjugate of $(a_2 + jb_2)$ *i.e.* by $(a_2 - jb_2)$

$$\therefore \quad \frac{\mathbf{A}}{\mathbf{B}} = \frac{(a_1a_2 + b_1b_2) + j(b_1a_2 - a_1b_2)}{a_2^2 + b_2^2} = \frac{a_1a_2 + a_1b_2}{a_2^2 + b_2^2} + j\frac{b_1a_2 + a_1b_2}{a_2^2 + b_2^2}$$

The magnitude and the angle with respect X-axis can be found in the same way as given above.

As will be noted, both the results are somewhat awkward but unfortunately, there is no easier way to perform multiplication in rectangular form.

(iii) **Multiplication** – *Polar Form*

Let $\mathbf{A} = a_1 + jb_1 = A\angle\alpha = A\,e^{j\alpha}$ where $\alpha = \tan^{-1}(b_1/a_1)$

$\mathbf{B} = a_2 + jb_2 = B\angle\beta = B\,e^{j\beta}$ where $\beta = \tan^{-1}(b_2/a_2)$

$\therefore$ $\mathbf{AB} = A\angle\alpha \times B\angle\beta = AB\angle(\alpha + \beta)$* or $AB = Ae^{j\alpha} \times Be^{j\beta} = ABe^{j(\alpha+\beta)}$

Hence, product of any two vector **A** and **B** is given by another vector equal in length to **A** × **B** and having a phase angle equal to the sum of the angles of **A** and **B**.

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{A\angle\alpha}{B\angle\beta} = \frac{A}{B}\angle(\alpha - \beta)$$

Hence, the quotient **A** ÷ **B** is another vector having a magnitude of **A** ÷ **B** and phase angle equal to angle of **A** minus the angle of **B**.

$$\text{Also} \qquad \frac{\mathbf{A}}{\mathbf{B}} = \frac{Ae^{j\alpha}}{Be^{j\beta}} = \frac{A}{B}e^{j(\alpha-\beta)}$$

As seen, the division and multiplication become extremely simple if vectors are represented in their polar or exponential form.

Example 12.3. *Add the following vectors given in rectangular form and illustrate the process graphically.*

$$A = 16 = j\,12, \quad B = -6 + j\,10.4$$

---

* $\mathbf{A} = A(\cos\alpha + j\sin\alpha)$ and $\mathbf{B} = B(\cos\beta + j\sin\beta)$

$\therefore$ $\mathbf{AB} = AB(\cos\alpha\cos\beta + j\sin\alpha\cos\beta + j\cos\alpha\sin\beta + j^2\sin\alpha\sin\beta)$

$= AB[\cos\alpha\cos\beta - \sin\alpha\sin\beta) + j(\sin\alpha\cos\beta + \cos\alpha\sin\beta)]$

$= AB[\cos(\alpha + \beta) + j\sin(\alpha + \beta)] = AB\angle(\alpha + \beta)$

Solution. $\mathbf{A} + \mathbf{B} = C = (16 + j\,12) + (-6 + j\,10.4) = 10 + j\,22.4$

$\therefore$ Magnitude of $\mathbf{C} = \sqrt{(10^2 + 22.4^2)} = 25.5$ units

Slope of $\mathbf{C} = \theta = \tan^{-1}\left(\frac{22.4}{10}\right) = 65.95°$

The vector addition is shown in Fig. 12.8.

$\alpha = \tan^{-1}(12/16) = 36.9°$

$\beta = \tan^{-1}(-10.4/6) = -240°$ or $120°$

The resultant vector is found by using parallelogram law of vectors (Fig. 12.8).

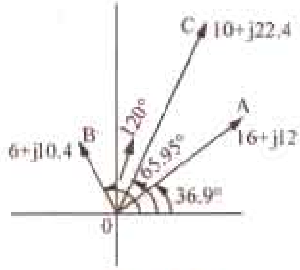


Fig. 12.8

Example 12.4. *Perform the following operation and express the final result in the polar form : $5\angle 30° + 8\angle -30°$.* **(Elect. Engg. & Electronics Bangalore Univ. 1989)**

Solution. $5\angle 30° = 5\,(\cos 30° + j\sin 30°) = 4.33 + j\,2.5$

$8\angle -30° = 8\,[\cos(-30°) + j\sin(-30°)] = 8\,(0.866 - 0.5) = 6.93 - j\,4$

$\therefore 5\angle 30° + 8\angle -30° = 4.33 + j2.5 + 6.93 - j4 = 11.26 - j1.5 = \sqrt{11.26^2 + 1.5^2}\,\angle\tan^{-1}(-1.5/11.26) = 11.35$

$\tan^{-1}(-0.1332) = 11.35\angle 7.6°$

Example 12.5. *Subtract the following given vectors from one another :*

$A = 30 + j\,52$ and $B = -39.5 - j\,14.36$

Solution. $\mathbf{A} - \mathbf{B} = \mathbf{C} = (30 + j\,52) - (-39.5 - j\,14/36) = 69.5 + j\,66.36$

$\therefore$ Magnitude of $C = \sqrt{(66.5^2 + 66.36^2)} = 96$

Slope of $C = \tan^{-1}(66.36/69.5) = 43.6°$ $\therefore$ $\mathbf{C} = 96\angle 43.6°$.

Similarly $\mathbf{B} - \mathbf{A} = -69.5 - j\,66.36 = 96\angle 223.6°$* or $= 96\angle -136.4°$

Example 12.6. *Given the following two vectors :*

$A = 20\angle 60°$ and $B = 5\angle 30°$

Perform the following indicated operations and illustrate graphically (i) $A \times B$ and (ii) A/B.

Solution. (*i*) $\mathbf{A} \times \mathbf{B} = \mathbf{C} = 20\angle 60° \times 5\angle 30° = 100\angle 90°$

Vectors are shown in Fig. 12.9.

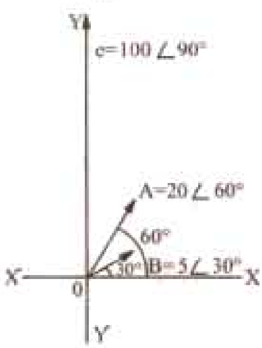


Fig. 12.9

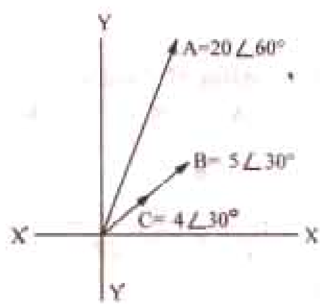


Fig. 12.10

* $\tan\theta = 66.36/69.5$ or $\theta = \tan^{-1}(66.36/69.5) = 43.6°$. Since both components are negative, the vector lies in third quadrant. Hence, the angle measured from + ve direction of X-axis and in the CCW direction is $= (180 + 43.6) = 223.6°$

(*ii*) $$\frac{\mathbf{A}}{\mathbf{B}} = \frac{20 \angle 60^\circ}{5 \angle 30^\circ} = 4 \angle 30^\circ$$ —Fig. 12.10

Example 12.7. *Perform the following operation and the final result may be given in the polar form : (8 + j6) × (−10 − j 7.5)* **(Elect. Engg. & Electronics Bangalore Univ. 1990)**

Solution. We will use the following two methods to solve the above question.

Method No. 1

We know that multiplication of $(A + B)$ and $(C + D)$ can be found as under :

$$\begin{array}{l} A + B \\ \times C + D \\ \hline CA \; + \; CB \\ \qquad + \; DA + DB \\ \hline CA + CD + DA + DB \\ \hline \end{array}$$

Similarly, the required multiplication can be carried out as follows :

$$\begin{array}{l} 8 + j\,6 \\ \times - 10 \; - j\,7.5 \\ \hline - 80 - \; j\,60 \\ - j\,60 - \; j^2\,45 \\ \hline - 80 - j120 + 45 \end{array}$$

or $\quad -35 - j120 = \sqrt{(-35)^2 + (-120)^2}\ \underline{|\tan^{-1}(120/35)} = 125 \tan^{-1} 3.42 = 125\ \underline{|73.8^\circ}$

Since both the components of the vector are negative, it obviously lies in the third quadrant. As measured from the *X*-axis in the *CCW* direction, its angle is = 180° + 73.8° = 253.8°. Hence, the product vector can be written as $125 \angle 53.8°$.

Method No. 2 $\quad 8 + j\,6 = 10 \angle 36.9°, -10 - j\,7.5 = 12.5 \tan^{-1} 0.75 = 12.5 \angle 36.9°$.

Again as explained in Method 1 above, the actual angle of the vector is 180° + 36.9° = 216.9°

$\therefore \quad -10 - j\,7.5 = 12.5 \angle 216.9 \quad \therefore \quad 10 \angle 36.9° \times 12.5 \angle 216.9° = 125 < 253.8°$

Example 12.8. *The following three vectors are given :*

$$A = 20 + j20,\ B = 30 \angle -120 \text{ and } c = 10 + j0$$

Perform the following indicated operations :

(*i*) $\dfrac{AB}{C}$ and (*ii*) $\dfrac{BC}{A}$.

Solution. Rearranging all three vectors in polar form, we get

$$\mathbf{A} = 28.3 \angle 45°\ ,\ \mathbf{B} = 30 \angle -120°,\ \mathbf{C} = 10 \angle 0°$$

(*i*) $$\frac{\mathbf{AB}}{\mathbf{C}} = \frac{28.3 \angle 45° \times 30 \angle -120°}{10 \angle 0°} = 84.9 \angle -75°$$

(*ii*) $$\frac{\mathbf{BC}}{\mathbf{A}} = \frac{30 \angle -120 \times 10 \angle 0°}{28.3 \angle 45°} = 10.6 \angle -165°$$

Example 12.9. *Given two current $i_1 = 10 \sin(\omega t + \pi/4)$ and $i_2 = 5 \cos(\omega t - \pi/2)$, find the r.m.s. value of $i_1 + i_2$ using the complex number representation.* **[Elect. Circuit Theory, Kerala Univ. 1985]**

Solution. The maximum value of first current is 10 A and it leads the reference quantity by 45°. The second current can be written as

$$i_2 = 5 \cos(\omega t - \pi/2) = 5 \sin[90 + (\omega t - \pi/2)] = 5 \sin \omega t$$

Hence, its maximum value is 5 A and is in phase with the reference quantity.

$$\therefore \quad \mathbf{I}_{m1} = 10(\cos 45° + j \sin 45°) = (7.07 + j\,7.07)$$

$$\mathbf{I}_{m2} = 5(\cos 0° + j \sin 0°) = (5 + j\,0)$$

The maximum value of resultant current is

$$\mathbf{I}_m = (7.07 + j\,7.07) + (5 + j\,0) = 12.07 + j\,7.07 = 14 \angle 30.4°$$

$\therefore$ R.M.S. value $= 14/\sqrt{2} = \mathbf{10\ A}$

12.10. Power and Roots of Vectors

(a) Powers

Suppose it is required to find the cube of the vector $3 \angle 15°$. For this purpose, the vector has to be multiplied by itself three times.

$\therefore$ $(3 \angle 15)^3 = 3 \times 3 \times 3.\ \angle (15° + 15° + 15°) = 27 \angle 45°$. In general, $\mathbf{A}^n = \mathbf{A}^n \angle n\alpha$

Hence, *n*th power of vector **A** is a vector whose magnitude is A^n and whose phase angle with respect to *X*-axis is $n\alpha$.

It is also clear that $\mathbf{A}^n\mathbf{B}^n = A^nB^n \angle (n\alpha + n\beta)$

(b) Roots

It is clear that $\sqrt[3]{(8 \angle 45°)} = 2 \angle 15°$

In general, $\sqrt[n]{\mathbf{A}} = \sqrt[n]{A} = \angle \alpha/n$

Hence, *n*th root of a vector **A** is a vector whose magnitude is $n\sqrt{A}$ and whose phase angle with respect to *X-axis is* α/n.

12.11. The 120° Operator

In three-phase work where voltage vectors are displaced from one another by 120°, it is convenient to employ an operator with rotates a vector through 120° forward or backwards without changing its length. This operator is '*a*'. Any operator which is multiplied by '*a*' remains unchanged in magnitude but is rotated by 120° in the counter-clockwise (ccw) direction.

$\therefore$ $\alpha = 1 \angle 120°$

This, when expressed in the cartesian form, becomes

$$a = \cos 120° + j \sin 120° = -0.5 + j\,0.866$$

Similarly, $a^2 = 1 \angle 120° \times 1 \angle 120° = 1 \angle 240° = \cos 240° + j \sin 240° = -0.5 - j\,0.866$

Hence, operator 'a^2' will rotate the vector in ccw by 240°. This is the same as rotating the vector in *clockwise* direction by 120°.

$\therefore$ $a^2 = 1 \angle -120°$. Similarly, $a^3 = 1 \angle 360° = 1$*

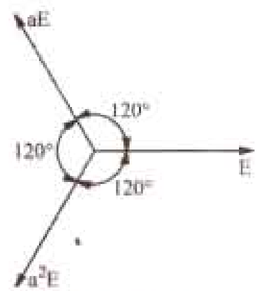


Fig. 12.11

As shown in Fig. 12.11, the 3-phase voltage vectors with standard phase sequence may be represented as E, a^2E and aE or as E, $E\,(-0.5 - j\,0.866)$ and $E\,(-0.5 + j\,0.866)$

It is easy to prove that

(*i*) $a^2 + a = -1$ (*ii*) $a^2 + a + 1 = 0$ (*iii*) $a^3 + a^2 + a = 0$

Note. We have seen in Art. 12.3 that operator $-j$ turns a vector through $-90°$ *i.e.* through 90° in *clockwise* direction. But it should be clearly noted that operator '$-a$' *does not* turn a vector through $-120°$. Rather '$-a$' turns a vector through $-60°$ as shown below.

Example 12.10. *Evaluate the following expressions in the polar form (i)* $a^2 - 1$, *(ii)* $1 - a - a^2$ *(iii)* $2a^2 + 3 + a$ *(iv)* ja. **[Elect. Meas and Meas. Inst., Madras Univ. 1984]**

Solution. (*i*) $a^2 = a \times a = 1 \angle 120° = 1 \angle 240° = 1 \angle -120° = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$

* Numerically, *a* is equivalent to the cube root of unity.

$$\therefore \quad a^2 - 1 = -\frac{1}{2} - j\frac{\sqrt{3}}{2} - 1 - j\frac{\sqrt{3}}{2}\sqrt{3}\left(-\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) = \sqrt{3}\angle 30^\circ + 180^\circ = \sqrt{3}\angle 210^\circ$$

(ii) $$a = 1\angle 120^\circ = -\frac{1}{2} + j\frac{\sqrt{3}}{2}\,;\; a^2 = 1\angle 240^\circ = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$$

$$\therefore \quad 1 - a - a^2 = 1 + \frac{1}{2} - j\frac{\sqrt{3}}{2} + \frac{1}{2} + j\frac{\sqrt{3}}{2} = 2 + j0 = 2\angle 0^\circ$$

(iii) $$2a^2 = 2\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) = -1 - j\sqrt{3}$$

$$2a = 2\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = -1 + j\sqrt{3}$$

$$\therefore \quad 2a^2 + 3 + 2a = 3 - 1 - j\sqrt{3} - 1 + j\sqrt{3} = 1\angle 0^\circ$$

(iv) $$ja = j \times a = 1\angle 90^\circ \times 1\angle 120^\circ = 1\angle 210^\circ$$

Tutorial Problem No. 12.1

1. Perform the following indicated operations :
   (a) $(60 + j\,80) + (30 - j\,40)$ (b) $(12 - j\,6) - (40 - j\,20)$ (c) $(6 + j\,8)\,(3 - j\,4)$ (d) $16 + j8) \div (3 - j\,4)$
   [(a) $(90 + j\,40)$ (b) $(-28 + j\,14)$ (c) $(50 + j\,0)$ (d) $(-0.56 + j\,1.92)$]
2. Two impedances $\mathbf{Z}_1 = 2 + j\,6\ \Omega$ and $\mathbf{Z}_2 = 6 - 12\ \Omega$ are connected in a circuit so that they are additive. Find the resultant impedance in the polar form. **[$10\angle -36.9^\circ$]**
3. Express in rectangular form and polar form a vector, the magnitude of which is 100 units and the phase of which with respect to reference axis is
   (a) $+30^\circ$ (b) $+180^\circ$ (c) -60° (d) $+120^\circ$ (e) -120° (f) -210°.
   [(a) $86.6 + j50\ \angle 30^\circ$ (b) $(-100 + j0)$, $100\angle 180^\circ$ (c) $50 - j86.6$, $100\angle -60^\circ$ (d) $(-50 + j86.6)$, $100\angle -120^\circ$ (e) $(-50 - j\,86.6)$, $100\angle -120^\circ$ (f) $(-50 + j\,86.6)$, $100\angle -210^\circ$]
4. In the equation $\mathbf{V}_m = V - \mathbf{ZI}$, $\mathbf{V} = 100\angle 0^\circ$ volts, $\mathbf{Z} = 10\angle 60^\circ\ \Omega$ and $\mathbf{I} = 8\angle -30^\circ$ amperes. Express $\mathbf{V}_m$ in polar form. **[$50.5\angle -52^\circ$]**
5. A voltage $\mathbf{V} = 150 + j\,180$ is applied across an impedance and the current flowing is found to be $\mathbf{I} = 5 - j\,4$. Determine (i) scalar impedance (ii) resistance (iii) reactance (iv) power consumed.
   [(i) $3.73\ \Omega$ (ii) $0.75\ \Omega$ (iii) $36.6\ \Omega$ (iv) 30 W]

OBJECTIVE TESTS – 12

1. The symbol j represents counterclockwise rotation of a vector thorugh—degrees.
   (a) 180 (b) 90
   (c) 360 (d) 270
2. The operator j has a value of
   (a) $+1$ (b) -1
   (c) $\sqrt{-1}$ (d) $\sqrt{+1}$
3. The vector $j^5\,E$ is the same as vector
   (a) jE (b) $j^2\,E$
   (c) $j^3\,E$ (d) $j^4\,E$
4. The conjugate of $(-a + jb)$ is
   (a) $(a - jb)$ (b) $(-a - jb)$
   (c) $(a + jub)$ (d) $(jb - a)$
5. The operator '$-a$' turns a vector through—degrees.
   (a) -120° (b) 120
   (c) 60 (d) -60
6. The polar form of the expression ja is
   (a) $2\angle 0^\circ$ (b) $\sqrt{3}\angle 210^\circ$
   (c) $1\angle 210^\circ$ (d) $1\angle 0^\circ$

ANSWERS

1. b 2. c 3. a 4. b 5. d 6. c

13 SERIES A.C. CIRCUITS

13.1. A.C. Through Resistance and Inductance

A pure resistance R and a pure inductive coil of inductance L are shown connected in series in Fig. 13.1.

Let V = r.m.s. value of the applied voltage, I = r.m.s. value of the resultant current

$V_R = IR$ – voltage drop across R (in phase with I), $V_L = I \,.\, X_L$ –voltage drop across coil (ahead of I by 90°)

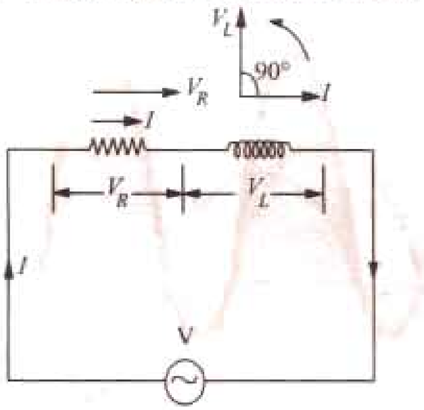


Fig. 13.1

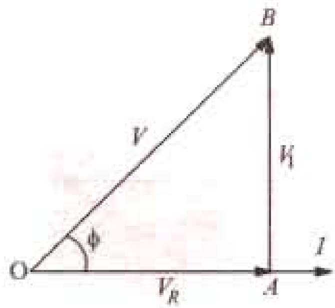


Fig. 13.2

These voltage drops are shown in voltage triangle OAB in Fig. 13.2. Vector OA represents ohmic drop V_R and AB represents inductive drop V_L. The applied voltage V is the vector sum of the two *i.e.* OB.

$$\therefore \quad V = \sqrt{(V_R^2 + V_L^2)} = \sqrt{[(IR)^2 = (I \,.\, X_L)^2]} = I\sqrt{R^2 + X_L^2} = \frac{V}{\sqrt{(R^2 + X_L^2)}}$$

The quantity $\sqrt{(R^2 + X_L^2)}$ is known as the *impedance* (Z) of the circuit. As seen from the impedance triangle ABC (Fig. 13.3) $Z^2 = R^2 + X_L^2$.

i.e. $(Impedance)^2 = (resistance)^2 + (reactance)^2$

From Fig. 13.2, it is clear that the applied voltage V leads the current I by an angle ϕ such that

$$\tan\phi = \frac{V_L}{V_R} = \frac{I \,.\, X_L}{I \,.\, R} = \frac{X_L}{R} = \frac{\omega L}{R} = \frac{\text{reactance}}{\text{reactance}} \quad \therefore \ \phi = \tan^{-1}\frac{X_L}{R}$$

The same fact is illustrated graphically in Fig. 13.4.

In other words, current I lags behind the applied voltage V by an angle ϕ.

Hence, if applied voltage is given by $v = V_m \sin \omega t$, then current equation is

$$i = I_m \sin(\omega t - \phi) \text{ where } I_m = V_m / Z$$

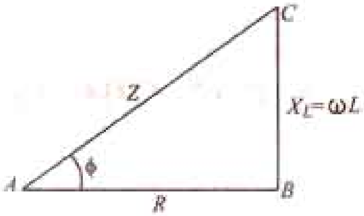


Fig. 13.3

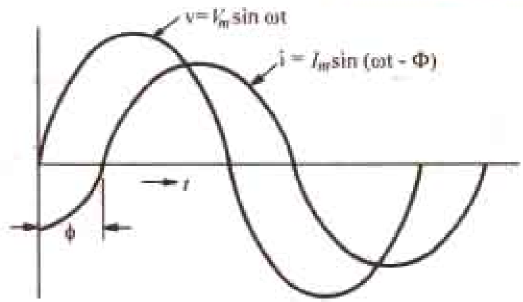


Fig. 13.4

In Fig. 13.5, I has been resolved into its two mutually perpendicular components, $I \cos \phi$ along the applied voltage V and $I \sin \phi$ in quadrature (*i.e.* perpendicular) with V.

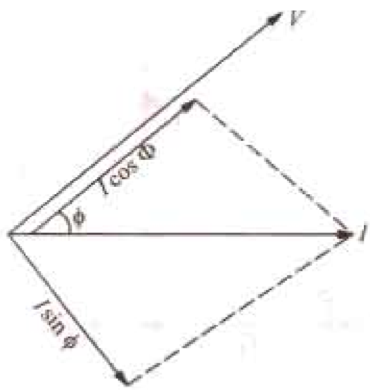


Fig. 13.5

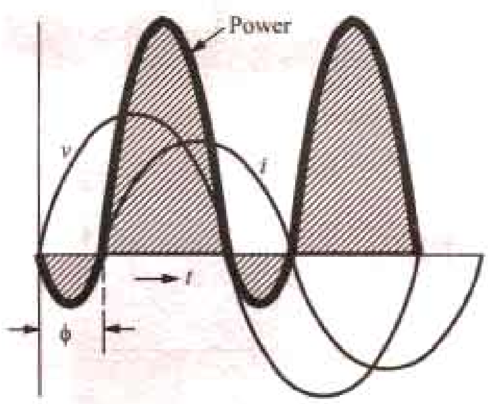


Fig. 13.6

The mean power consumed by the circuit is given by the product of V and *that component of the current I which is in phase with V.*

So $P = V \times I \cos \phi$ = r.m.s. voltage × r.m.s. current × cos ϕ

The term 'cos ϕ' is called the power factor of the circuit.

Remember that in an a.c. circuit, the product of r.m.s. volts and r.m.s. amperes gives volt-amperes (VA) and **not** *true power in watts.* True power (W) = volt-amperes (VA) × power factor.

or **Watts = VA × cos ϕ***

It should be noted that power consumed is due to ohmic resistance only because pure inductance does not consume any power.

Now $P = VI \cos \phi = VI \times (R/Z) = (V/Z) \times I \cdot R = I^2 R$ ($\because \cos \phi = R/Z$) or $P = I^2 R$ watt

Graphical representation of the power consumed is shown in Fig. 14.6.

Let us calculate power in terms of instantaneous values.

Instantaneous power is $= v\,i = V_m \sin \omega t \times I_m \sin (\omega t - \phi) = V_m I_m \sin \omega t \sin (\omega t - \phi)$

$$= \frac{1}{2} V_m\, I_m [\cos \phi - \cos (2\omega t - \Phi)]$$

* While dealing with large supplies of electric power, it is convenient to use kilowatt as the unit kW = kVA × cos ϕ

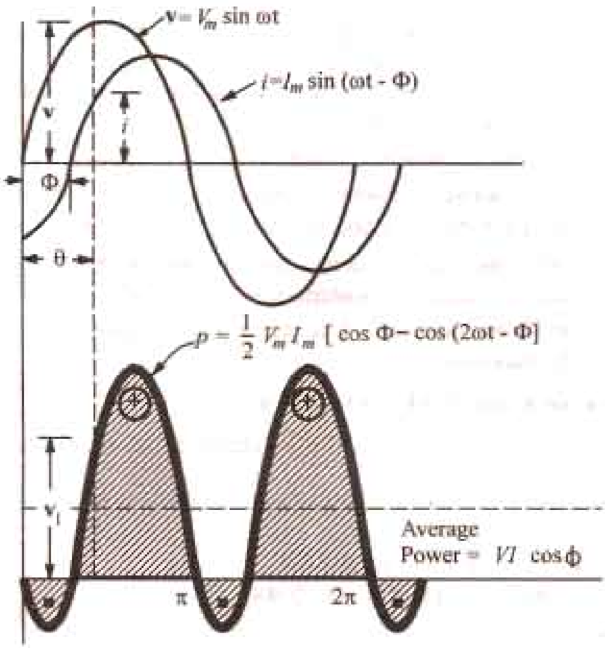


Fig. 13.7

Obviously, this power consists of two parts (Fig. 13.7).

(*i*) a constant part $\frac{1}{2} V_m I_m \cos \phi$ which contributes to real power.

(*ii*) a pulsating component $\frac{1}{2} V_m I_m \cos(2\omega t - \phi)$ which has a frequency twice that of the voltage and current. It does not contribute to actual power since its average value over a complete cycle is zero.

Hence, average power consumed $= \frac{1}{2} V_m I_m \cos \phi = \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} \cos \phi = VI \cos \phi$, where V and I represent the r.m.s values.

Symbolic Notation. $Z = R + jX_L$

Impedance vector has numerical value of $\sqrt{(R^2 + X_L^2)}$.

Its phase angle with the reference axis is $\phi = \tan^{-1}(X_L/R)$

It may also be expressed in the polar form as $\mathbf{Z} = \mathbf{Z} \angle \phi°$

(*i*) Assuming $\mathbf{V} = V \angle 0°$; $\mathbf{I} = \frac{V}{Z} = \frac{V \angle 0°}{Z \angle \phi°} = \frac{V}{Z} \angle -\phi°$ (Fig. 13.8)

It shows that current vector is lagging behind the voltage vector by $\phi°$. The numerical value of current is V/Z.

(*ii*) However, if we assumed that

$\mathbf{I} = I \angle 0$, then

$\mathbf{V} = \mathbf{IZ} = I \angle 0° \times Z \angle \phi°$
$= IZ \angle \phi°$

It shows that voltage vector is $\phi°$ ahead of current vector in ccw direction as shown in Fig. 13.9.

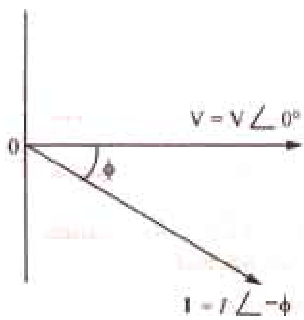


Fig. 13.8

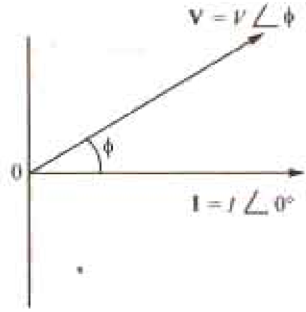


Fig. 13.9

13.2. Power Factor

It may be defined as

(*i*) consine of the angle of lead or lag

(*ii*) the ratio $\frac{R}{Z} = \frac{\text{resistance}}{\text{impedance}}$ (...Fig. 13.3) (*iii*) the ratio $\frac{\text{true power}}{\text{apparent power}} = \frac{\text{watts}}{\text{volt} - \text{amperes}} = \frac{W}{VA}$

13.3. Active and Reactive Components of Circuit Current I

Active component is that which is in phase with the applied voltage V *i.e.* $I \cos \phi$. It is also known as 'wattful' component.

Reactive component is that which in quadrature with V *i.e.* $I \sin \phi$. It is also known as 'wattless'

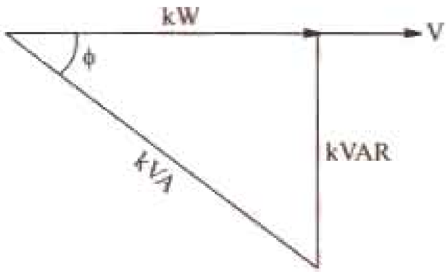


Fig. 13.10

or 'idle' component.

It should be noted that the product of volts and amperes in an a.c. circuit gives voltamperes (VA). Out of this, the actual power is $VA \cos \phi = W$ and reactive power is $VA \sin \phi$. Expressing the values in kVA, we find that it has two rectangular components :

(*i*) active component which is obtained by multiplying kVA by cos ϕ and this gives power in kW.

(*ii*) the reactive component known as reactive kVA and is obtained by multiplying kVa by sin ϕ. It is written as kVAR (kilovar). The following relations can be easily deduced.

$\text{kVA} = \sqrt{\text{kW}^2 + \text{kVAR}^2}$; kW = kVa cos ϕ and kVAR = kVA sin ϕ

These relationships can be easily understood by referring to the kVA triangle of Fig. 13.10 where it should be noted that lagging kVAR has been taken as negative.

For example, suppose a circuit draws a current of 1000 A at a voltage of 20,000 V and has a power factor of 0.8. Then

input = $1,000 \times 20,000/1000 = 20,000$ kVA; $\cos \phi = 0.8$; $\sin \phi = 0.6$

Hence kW = $20,000 \times 0.8 = 16,000$; kVAR = $20,000 \times 0.6 = 12,000$

Obviously, $\sqrt{16000^2 + 12000^2} = 20,000$ *i.e.* $\text{kVA} = \sqrt{\text{kW}^2 + \text{kVAR}^2}$

13.4. Active, Reactive and Apparent Power

Let a series *R-L* circuit draw a current of *I* when an alternating voltage of r.m.s. value *V* is applied to it. Suppose that current lags behind the applied voltage by ϕ. The three powers drawn by the circuit are as under :

(*i*) apparent power (*S*)

It is given by the product of r.m.s. values of applied voltage and circuit current.

$\therefore \quad S = VI = (IZ) \,.\, I = I^2 Z$ volt-amperes (*VA*)

(*ii*) active power (*P* or *W*)

It is the power which is actually dissipated in the circuit resistance. $P = I^2 R = VI \cos \phi$ watts

(*iii*) reactive power (Q)

It is the power developed in the inductive reactance of the circuit.

$Q = I^2 X_L = I^2 \,.\, Z \sin \phi = I \,.\, (IZ) .\, \sin \phi = VI \sin \phi$ volt-amperes-reactive (VAR)

These three powers are shown in the power triangle of Fig. 13.11 from where it can be seen that

$S^2 = P^2 + Q^2$ or $S = \sqrt{P^2 + Q^2}$

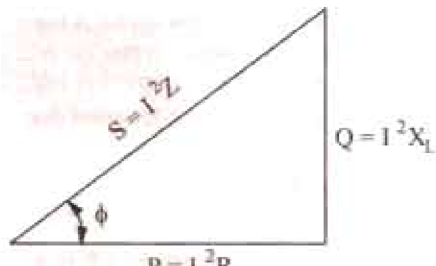


Fig. 13.11

13.5. Q-factor of a Coil

Reciprocal of power factor is called the *Q*-factor of a coil or its figure of merit. It is also known as quality factor of the coil.

$$Q \text{ factor} = \frac{1}{\text{power factor}} = \frac{1}{\cos \phi} = \frac{Z}{R}$$

If *R* is small as compared to reactance, then *Q*-factor = $Z/R = \omega L/R$

Also, $$Q = 2\pi \frac{\text{maximum energy stored}}{\text{energy dissipated per cycle}} \quad \text{—in the coil}$$

Example 13.1. *In a series circuit containing pure resistance and a pure inductance, the current and the voltage are expressed as :*

$$i(t) = 5 \sin(314t + 2\pi/3) \text{ and } v(t) = 15 \sin(314t + 5\pi/6)$$

(a) What is the impedance of the circuit ? (b) What is the value of the resistance ? (c) What is the inductance in henrys ? (d) What is the average power drawn by the circuit ? (e) What is the power factor ? **[Elect. Technology, Indore Univ. 1986]**

Solution. Phase angle of current = $2\pi/3 = 2\pi\ 180°/3 = 120°$ and phase angle of voltage = $5\pi/6 = 5 \times 180°/6 = 150°$. Also, $Z = V_m/I_m = 3\ \Omega$.

Hence, current *lags* behind voltage by 30°. It means that it is an *R-L* circuit. Also $314 = 2\pi f$ or $f = 50$ Hz. Now, $R/Z = \cos 30° = 0.866$; $R = 2.6\ \Omega$; $X_L/Z = \sin 30° = 0.5$

$\therefore\ X_L = 1.5\ \Omega \qquad 314\,L = 1.5, \qquad L = 4.78$ mH

(*a*) $Z =$ **3 Ω** (*b*) $R =$ **2.6 Ω** (*c*) $L =$ **4.78 mH**

(*d*) $P = I^2 R = (5/\sqrt{2})^2 \times 2.6 =$ **32.5 W** (*e*) p.f. = cos 30° = **0.866 (lag).**

Example 13.2. *In a circuit the equations for instantaneous voltage and current are givn by* $v = 141.4 \sin\left(\omega t - \frac{2\pi}{3}\right)$ *volt and* $i = 7.07 \sin\left(\omega t - \frac{\pi}{2}\right)$ *amp, where* $\omega = 314$ *rad/sec.*

(i) Sketch a neat phasor diagram for the circuit. (ii) Use polar notation to calculate impedance with phase angle. (iii) Calculate average power & power factor. (iv) Calculate the instantaneous power at the instant t = 0. **(F.Y. Engg. Pune Univ. Nov. 1988)**

Solution. (*i*) From the voltage equation, it is seen that the voltage lags behind the reference quantity by $2\pi/3$ radian or $2 \times 180/3 = 120°$. Similarly, current lags behind the reference quantity by $\pi/2$ radian or $180/2 = 90°$. Between themselves, voltage lags behind the current by $(120 - 90) = 30°$ as shwon in Fig. 13.12 (*b*).

(*ii*) $V = V_m/\sqrt{2} = 141.4\sqrt{2} = 100\ V$; $I = I_m/\sqrt{2} = 7.07/\sqrt{2} =$ **5 A.**

$\therefore\ V = 100 \angle -120°$ and $I = 5 \angle -90°$ $\therefore Z = \dfrac{100 \angle -120°}{5 \angle -90°} = 20 \angle -30°$

(*iii*) Average power = $VI \cos\phi = 100 \times 5 \times \cos 30° =$ **433 W**

(*iv*) At $t = 0$; $v = 141.4 \sin(0 - 120°) = -122.45\ V$; $i = 7.07 \sin(0 - 90°) = -7.07$ A.

$\therefore$ instantaneous power at $t = 0$ is given by $vi = (-122.45) \times (-7.07) =$ **865.7 W.**

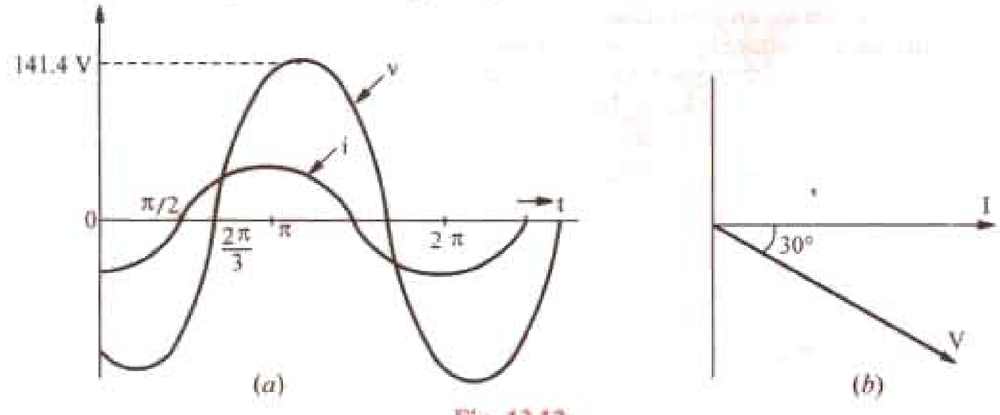


Fig. 13.12

Example 13.3. *The potential difference measured across a coil is 4.5 V, when it carries a direct current of 9 A. The same coil when carries an alternating current of 9 A at 25 Hz, the potential difference is 24 V. Find the current, the power and the power factor when it is supplied by 50 V, 50 Hz supply.* **(F.Y. Pune Univ. May 1989)**

Solution. Let R be the *d.c.* resistance and L be the inductance of the coil.

$\therefore \qquad R = V/I = 4.5/9 = 0.5\ \Omega$;

With a.c. current of 25 Hz, $Z = V/I = 24/9 = 2.66\ \Omega$.

$$X_L = \sqrt{Z^2 - R^2} = \sqrt{2.66^2 - 0.5^2} = 2.62\ \Omega \quad \text{Now } X_L = 2\pi \times 25 \times L\ ;\ L = 0.0167\ \Omega$$

At 50 Hz

$$X_L = 2.62 \times 2 = 5.24\ \Omega\ ;\ Z = \sqrt{0.5^2 + 5.24^2} = 5.26\ \Omega$$

Current $I = 50/5.26 =$ **9.5 A** ; Power $= I^2 R = 9.5^2 \times 0.5 =$ **45 W.**

Example 13.4. *In a particular R-L series circuit a voltage of 10 V at 50 Hz produces a current of 700 mA while the same voltage at 75 Hz produces 500 mA. What are the values of R and L in the circuit ?* **(Network Analysis A.M.I.E. Sec. B, S 1990)**

Solution. (*i*) $Z = \sqrt{R^2 + (2\pi \times 50\,L)^2} = \sqrt{R^2 + 98696\,L^2}$; $V = IZ$ or $10 = 700 \times 10^{-3}\sqrt{(R^2 + 98696\,L^2)}$

$$\sqrt{(R^2 + 98696\,L^2)} = 10/700 \times 10^{-3} = 100/7 \text{ or } R^2 + 98696\,L^2 = 10000/49 \quad ...(i)$$

(*ii*) In the second case $Z = \sqrt{R^2 + (2\pi \times 75L)^2} = \sqrt{(R^2 + 222066\,L^2)}$

$\therefore\ 10 = 500 \times 10^{-3}\sqrt{(R^2 + 222066\,L^2)}$ *i.e.* $\sqrt{(R^2 + 222066\,L^2)} = 20$ or $R^2 + 222066L^2 = 400$ (*ii*)

Substracting Eq. (*i*) from (*ii*), we get

$222066\,L^2 - 98696\,L^2 = 400 - (10000/49)$ or $123370\,L^2 = 196$ or $L = 0.0398\ H =$ **40 mH.**

Substituting this value of L in Eq. (*ii*), we get, $R^2 + 222066\,(0.398)^2 = 400\ \therefore$ **$R = 6.9\ \Omega$.**

Example 13.5. *A series circut consists of a resistance of 6 Ω and an inductive reactance of 8 Ω. A potential difference of 141.4 V (r.m.s.) is applied to it. At a certain instant the applied voltage is + 100 V. and is increasing. Calculate at this current, (i) The current (ii) the voltage drop across the resistance and (iii) Voltage drop across inductive reactance.* **(F.E. Pune Univ. May 1989)**

Solution. $Z = R = jX = 6 + j8 = 10\angle 53.1°$

It shows that current lags behind the applied voltage by 53.1°. Let V be taken as the reference quantity. Then $v = (141.4 \times \sqrt{2})\sin \omega t = 200 \sin \omega t$; $i = (V_m/Z \sin(\omega t) - 30° = 20 \sin(\omega t - 53.1°)$.

(*i*) When the voltage is + 100 Vand increasing; $100 = 200 \sin \omega t$; $\sin \omega t = 0.5$; $\omega t = 30°$

At this instant, the current is given by $i = 20 \sin(30° - 53.1°) = -20 \sin 23.1° =$ **− 7.847 A.**

(*ii*) drop across resistor $= iR = -7.847 \times 6 = -47$ V.

(*iii*) Let us first find the equation of the voltage drop V_L across the indcutive reactance. Maximum value of the voltage drop $= I_m X_L = 20 \times 8 = 160$ V. It leads the current by 90°. Since current itself lags the applied voltage by 53.1°, the reactive voltage drop across the applied voltage by (90° − 53.1°) = 36.9°. Hence, the equation of this voltage drop at the instant when $\omega t = 30°$ is

$$V_L = 160 \sin(30° + 36.9°) = 160 \sin 66.9° = \mathbf{147.2\ V.}$$

Example 13.6. *A 60 Hz sinusoidal voltage v = 141 sin ω t is applied to a series R-L circuit. The values of the resistance and the inductance are 3 Ω and 0.0106 H respectively.*

(i) Compute the r.m.s. value of the current in the circuit and its phase angle with respect to the voltage.

(ii) Write the expression for the instantaneous current in the circuit.

(iii) Compute the r.m.s. value and the phase of the voltages appearing across the resistance and the inductance.

(iv) Find the average power dissipated by the circuit.

(v) Calculate the p.f. of the circuit. **(F.E. Pune Univ. Nov. 1989)**

Solution. $V_m = 141$ V; $V = 141/\sqrt{2} = 100\,V \quad \therefore \quad V = 100 + j0$

$$X_L = 2\pi \times 60 \times 0.0106 = 4\ \Omega.\ \ Z = 3 + j4 = 5\angle 53.1°$$

(*i*) $I = V/Z = 100\angle 0°/5\angle 53.1° = 20\angle -53.1°$

Since angle is minus, the current lags behind the voltage by 53.1

(*ii*) $I_m = \sqrt{2} \times 20 = 28.28$; $\therefore$ $i = 28.28 \sin(\omega t - 53.1°)$

(*iii*) $VR = IR = 20\angle -53.1° \times 3 = 60\angle -53.1°$ volt.

$V_L = jIX_L = 1\angle 90° \times 20\angle -53.1° \times 4 = 80\angle 36.9°$

(*iv*) $P = VI \cos \phi = 100 \times 20 \times \cos 53.1° =$ **1200 W.**

(*v*) p.f. $= \cos \phi = \cos 53.1° =$ **0.6.**

Example 13.7. *In a given R-L circuit, R = 3.5 Ω and L = 0.1 H. Find (i) the current through the circuit and (ii) power factor if a 50-Hz voltage V = 220 ∠ 30° is applied across the circuit.*

Solution. The vector diagram is shown in Fig. 13.13.

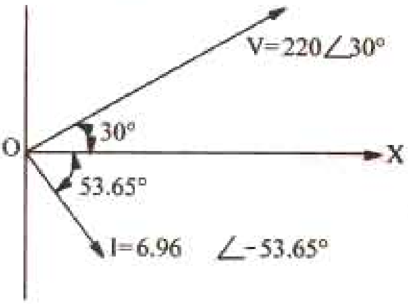


Fig. 13.13

$$X_L = 2\pi fL = 2\pi \times 50 \times 0.1 = 31.42\ \Omega$$

$$Z = \sqrt{(R^2 + X_L^2)} = \sqrt{3.5^2 = 31.42^2} = 31.6\ \Omega$$

$$\therefore \mathbf{Z} = 31.6 \angle \tan^{-1} (31.42/3.5) = 31.6 \angle 83.65°$$

(*i*) $\mathbf{I} = \dfrac{\mathbf{V}}{\mathbf{Z}} = \dfrac{220 \angle 30°}{31.6 \angle 83.65°} = 6.96 \angle -53.65°$

(*ii*) Phase angle between voltage and current is

$= 53.65° + 30° = 83.65°$ with current lagging.

p.f. $= \cos 83.65° =$ **0.11 (lag).**

Example 13.8. *In an alternating circuit, the impressed voltage is given by V = (100 − j50) volts and the current in the circuit is I = (3 - j4) A. Determine the real and reactive power in the circuit.*

(Electrical Engg., Calcutta Univ. 1991)

Solution. Power will be found by the conjugate method. Using current conjugate, we have

$$P_{VA} = (100 - j\,50)\,(3 + j\,4) = 300 + j\,400 - j\,150 + 200 = 500 + j\,250$$

Hence, real power is **500 W** and reactive power of VAR is **250**. Since the second term in the above expression is positive, the reactive volt-amperes of 250 are inductive.*

Example 13.9. *In the circuit of Fig. 14.14, applied voltage V is given by (0 + j10) and the current is (0.8 + j 0.6) A. Determine the values of R and X and also indicate if X is inductive or capacitive.* **(Elect. Technology, Nagpur Univ. 1991)**

Solution. $\mathbf{V} = 0 + j10 = \angle 90°$; $\mathbf{I} = 0.8 + j0.6 = 1\angle 36.9°$

As seen, V leads the reference quantity by 90° whereas I leads by 36.9°. In other words, *I lags* behind the applied voltage by (90° − 36.9°) = 53.1°

Hence, the circuit of Fig. 13.14 is an *R-L* circuit.

Now, $\mathbf{Z} = \mathbf{V}/\mathbf{I} = 10 \angle 90°/1 \angle 36.9° = 10 \angle 53.1° = 6 + j8$

Hence, $R = \mathbf{6\ \Omega}$ and $X_L = \mathbf{8\ \Omega}$.

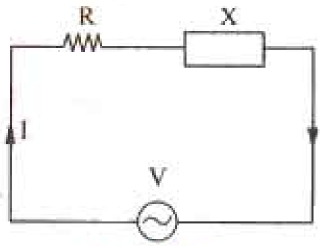


Fig. 13.14

Example 13.10. *A two-element series circuit is connected across an a.c. source e = 200√2 sin (ωt + 20°) V. The current in the circuit then is found to be i = 10√2 cos (314 t − 25°) A. Determine the parameters of the circuit.*

(Electromechanic Allahabad Univ. 1991)

Solution. The current can be written as $i = 10\sqrt{2} \sin (314\,t - 25° + 90°) = 10\sqrt{2} \sin (314\,t + 65°)$. It is seen that applied voltage leads by 20° and current leads by 65° with regards to the reference quantity, their mutual phase difference is = 65° − (20°) = 45°. Hence, p.f. = cos 45° = **$10\sqrt{2}$ (lead).**

Now, $V_m = 200\sqrt{2}$ and $I_m = 10\sqrt{2}$ $\therefore$ $Z = V_m/I_m = 200\sqrt{2}/10\sqrt{2} = 20\ \Omega$

$R = Z \cos \phi = 20/\sqrt{2}\ \Omega =$ **14.1 Ω** ; $X_c = Z \sin \phi = 20/\sqrt{2} =$ **14.1 Ω**

Now, $f = 314/2\pi = 50$ Hz. Also, $X_c = 1/2\pi fC$ $\therefore$ $C = 1/2\pi \times 50 \times 14.1 =$ **226 μF**

Hence, the given circuit is an *R-C* circuit.

* If voltage conjugate is used, then capacitive VARs are positive and inductive VARs negative. It current conjugate is used, then capacitive VARs are negative and inductive VARs are positive.

Example 13.11. *Transform the following currents to the time domain : (i) 6 – j 8 (ii) – 6 + j8 (iii) – j5.*

Solution. (*i*) Now, (6 – *j*8) when expressed in the polar form is $\sqrt{6^2+8^2}\ \angle -\tan^{-1} 8/6 = 10 \angle -53.1°$. The time domain representation of this current is $i(t) = 10 \sin(\omega t - 53.1°)$

(*ii*) $-6 + j8 = \sqrt{6^2+8^2}\ \angle \tan^{-1} 8/-6 = 10 \angle 126.9°$

$\therefore\ i(t) = 10 \sin(\omega t + 126.9°)$

(*iii*) $-j5 = 10 \angle -90°\ \therefore\ i(t) = 10 \sin(\omega t - 90°)$

Example 13.12. *A choke coil takes a current of 2 A lagging 60° behind the applied voltage of 200 V at 50 Hz. Calculate the inductance, resistance and impedance of the coil. Also, determine the power consumed when it is connected across 100-V, 25-Hz supply.*

(Elect. Engg. & Electronics, Bangalore Univ. 1989)

Solution. (*i*) $Z_{coil} = 200/2 = 100\ \Omega$; $R = Z \cos\phi = 100 \cos 60° = 50\ \Omega$

$X_L = Z \sin\phi = 100 \sin 60° = 86.6\ \Omega\ X_L = 2\pi fL = 86.6\ \therefore\ L = 86.6/2\pi \times 50 = 0.275$ H

(*ii*) Now, the coil will have different impedance because the supply frequency is different but its resistance would remain the same *i.e.* 50 Ω. Since the frequency has been halved, the inductive reactance of the coil is also halved *i.e.* it becomes 86.6/2 = 43.3 Ω.

$$Z_{coil} = \sqrt{50^2 + 43.3^2} = 66.1\ \Omega$$

$$I = 100/66.1 = 1.5 \text{ A, p.f. } \cos\phi = 50/66.1 = 0.75$$

Power consumed by the coil $= VI \cos\phi = 100 \times 1.5 \times 0.75 =$ **112.5 W**

Example 13.13. *An inductive circuit draws 10 A and 1 kW from a 200-V, 50 Hz a.c. supply. Determine :*

(i) the impedance in cartesian from (a + jb) (ii) the impedance in polar from Z ∠ θ (iii) the power factor (iv) the reactive power (v) the apparent power.

Solution. $Z = 200/10 = 20\ \Omega$; $P = I^2R$ or $1000 = 10^2 \times R$; $R = 10\ \Omega\ X_L = \sqrt{20^2 - 10^2} = 17.32\Omega$.

(*i*) $\mathbf{Z} = 10 + j17.32$ (*ii*) $|Z| = \sqrt{(10^2 + 17.32^2)} = 20\ \Omega$; $\tan\phi = 17.32/10 = 1.732$; $\phi = \tan^{-1}(1.732) = 60°\ \therefore\ \mathbf{Z = 20 \angle 60°}$. (*iii*) p.f. $\cos\phi = \cos 60° =$ **0.5 lag** (*iv*) reactive power $= VI \sin\phi = 200 \times 10 \times 0.866 =$ **1732 VAR** (*v*) apparent power $= VI = 200 \times 10 =$ **2000 VA.**

Example 13.14. *When a voltage of 100 V at 50 Hz is applied to a choking coil A, the current taken is 8 A and the power is 120 W. When applied to a coil B, the current is 10 A and the power is 500 W. What current and power will be taken when 100 V is applied to the two coils connected in series ?*

(Elements of Elect. Engg., Bangalore Univ. 1985)

Solution.

$$Z_1 = 100/8 = 12.5\ \Omega\ ;\ P = I^2 . R_1 \text{ or } 120 = 8^2 \times R_1\ ;\ R_1 = 15/8\ \Omega$$

$$X_1 = \sqrt{Z_1^2 - R_1^2} = \sqrt{12.5^2 - (15/8)^2} = 12.36\ \Omega$$

$$Z_2 = 100/10 = 10\ \Omega\ ;\ 500 = 10^2 \times R_2 \quad \text{or} \quad R_2 = 5\ \Omega$$

$$X_2 = \sqrt{10^2 - 5^2} = 8.66\ \Omega$$

With Joined in Series

$$R = R_1 + R_2 = (15/8) + 5 = 55/9\ \Omega;\ X = 12.36 + 8.66 = 21.02\ \Omega$$

$$Z = \sqrt{(55/8)^2 + (21.02)^2} = 22.1\ \Omega,\ I = 100/22.1 = 4.52 \text{ A},\ P = I^2R = 4.52^2 \times 55/8 = \mathbf{140\ W}$$

Example 13.15. *A coil takes a current of 6 A when connected to a 24-V d.c. supply. To obtain the same current with a 50-Hz a.c. supply, the voltage required was 30 V.*

Calculate (i) the inductance of the coil (ii) the power factor of the coil.

(F.Y. Engg. Pune Univ. 1989)

Solution. It should be kept in mind the coil offers only resistance to direct voltage whereas it offers impedance to an alternating voltage.

$\therefore \quad R = 24/6 = 4\ \Omega;\ Z = 30/6 = 5\ \Omega$

(*i*) $\therefore \quad X_L = \sqrt{Z^2 - R^2} = \sqrt{5^2 - 4^2}\ \ 3\ \Omega$ (*iii*) p.f. $= \cos\phi = R/Z = 4/5 =$ **0.8 (lag)**

Example 13.16. *A resistance of 20 ohm, inductance of 0.2 H and capacitance of 150 μF are connected in series and are fed by a 230 V, 50 Hz supply. Find X_L, X_C, Z, Y, p.f., active power and reactive power.* **(Elect. Science-I, Allahabad Univ. 1992)**

Solution. $X_L = 2\pi fL = 2\pi \times 50 \times 0.2 = 62.8\Omega$; $X_C = 1/2\pi fC$

$= 10^{-6}\,2\pi \times 50 \times 150 = 21.2\ \Omega$; $X = (X_L - X_C) = 41.6\ \Omega$;

$Z = \sqrt{R^2 + X^2} = \sqrt{20^2 + 41.6^2} = 46.2\ \Omega$; $I = V/Z = 230/46.2 = 4.98\ A$

Also, $Z = R + jX = 20 + j\,41.6 = 46.2\ \angle 64.3°$ ohm

$\therefore \quad Y = 1/Z = 1/46.2\ \angle 64.3° = 0.0216\ \angle -64.3°$ siemens

p.f. $= \cos 64.3° = 0.4336$ (lag)

Active power $= VI\cos\phi = 230 \times 4.98 \times 0.4336 = 497$ W

Reactive power $= VI\sin\phi = 230 \times 4.98 \times \sin 64.3° = 1031$ VAR

Example 13.17. *A 120-V, 60-W lamp is to be operated on 220-V, 50-Hz supply mains. Calculate what value of (a) non-inductive resistance (b) pure inductance would be required in order that lamp is run on correct voltage. Which method is preferable and why ?*

Solution. Rated current of the bulb = 60/120 = 0.5 A

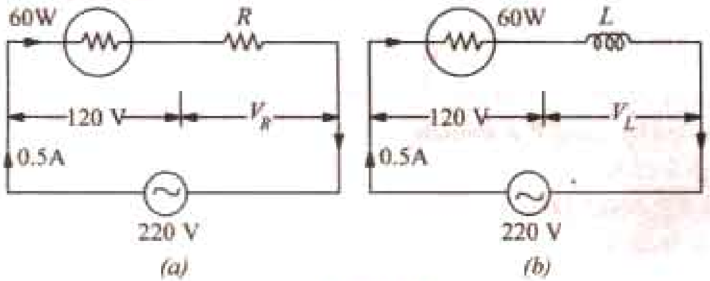


Fig. 13.15

(*a*)Resistor has been shown connected in series with the lamp in Fig. 13.15 (*a*).

P.D. across *R* is $V_R = 220 - 120 = 100$ V

It is in phase with the applied voltage, $\therefore R = 100/0.5 =$ **200 Ω**

(*b*) P.D. across bulb = 120 V

P.D. across *L* is

$V_L = \sqrt{(220^2 - 120^2)} = 184.4$ V

(Remember that V_L is in quadrature with V_R –the voltage across the bulb).

Now, $V_L = 0.5 \times X_L$ or $184.4 = 0.5 \times L \times 2\pi \times 50 \quad \therefore \quad L = 184.4/0.5 \times 3.14 =$ **1.17 H**

Method (*b*) is preferable to (*a*) because in method (*b*), there is no loss of power. Ohmic resistance of 200 Ω itself dissipates large power (*i.e.* $100 \times 0.5 =$ **50 W**).

Example 13.18. *A non-inductive resistor takes 8 A at 100 V. Calculate the inductance of a choke coil of negligible resistance to be connected in series in order that this load may be supplied from 220-V, 50-Hz mains. What will be the phase angle between the supply voltage and current ?*

(Elements of Elect. Engg.-I, Bangalore Univ. 1987)

Solution. It is a case of pure resistance in series with pure inductance as shown in Fig. 13.16 (*a*).

Here $V_R = 100$ V,

$V_L = \sqrt{(220^2 - 100^2)} = 196$ V

Now, $V_L = I \cdot X_L$

or $196 = 8 \times 2\pi \times 50 \times L =$ **0.078 H**

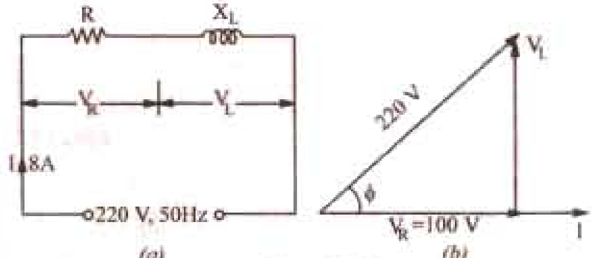


Fig. 13.16

Example 13.19. *A current of 5 A flows through a non-inductive resistance in series with a choking coil when supplied at 250-V, 50-Hz. If the voltage across the resistance is 125 V and across the coil 200 V, calculate (a) impedance, reactance and resistance of the coil (b) the power absorbed by the coil and (c) the total power. Draw the vector diagram.* **(Elect. Engg., Madras Univ. 1988)**

Solution. As seen from the vector diagram of Fig. 13.17 (*b*).

$$BC^2 + CD^2 = 200^2 \quad ...(i) \quad (125 + BC)^2 + CD^2 = 250^2 \quad ...(ii)$$

Substracting Eq. (*i*) from (*ii*), we get, $(125 + BC)^2 - BC^2 = 250^2 - 200^2$

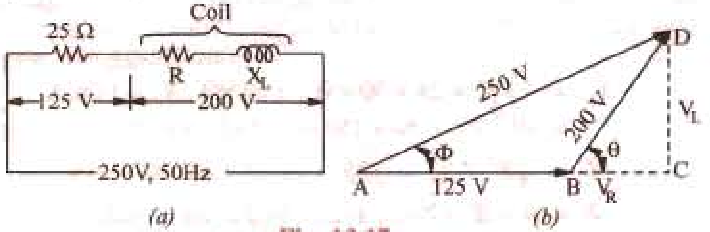


Fig. 13.17

$$\therefore \quad BC = 27.5 \text{ V}; \; CD = \sqrt{200^2 - 27.5^2} = 198.1 \text{ V}$$

(*i*) Coil impedance $= 200/5 =$ **40 Ω**

$V_R = IR = BC$ or $5R = 27.5 \quad \therefore \quad$ P = 27.5/5 = **5.5 Ω**

Also $V_L = I \cdot X_L = CD = 198.1 \quad \therefore \quad X_L = 198.1/5 =$ **39.62 Ω**

or $X_L = \sqrt{40^2 - 5.5^2} =$ **39.62 Ω**

(*ii*) Power absorbed by the coil is $= I^2 R = 5^2 \times 5.5 = 137.5$ W

Also $P = 200 \times 5 \times 27.5/200 =$ **137.5 W**

(*iii*) Total power $= VI \cos \phi = 250 \times 5 \times AC/AD = 250 \times 5 \times 152.5/250 =$ **762.5 W**

The power may also be calculated by using $I^2 R$ formula.

Series resistance $= 125/5 = 25\ \Omega$

Total *circuit* resistance $= 25 + 5.5 = 30.5\ \Omega$

$\therefore$ Total power $= 5^2 \times 30.5 =$ **762.5 W**

Example 13.20. *Two coils A and B are connected in series across a 240-V, 50-Hz supply. The resistance of A is 5 Ω and the inductance of B is 0.015 H. If the input from the supply is 3 kW and 2 kVAR, find the inductance of A and the resistance of B. Calculate the voltage across each coil.*

(Elect. Technology Hyderabad Univ. 1991)

Solution. The kVA triangle is shown in Fig. 13.18 (*b*) and the circuit in Fig. 13.18(*a*). The circuit kVA is given by, kVA $= \sqrt{(3^2 + 2^2)} = 3.606$ or $VA = 3{,}606$ voltmeters

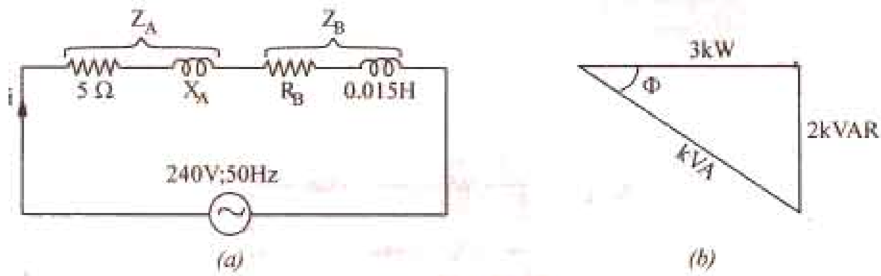


Fig. 13.18

Circuit current $= 3{,}606/240 = 15.03$ A $\quad \therefore \quad 15.03^2 (R_A + R_B) = 3{,}000$

$\therefore \quad R_A + R_B = 3{,}000/15.03^2 = 13.3\ \Omega \quad \therefore \quad R_B = 13.3 - 5 =$ **8.3 Ω**

Now, impedance of the whole circuit is given by $Z = 240/15.03 = 15.97\ \Omega$

$$\therefore \quad X_A + X_B = \sqrt{(Z^2 - (R_A + R_B)^2} = \sqrt{15.97^2 - 13.3^3} = 8.84\ \Omega$$

Now $X_B = 2\pi \times 50 \times 0.015 = 4.713\ \Omega \quad \therefore \quad X_A = 8.843 - 4.713 = 4.13\ \Omega$

or $2\pi \times 50 \times L_A = 4.13 \quad \therefore \quad L_A =$ **0.0132 H (approx)**

Now $Z_A = \sqrt{R_A^2 + X_A^2} = \sqrt{5^2 + 4.13^2} = 6.585\ \Omega$

P.D. across coil $A = I \cdot Z_A = 15.03 \times 6.485 =$ **97.5 V**; $Z_b = \sqrt{8.3^2 + 4.713^2} = 9.545\ \Omega$

$\therefore$ p.d. across coil $B = I \cdot Z_B = 15.03 \times 9.545 =$ **143.5 V**

Example 13.21. *An e.m.f. $e_0 = 141.4 \sin (377\ t + 30°)$ is impressed on the impedance coil having a resistance of 4 Ω and an inductive reactance of 1.25 Ω, measured at 25 Hz. What is the equation of the current ? Sketch the waves for i, e_R, e_L and e_0.*

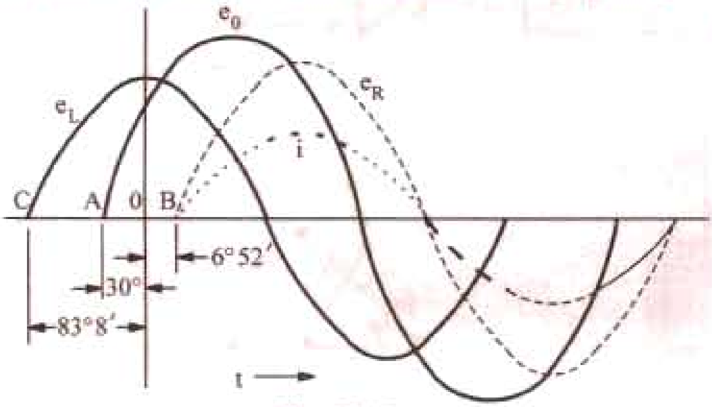


Fig. 13.19

Solution. The frequency of the applied voltage is $f = 377/2\pi = 60$ Hz

Since coil reactance is 1.25 Ω at 25 Hz, its value at 60 Hz = 1.25 × 60/25 = 3 Ω

Coil impedance, $Z = \sqrt{4^2 + 3^2} = 5\ \Omega$; $\phi = \tan^{-1} (3/4) = 36°52'$

It means that circuit current lags behind the applied voltage by 36°52′. Hence, equation of the circuit current is

$i = (141.4/5) \sin (377\ t + 30° - 36°52') = 28.3 \sin (377\ t - 6°52')$

Since, resistance drop is in phase with current, its equation is $e_R = iR = 113.2 \sin (377\ t - 6°52')$

The inductive voltage drop *leads the current* by 90°, hence its equation is

$$e_L = iX_L = 3 \times 28.3 \sin (377\ t - 6°52' + 90°) = 54.9 \sin (377\ t + 83°8')$$

The waves for i, e_R, e_L and e_0 have been drawn in Fig. 13.19.

Example 13.22. *A single phase, 7.46 kW motor is supplied from a 400-V, 50-Hz a.c. mains. If its efficiency is 85% and power factor 0.8 lagging, calculate (a) the kVA input (b) the reactive components of input current and (c) kVAR.*

Solution. Efficiency $= \dfrac{\text{output in watts}}{\text{input in watts}} \quad \therefore \quad 0.85 = \dfrac{7.46 \times 1000}{VI + \cos\phi} = \dfrac{7,460}{VI \times 0.8}$

$\therefore \quad VI = \dfrac{7460}{0.85 \times 0.8} = 10,970$ voltamperes

(*a*) $\therefore$ Input $= 10,970/1000 =$ **10.97 kVA**

(*b*) Input current $I = \dfrac{\text{voltamperes}}{\text{volts}} = \dfrac{10,970}{400} = 27.43$ A

Active component of current = $I \cos\phi$ = **27.43** × 0.8 = **21.94 A**

Reactive component of current = $I \sin\phi$ = 27.43 × 0.6 = **16.46 A** ($\because \sin\phi = 0.6$)

(Reactive component = $\sqrt{27.43^2 - 21.94^2} = 16.46$ A)

(*c*) kVAR = kVA sin φ = 10.97 × 0.6 = **6.58** (or kVAR = $VI \sin\phi \times 10^{-3} = 400 \times 16.46 \times 10^{-3}$ = 6.58)

Example 13.23. *Draw the phasor diagram for each of the following combinations :*

(i) R and L in series and combination in parallel with C.

(ii) R, L and C in series with $X_C > X_L$ when ac voltage source is connected to it.

[Nagpur University—Summer 2000]

Solution. *(i)*

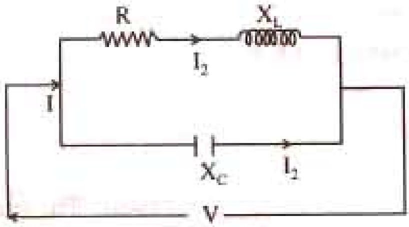


Fig. 13.20 *(a)* Circuit

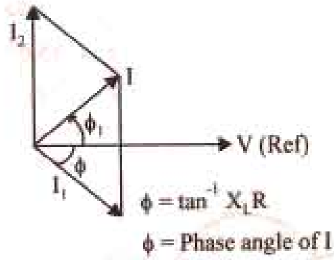


Fig. 13.20 *(b)* Phasor diagram

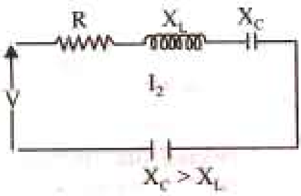


Fig. 13.20 *(c)* Circuit

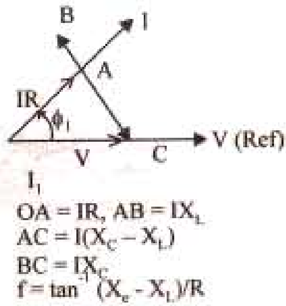


Fig . 13.20 *(d)* Phasor diagram

Example 13.24. *A voltage $v(t) = 141.4 \sin (314\, t + 10°)$ is applied to a circuit and the steady current given by $i(t) = 14.14 \sin (314\, t - 20°)$ is found to flow through it.*

Determine

(i) The p.f. of the circuit *(ii) The power delivered to the circuit*

(iii) Draw the phasor diagram. **[Nagpur University Summer 2000]**

Solution. $v(t) = 141.4 \sin (314\, t + 10°)$

This expression indicates a sinusoidally varying alternating voltage at a frequency

$\omega = 314$ rad/sec, $f = 50$ Hz

V = RMS voltage (Peak voltage)/ $\sqrt{2}$ = 100 volts

The expression for the current gives the following data :

I = RMS value = $14.14/\sqrt{2} = 10$ amp

frequency = 50 Hz, naturally.

Phase shift between I and $V = 30°$, I lags behind V.

(i) Power factor of the circuit = cos 30° = 0.866 lag

(ii) $P = VI \cos \phi = 100 \times 10 \times 0.866 = 866$ watts

(iii) Phasor diagram as drawn below, in Fig. 13.21 *(a)*.

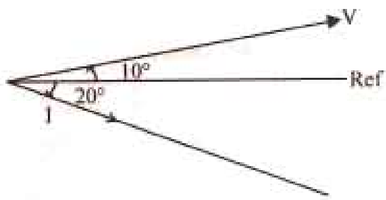


Fig. 13.21 *(a)* Phasor diagram

(ii) R, L and C in series with $X_C > X_L$ when ac voltage source is connected to it.

[Nagpur University—Summer 2000]

Solution. *(i)*

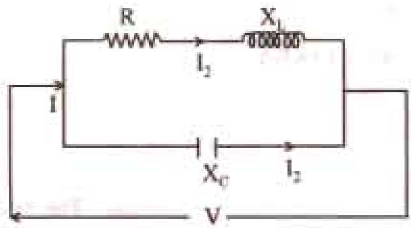


Fig. 13.20 *(a)* Circuit

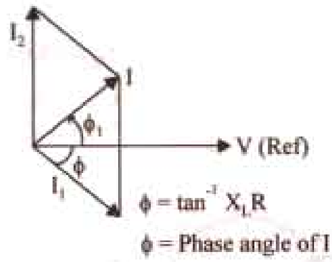


Fig. 13.20 *(b)* Phasor diagram

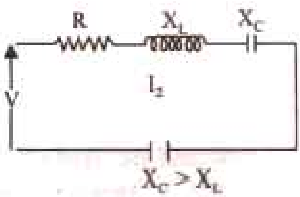


Fig. 13.20 *(c)* Circuit

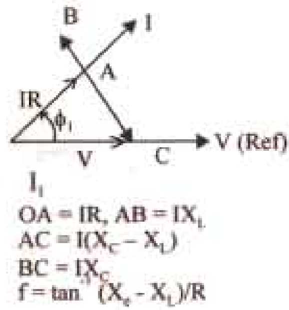


Fig . 13.20 *(d)* Phasor diagram

Example 13.24. *A voltage $v(t) = 141.4 \sin(314\,t + 10^\circ)$ is applied to a circuit and the steady current given by $i(t) = 14.14 \sin(314\,t - 20^\circ)$ is found to flow through it.*

Determine

(i) The p.f. of the circuit *(ii) The power delivered to the circuit*

(iii) Draw the phasor diagram. **[Nagpur University Summer 2000]**

Solution. $v(t) = 141.4 \sin(314\,t + 10^\circ)$

This expression indicates a sinusoidally varying alternating voltage at a frequency

$\omega = 314$ rad/sec, $f = 50$ Hz

V = RMS voltage (Peak voltage)/$\sqrt{2}$ = 100 volts

The expression for the current gives the following data :

I = RMS value = $14.14/\sqrt{2} = 10$ amp

frequency = 50 Hz, naturally.

Phase shift between I and $V = 30^\circ$, I lags behind V.

(i) Power factor of the circuit = cos 30° = 0.866 lag

(ii) $P = VI \cos\phi = 100 \times 10 \times 0.866 = 866$ watts

(iii) Phasor diagram as drawn below, in Fig. 13.21 *(a)*.

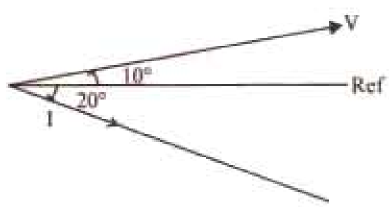


Fig. 13.21 *(a)* Phasor diagram

Example 13.25. *A coil of 0.8 p.f. is connected in series with 110 micro-farad capacitor. Supply frequency is 50 Hz. The potential difference across the coil is found to be equal to that across the capacitor. Calculate the resistance and the inductance of the coil. Calculate the net power factor.*

[Nagpur University, November 1997]

Solution. $X_C = 1/(3.14 \times C) = 28.952$ ohms

$\therefore$ Coil Impedance, $Z = 28.952\ \Omega$

Coil resistance = $28.952 \times 0.8 = 23.162\ \Omega$

Coil reactance = 17.37 ohms

Coil-inductance = 17.37/314 = 55.32 milli-henrys

Total impedance, $Z_T = 23.16 + j\,17.37 - j\,28.952 = 23.162 - j\,11.582 = 25.9$ ohms

Net power-factor = 23.162/25.9 = 0.8943 leading

Example 13.26. *For the circuit shown in Fig. 13.21 (c), find the values of R and C so that $V_b = 3V_a$, and V_b and V_a are in phase quadrature. Find also the phase relationships between V_s and V_b, and V_b and I.* **[Rajiv Gandhi Technical University, Bhopal, Summer 2001]**

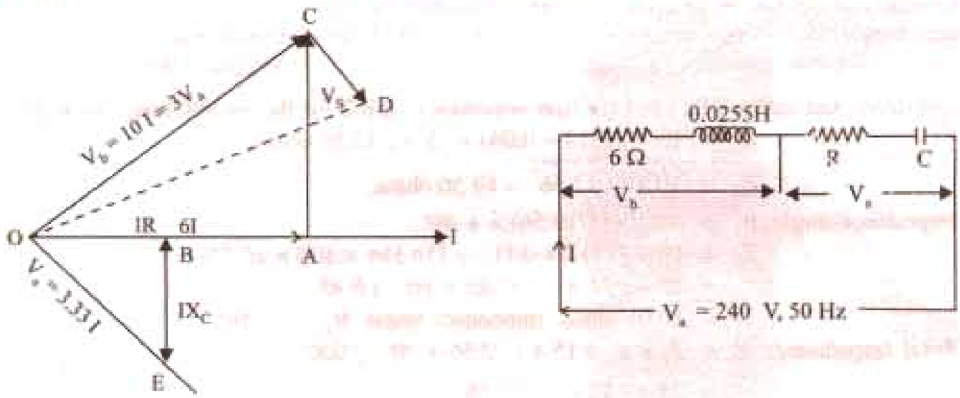


Fig. 13.21 (*b*) **Fig . 13.21 (*c*)**

Solution. $\angle COA = \phi = 53.13°$

$\angle BOE = 90° - 53.13° = 36.87°$

$\angle DOA = 34.7°$ Angle between V and I

Angle between V_s and $V_b = 18.43°$

$X_L = 314 \times 0.0255 = 8$ ohms

$Z_b = 6 + j\,8 = 10 \angle 53.13°$ ohms

$V_b = 10\,I = 3\,V_a$, and hence $V_a = 3.33\,I$

In phasor diagram, I has been taken as reference. V_b is in first quadrant. Hence V_a must be in the fourth quadrant, since Z_a consists of R and X_c. Angle between V_a and I is then 36.87°. Since Z_a and Z_b are in series, V is represented by the phasor OD which is at angle of 34.7°.

$$|V| = \sqrt{10}\,V_a = 10.53\,I$$

Thus, the circuit has a total effective impedance of 10.53 ohms.

In the phasor diagram, $OA = 6\,I$, $AC = 8\,I$, $OC = 10\,I = V_b = 3\,V_a$

Hence, $V_a = OE = 3.33\,I$,

Since $BOE = 36.87°$, $OB - RI = OE \times \cos 36.87° = 3.33 \times 0.8 \times I = 2.66\,I$.

Hence, $R = 2.66$

And $BE = OE \sin 36.87° = 3.33 \times 0.6 \times I = 2\,I$

Hence X_c = 2 ohms. For X_c = 2 ohms, $C = 1/(314 \times 2) = 1592\ \mu F$

Horizontal component of $OD = OB + OA = 8.66\ I$

Vertical component of $OD = AC - BE = 6\ I$

$$OD = 10.54\ I = V_s$$

Hence, the total impedance = 10.54 ohms = 8.66 + j 6 ohms

Angle between V_s and $I = \angle DOA = \tan^{-1}$ (6/866) = 34.7°

Example 13.27. *A coil is connected in series with a pure capacitor. The combination is fed from a 10 V supply of 10,000 Hz. It was observed that the maximum current of 2 Amp flows in the circuit when the capacitor is of value 1 microfarad. Find the parameters (R and L) of the coil.*

[Nagpur University April 1996]

Solution. This is the situation of resonance in A.C. Series circuit, for which $X_L = X_C$

$$Z = R = V/I = 10/2 = 5 \text{ ohms}$$

If ω_0 is the angular frequency, at resonance, L and C are related by $\omega_0^2 = 1/(LC)$,

which gives $L = 1/(\omega_0^2 C) = 2.5 \times 10^{-4}$ H = 0.25 mH

Example 13.28. *Two impedances consist of (resistance of 15 ohms and series-connected inductance of 0.04 H) and (resistance of 10 ohms, inductance of 0.1 H and a capacitance of 100 μF, all in series) are connectd in series and are connected to a 230 V, 50 Hz a.c. source. Find : (i) Current drawn, (ii) Voltage across each impedance, (iii) Individual and total power factor. Draw the phasor diagram.*

[Nagpur University, Nov. 1996]

Solution. Let suffix 1 be used for first impedance, and 2 for the second one. At 50 Hz,

$$Z_1 = 15 + j\,(314 \times 0.04) = 15 + j\,12.56 \text{ ohms}$$

$$Z_1 = \sqrt{15^2 + 12.56^2} = 19.56 \text{ ohms},$$

Impedance-angle, $\theta_1 = \cos^{-1}$ (15/19.56) = + 40°,

$$Z_2 = 10 + j\,(314 \times 0.1) - j\,\{1/(314 \times 100 \times 16^{-6})\}$$

$$= 10 + j\,31.4 - j\,31.85 = 10 - j\,0.45$$

$$= 10.01 \text{ ohms, Impedance angle, } \theta_2 = -\ 2.56°,$$

Total Impedance, $Z = Z_1 = Z_2 = 15 + j\,12.56 + 10 - j\,0.85$

$$= 25 + j\,12.11 = 27.78\ \underline{|25.85°}$$

For this, Phase-angle of + 25.85°, the power-factor of the total impedance

= cos 25.85° = 0.90, Lag.

Current drawn = 230/27.78 = 8.28 Amp, at 090 lagging p.f.

$$V_1 = 8.28 \times 19.56 = 162 \text{ Volts}$$

$$V_2 = 8.28 \times 10.01 = 82.9 \text{ Volts}$$

Individual Power-factor

$$\cos\theta_1 = \cos 40° = 0.766 \text{ Lagging}$$

$$\cos\theta_2 = \cos 2.56° = 0.999 \text{ leading}$$

Phasor diagram : In case of a series circuit, it is easier to treat the current as a reference. The phasor diagram is drawn as in Fig. 13.22.

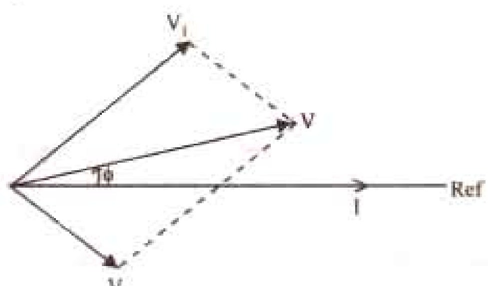


Fig. 13.22

Example 13.29. *Resistor (= R), choke-coil (r, L), and a capacitor of 25.2 μF are connected in series. When supplied from an A.C. source, in takes 0.4 A. If the voltage across the resistor is 20 V, voltage across the resistor and choke is 45 volts, voltage across the choke is 35 volts, and voltage across the capacitor is 50 V.*

Find : *(a) The values of r, L (b) Applied voltage and its frequency, (c) P.F. of the total circuit and active power consumed. Draw the phasor diagram.* **[Nagpur Univ. April 1998]**

Solution.

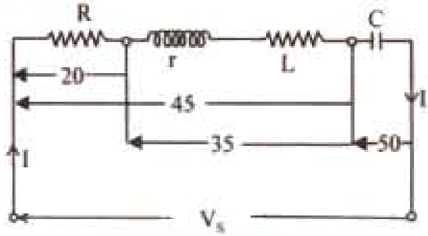


Fig. 13.23 (*a*)

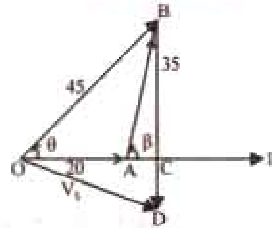


Fig. 13.23 (*b*)

(*b*) Since the current I is 0.4 amp. and voltage drop across R is 20 V,

$$R = 20/0.4 = 50 \text{ ohms}$$

Similarly, Impedance of the coil, $Z_1 = 35/0.4 = 87.5$ ohms

Capacitive reactance $X_c = 50/0.4 = 125$ ohms

With a capacitor of 25.5 μF. and supply angular frequency of ω radians/sec

$$\frac{1}{\omega} = X_C \cdot C = 125 \times 25.5 \times 10^{-6}, \text{ which gives } \omega = 314 \text{ ra./sec.}$$

The corresponding source frequency, $f = 50$ Hz

(*c*) The phasor diagram is drawn in Fig. 13.23 (*b*), taking I as the reference.

Solving triangle OAB,

$$\cos\phi = \frac{45^2 + 20^2 - 35^2}{2 \times 45 \times 20} = 0.667, \text{ and hence } \phi = 48.2°$$

$$\text{Similarly, } \cos(180° - \beta) = \frac{400 + 1225 - 2025}{2 \times 20 \times 35} = 106.6°$$

This gives $\beta = 73.4°$. From the phasor diagram is Fig. 13.23 (*b*).

$$OC = OA + AC = 20 + 35 \cos\beta = 30$$

$$BC = 35 \sin\beta = 33.54.$$ The capacitive–reactance drop is BD.

Since $BD = 50$, $CD = 16.46$ volts.

$$V_s = \sqrt{OC^2 + CD^2} = 34.22 \text{ volts}$$

$$\angle COD = \phi = \cos^{-1}(OC/OD) = 28.75°$$

The power-factor of the total circuit = cos φ = 0.877, Leading.

Since I Leads V_s in Fig. 13.23 (*b*).

(*a*) For the coil, ACB part of the phasor diagram is to be observed.

$$r = AC/I = 10/0.4 = 25 \text{ ohms}$$

$$X_L = BC/I = 33.54/0.4 = 83.85 \text{ ohms}$$

Hence, coil-inductance, $L = 83.85/314 = 267$ milli-henrys.

$$P = \text{Active Power Consumed} = V_s I \cos\phi = 12 \text{ watts}$$

or $$P = (0.4)^2 \times (R + r) = 12 \text{ watts}$$

13.6. Power in an Iron-cored Choking Coil

Total power, P taken by an iron-cored choking coil is used to supply

(*i*) power loss in ohmic resistance *i.e.* $I^2 R$. (*ii*) iron-loss in core, P_i

$\therefore$ $P^* = I^2 R + P_i$ or $\dfrac{P}{I^2} = R + \dfrac{P_i}{I^2}$ is known as the **effective resistance** of the choke.

$\therefore$ effective resistance = true resistance = equivalent resistance $\dfrac{P_i}{I^2}$ $\therefore$ $R_{eff} = \dfrac{P}{I^2} = R + \dfrac{P_i}{I^2}$

Example 13.30. *An iron-cored choking coil takes 5 A when connected to a 20-V d.c. supply and takes 5 A at 100 V a.c. and consumes 250 W. Determine (a) impedance (b) the power factor (c) the iron loss (d) inductance of the coil.* **(Elect. Engg. & M.A.S.I. June, 1991)**

Solution. (*a*) $Z = 100/5 = \mathbf{20\ \Omega}$

(*b*) $P = VI \cos \phi$ or $250 = 100 \times 5 \times \cos \phi$ $\therefore$ $\cos \phi = 250/500 = \mathbf{0.5}$

(*c*) Total loss = loss in resistance + iron loss $\therefore$ $250 = 20 \times 5 + P_i$ $\therefore$ $P_i = 250 - 100 = \mathbf{150\ W}$

(*d*) Effective resistance of the choke is $\dfrac{P}{I^2} = \dfrac{250}{25} = 10\ \Omega$

$\therefore$ $X_L = \sqrt{(Z^2 - R^2)} = \sqrt{(400 - 100)} = \mathbf{17.32\ \Omega}$

Example 13.31. *An iron-cored choking coil takes 5 A at a power factor of 0.6 when supplied at 100-V, 50 Hz. When the iron core is removed and the supply reduced to 15 V, the current rises to 6 A at power factor of 0.9.*

Determine (a) the iron loss in the core (b) the copper loss at 5 A (c) the inductance of the choking coil with core when carrying a current of 5 A.

Solution. When core is removed, then $Z = 15/6 = 2.5\ \Omega$

True resistance, $R = Z \cos \phi = 2.5 \times 0.9 = 2.25\ \Omega$

With Iron Core

Power input $= 100 \times 5 \times 0.6 = 300$ W

Power wasted in the true resistance of the choke when current is 5 A $= 5^2 \times 2.25 = 56.2$ W

(*a*) Iron loss $= 300 - 56.2 =$ **244 W (approx)** (*b*) Cu loss at 5 A = **56.2 W**

(*c*) $Z = 100/5 = 20\ \Omega$; $X_L = Z \sin \phi = 20 \times 0.8 = 16\Omega$ $\therefore$ $2\pi \times 50 \times L = 16$ $\therefore$ $L = \mathbf{0.0509\ N}$

Tutorial Problem No. 13.1

1. The voltage applied to a coil having $R = 200\ \Omega$, $L = 638$ mH is represented by $e = 20 \sin 100\,\pi\, t$. Find a corresponding expression for the current and calculate the average value of the power taken by the coil. **[$i = 0.707 \sin (100\,\pi t - \pi/4)$; 50 W]** (*I.E.E. London*)
2. The coil having a resistance of 10 Ω and an inductance of 0.2 *H* is connected to a 100-V, 50-Hz supply. Calculate (*a*) the impedance of the coil (*b*) the reactance of the coil (*c*) the current taken and (*d*) the phase difference between the current and the applied voltage. **[(*a*) 63.5 Ω (*b*) 62.8 Ω (*c*) 1.575 A (*d*) 80°57′]**
3. An inductive coil having a resistance of 15 Ω takes a current of 4 A when connected to a 100-V, 60 Hz suppy. If the coil is connected to a 100 -V, 50 Hz supply, calculate (*a*) the current (*b*) the power (*c*) the power factor. Draw to scale the vector diagram for the 50-Hz conditions, showing the component voltages. **[(*a*) 4.46 A (*b*) 298 W (*c*) 0.669]**
4. When supplied with current at 240-V, single-phase at 50 Hz, a certain inductive coil takes 13.62 A. If the frequency of supply is changed to 40 Hz, the current increases to 16.12 A. Calculate the resistance and inductance of the coil. **[17.2 W, 0.05 H]** (*London Univ.*)
5. A voltage $v(t) = 141.4 \sin (314\, t + 10°)$ is applied to a circuit and a steady current given by $i(t) = 14.4$

* At higher frequencies like radio frequencies, there is skin-effect loss also.

sin (314 t – 20°) is found to flow through it. Determine (*i*) the p.f. of the circuit and (*ii*) the power delivered to the circuit. **[0.866 (lag); 866 W]**

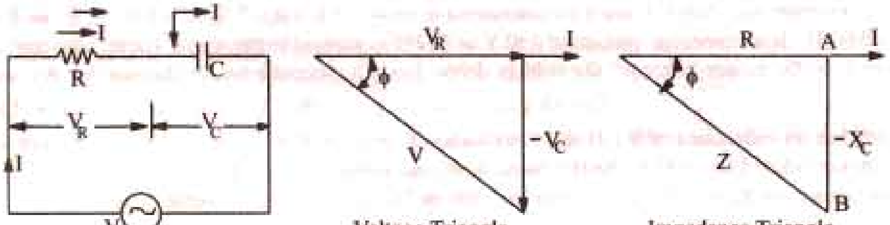
6. A circuit takes a current of 8 A at 100 V, the current lagging by 30° behind the applied voltage. Calculate the values of equivalent resistance and reactance of the circuit. **[10.81 Ω ; 6.25 Ω]**
7. Two inductive impedance *A* and *B* are connected in series. *A* has $R = 5\ \Omega$, $L = 0.01\ H$; B has $R = 3\ \Omega$, $L = 0.02$ H. If a sinusoidal voltage of 230 V at 50 Hz is applied to the whole circuit calculate (*a*) the current (*b*) the power factor (*c*) the voltage drops. Draw a complete vector diagram for the circuit. **[(*a*) 18.6 (*b*) 0.648 (*c*) V_A = 109.5 V, V_B = 129.5 V]** (*I.E.E. London*)
8. A coil has an inductance of 0.1 H and a resistance of 30 Ω at 20°C. Calculate (*i*) the current and (*ii*) the power taken from 100-V, 50-Hz mains when the temperature of the coil is 60° C, assuming the temperature coefficient of resistance to be 0.4% per°C from a basic temperature of 20°C. **[(*i*) 2.13 A (*ii*) 158.5 W]** (*London Univ.*)
9. An air-cored choking coil takes a current of 2 A and dissipates 200 W when connected to a 200-V, 50-Hz mains. In other coil, the current taken is 3 A and the power 270 W under the same conditions. Calculate the current taken and the total power consumed when the coils are in series and connected to the same supply. **[1.2, 115 W]** (*City and Guilds, London*)
10. A circuit consists of a pure resistance and a coil in series. The power dissipated in the resistance is 500 W and the drop across it is 100 V. The power dissipated in the coil is 100 W and the drop across it is 50 V. Find the reactance and resistance of the coil and the supply voltage. **[9.168Ω; 4Ω; 128.5V]**
11. A choking coil carries a current of 15 A when supplied from a 50-Hz, 230-V supply. The power in the circuit is measured by a wattmeter and is found to be 1300 watt. Estimate the phase difference between the current and p.d. in the circuit. **[0.3768]** (*I.E.E. London*)
12. An ohmic resistance is connected in series with a coil across 230-V, 50-Hz supply. The current is 1.8 A and p.ds. across the resistance and coil are 80 V and 170 V respectively. Calculate the resistance and inductance of the coil and the phase difference between the current and the supply voltage. **[61.1 Ω, 0.229 H, 34°20′]** (*App. Elect. London Univ.*)
13. A coil takes a current of 4 A when 24 V d.c. are applied and for the same power on a 50-Hz a.c. supply, the applied voltage is 40. Explain the reason for the difference in the applied voltage. Determine (*a*) the reactance (*b*) the inductance (*c*) the angle between the applied p.d. and current (*d*) the power in watts. **[(*a*) 8 Ω (*b*) 0.0255 H (*c*) 53°7′ (*d*) 96 W]**
14. An inductive coil and a non-inductive resistance *R* ohms are connected in series across an a.c. supply. Derive expressions for the power taken by the coil and its power factor in terms of the voltage across the coil, the resistance and the supply respectively. If $R = 12\ \Omega$ and the three voltages are in order, 110 V, 180 V and 240 V, calculate the power and the power factor of the coil. **[546 W; 0.331]**
15. Two coils are connected in series. With 2 A d.c. through the circuit, the p.ds. across the coils are 20 and 30 V respectively. With 2 A a.c. at 40 Hz, the p.ds. across the coils are 140 and 100 V respectively. If the two coils in series are connected to a 230-V, 50-Hz supply, calculate (*a*) the current (*b*) the power (*c*) the power factor. **[(*a*) 1.55 A (*b*) 60 W (*c*) 0.1684]**
16. It is desired to run a bank of ten 100-W, 10-V lamps in parallel from a 230-V, 50-Hz supply by inserting a choke coil in series with the bank of lamps. If the choke coil has a power factor of 0.2, find its resistance, reactance and inductance. **[R = 4.144 Ω, X = 20.35 Ω , L 0.065 H]** (*London Univ.*)
17. At a frequency for which ω = 796, an e.m.f. of 6 V sends a current of 100 mA through a certain circuit. When the frequency is raised so that ω = 2866, the same voltage sends only 50 mA through the same circuit. Of what does the circuit consist ? **[R = 52 Ω, L = 0.038 H in series]** (*I.E.E. London*)
18. An iron-cored electromagnet has a d.c. resistance of 7.5 Ω and when connected to a 400-V 50-Hz supply, takes 10 A and consumes 2 kW. Calculate for this value of current (*a*) power loss in iron core (*b*) the inductance of coil (*c*) the power factor (*d*) the value of series resistance which is equivalent to the effect of iron loss. **[1.25 kW, 0.11 H, 0.5; 12.5 Ω]** (*I.E.E. London*)

13.7. A.C. Through Resistance and Capacitance

The circuit is shown in Fig. 13.24 (*a*). Here $V_R = IR$ = drop across R –in phase when I

$V_C = IX_C$ = drop across capacitor –lagging I by $\pi/2$

As capacitive reactance X_C is taken negative, V_C is shown along negative direction of Y-axis in the voltage triangle [Fig. 13.24 (b)]



Voltage Triangle | Impedance Triangle

(a) (b) (c)

Fig. 13.24

Now $V = \sqrt{V_R^2 + (-V_C)^2} = \sqrt{(IR)^2 + (-IX_C)^2} = I\sqrt{R^2 + X_C^2}$ or $I = \dfrac{V}{\sqrt{R^2 + X_C^2}} = \dfrac{V}{Z}$

The denominator is called the *impedance* of the circuit. So, $Z = \sqrt{R^2 + X_C^2}$

Impedance triangle is shown in Fig. 13.24 (c)

From Fig. 13.24 (b) it is found that I leads V by angle ϕ such that $\tan\phi = -X_C/R$

Hence, it means that if the equation of the applied alternating voltage is $v = V_m \sin \omega t$, the equation of the resultant current in the R-C circuit is $i = I_m \sin(\omega t + \phi)$ so that current *leads* the applied voltage by an angle ϕ. This fact is shown graphically in Fig. 13.25.

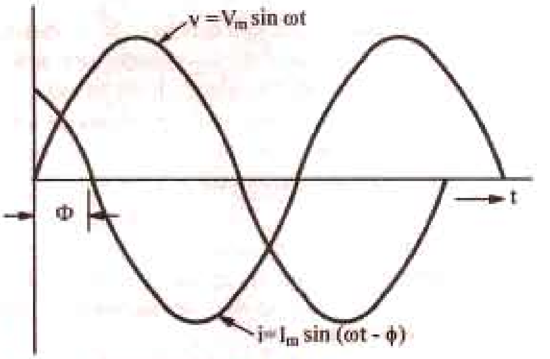


Fig. 13.25

Example 13.32. *An a.c. voltage (80 + j 60) volts is applied to a circuit and the current flowing is (– 4 + j 10) amperes. Find (i) impedance of the circuit (ii) power consumed and (iii) phase angle.* **[Elect. Technology, Indore, Univ. 1989, Bombay Univ. 1999]**

Solution. $\mathbf{V} = (80 + j\,60) = 100 \angle 36.9°$;

$\mathbf{I} = -4 + j\,10 = 10.77 \angle \tan^{-1}(-2.5) = 10.77 \angle (180° - 68.2°) = 10.77 \angle 111.8°$

(i) $\mathbf{Z} = \mathbf{V}/\mathbf{I} = 100 \angle 36.9°/10.77 \angle 111.8°$

$= 9.28 \angle -74.9°$

$= 9.28 (\cos 74.9° - j \sin 74.9°) = 2.42 - j\,8.96\ \Omega$

Hence $R = 2.42\ \Omega$ and $X_C = 8.96\ \Omega$ capacitive

(ii) $P = I^2 R = 10.77^2 \times 2.42 = \mathbf{2.81\ W}$

(iii) Phase angle between voltage and current = 74.9° with current *leading* as shown in Fig. 13.26.

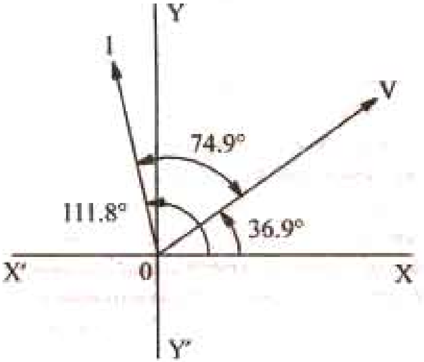


Fig. 13.26

Alternative Method for Power

The method of conjugates will be used to determine the real power and reactive volt-ampere. It is a convenient way of calculating these quantities when both voltage and current are expressed in cartesian form. If the conjugate of *current* is multiplied by the voltage in cartesian form, the result is a complex quantity, the real part of which gives the real power j part of which gives the reactive volt-amperes (VAR). It should, however, be noted that real power as obtained by this method of conjugates is the same regardless of whether V or I is reversed although sign of voltamperes will depend on the choice of V or I.*

Using current conjugate, we get $P_{VA} = (80 + j\,60)\,(-4 - j\,10) = 280 - j\,1040$

$\therefore$ Power consumed = **280 W**

Example 13.33. *In a circuit, the applied voltage is 100 V and is found to lag the current of 10 A by 30°. (i) Is the p.f. lagging or leading ? (ii) What is the value of p.f. ?*

(iii) Is the circuit inductive or capacitive ? (iv) What is the value of active and reactive power in the circuit ? **(Basic Electricity, Bombay Univ. 1985)**

Solution. The applied voltage lags behind the current which, in other words, means that current leads the voltage.

(*i*) $\therefore$ p.f. is **leading** (*ii*) p.f. = $\cos\phi = \cos 30° =$ **0.866 (lead)** (*iii*) Circuit is **capacitive** (*iv*) Active power = $VI\cos\phi = 100 \times 10 \times 0.866 =$ **866 W**

$$\text{Reactive power} = VI\sin\phi = 100 \times 10 \times 0.5 = \textbf{500 VAR (lead)}$$

or $$VAR = \sqrt{(VA)^2 - W^2} = \sqrt{(100 \times 10)^2 - 866^2} = \textbf{500 (lead)}$$

Example 13.34. *A tungsten filament bulb rated at 500-W, 100-V is to be connected to series with a capacitance across 200-V, 50-Hz supply. Calculate :*

(a) the value of capacitor such that the voltage and power consumed by the bulb are according to the rating of the bulb. (b) the power factor of the current drawn from the supply. (c) draw the phasor diagram of the circuit. **(Elect. Technology-1, Nagpur Univ. 1991)**

Solution. The rated values for bulb are : voltage = 100 V and current $I = W/V = 500/100 = 5\text{A}$. Obviously, the bulb has been treated as a pure resistance :

(*a*) $V_C = \sqrt{220^2 - 100^2} = 196\ V$

Now, $IX_C = 196$ or $5\,X_V = 196$, $X_C = 39.2\ \Omega$

$\therefore 1/\omega C = 39.2$ or $C = 1/314 \times 39.2 =$ **81 μF**

(*b*) p.f. = $\cos\phi = V_R/V = 100/200 =$ **0.455 (lead)**

(*c*) The phasor diagram is shown in Fig 13.27.

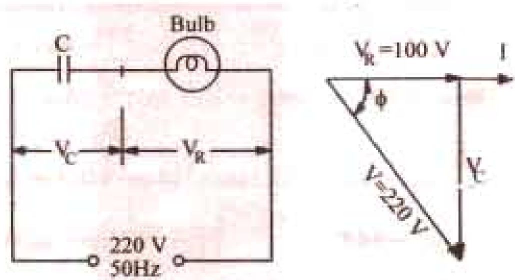


Fig 13.27

Example 13.35. *A pure resistance of 50 ohms is in series with a pure capacitance of 100 microfarads. The series combination is connected across 100-V, 50-Hz supply. Find (a) the impedance (b) current (c) power factor (d) phase angle (e) voltage across resistor (f) voltage across capacitor. Draw the vector diagram.* **(Elect. Engg.-1, JNT Univ. Warrangel 1985)**

Solution. $$X_C = 10^6\pi/2\pi \times 50 \times 100 = 32\ \Omega\ ;\ R = 50\ \Omega$$

(*a*) $Z = \sqrt{50^2 + 32^2}$ **59.4 Ω** (*b*) $I = V/Z = 100/59.4 =$ **1.684 A**

(*c*) p.f. = $R/Z = 50/59.4 =$ **0.842 (lead)** (*d*) $\phi = \cos^{-1}(0.842) =$ **32°36′**

(*e*) $V_R = IR = 50 \times 1.684 =$ **84.2 V** (*f*) $V_C = IX_C = 32 \times 1.684 =$ **53.9 V**

* If voltage conjugate is used, then capacitive VARs are positive and inductive VARs negative. If current conjugate is used, then capacitive VARs are negative and inductive VARs are positive.

Example 13.36. *A 240-V, 50-Hz series R-C circuit takes an r.m.s. current of 20 A. The maximum value of the current occurs 1/900 second before the maximum value of the voltage. Calculate (i) the power factor (ii) average power (iii) the parameters of the circuit.*

(Elect. Engg.-I, Calcutta Univ. 1987)

Solution. Time-period of the alternating voltage is 1/50 second. Now a time interval of 1/50 second corresponds to a phase difference of 2 π radian or 360°. Hence, a time interval of 1/900 second corresponds to a phase difference of 360 × 50/900 = 20°.

Hence, current leads the voltage by 20°.

(*i*) power factor = cos 20° = **0.9397 (lead)**

(*ii*) averge power = 240 × 20 × 0.9397 = **4,510 W**

(*iii*) $Z = 240/20 = 12\ \Omega$; $R = Z \cos \Phi = 12 \times 0.9397 =$ **11.28 Ω**

$X_C = Z \sin \Phi = 12 \times \sin 20° = 12 \times 0.342 = 4.1\ \Omega$

$C = 10^6/2\pi \times 50 \times 4.1 =$ **775 μF**

Example 13.37. *A voltage v = 100 sin 314 t is applied to a circuit consisting of a 25 Ω resistor and an 80 μF capacitor in series. Determine : (a) an expression for the value of the current flowing at any instant (b) the power consumed (c) the p.d. across the capacitor at the instant when the current is one-half of its maximum value.*

Solution. $X_C = 1/(314 \times 80 \times 10^{-6}) = 39.8\ \Omega$, $Z = \sqrt{25^2 + 39.8^2} = 47\ \Omega$

$I_m = V_m/Z = 100/47 = 2.13$ A

$\phi = \tan^{-1}(39.8/25) = 57°52' = 1.01$ radian (lead)

(*a*) Hence, equation for the instantaneous current

$i =$ **2.13 sin (314 t + 1.01)** (*b*) Power $= I^2 R = (2.13/\sqrt{2})^2 \times 25 =$ **56.7 W** (*c*) The voltage across the capacitor lags the circuit current by π/2 radians. Hence, its equation is given by

$$v_c = V_{cm} \sin\left(314\,t + 1.01 - \frac{\pi}{2}\right) \text{ where } V_{cm} = I_m \times X_C = 2.13 \times 39.8 = 84.8\,V$$

Now, when i is equal to half the maximum current (say, in the positive direction) then

$$i = 0.5 \times 2.13 \text{ A}$$

$\therefore\quad 0.5 \times 2.13 = 2.13 \sin(314\,t + 1.01)$ or $314\,t + 1.01 = \sin^{-1}(0.5) = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$ radian

$\therefore\quad v_C = 84.8 \sin\left(\dfrac{\pi}{6} - \dfrac{\pi}{2}\right) = 84.8 \sin(-\pi/3) = -73.5$ V

or $\quad v_c = 34.8 \sin\left(\dfrac{5\pi}{6} - \dfrac{\pi}{2}\right) = 84.4$ si $\pi/3 =$ **73.5 V**

Hence, p.d. Across the capacitor is **73.5 V**

Example 13.38. *A capacitor and a non-inductive resistance are connected in series to a 200-V, single-phase suppy. When a voltmeter having a non-inductive resistance of 13,500 Ω is connected across the resistor, it reads 132 V and the current then taken from the supply is 22.35 mA.*

Indicate on a vector diagram, the voltages across the two components and also the supply current (a) when the voltmeter is connected and (b) when it is disconnected.

Solution. The circuit and vector diagrams are shown in Fig. 13.28 (*a*) and (*b*) respectively.

(*a*) $$V_C = \sqrt{200^2 - 132^2} = \textbf{150 V}$$

It is seen that $\phi = \tan^{-1}(150/132) = 49°$ in Fig. 13.28 (*b*). Hence

(*i*) Supply voltage lags behind the current by 49°. (*ii*) V_R leads supply voltage by 49° (*iii*) V_C lags behind the supply voltage by (90° – 49°) = 41°

The supply current is, as given equal to **22.35 mA.** The value of unknown resistance R can be found as follows :

Current through voltmeter = 132/13,500 = 9.78 mA

$\therefore$ Current through R = 22.35 − 9.78 = 12.57 mA $\therefore$ $R = 132/12.57 \times 10^{-3} = 10{,}500\ \Omega$;

$X_C = 150/22.35 \times 10^{-3} = 6.711\ \Omega$

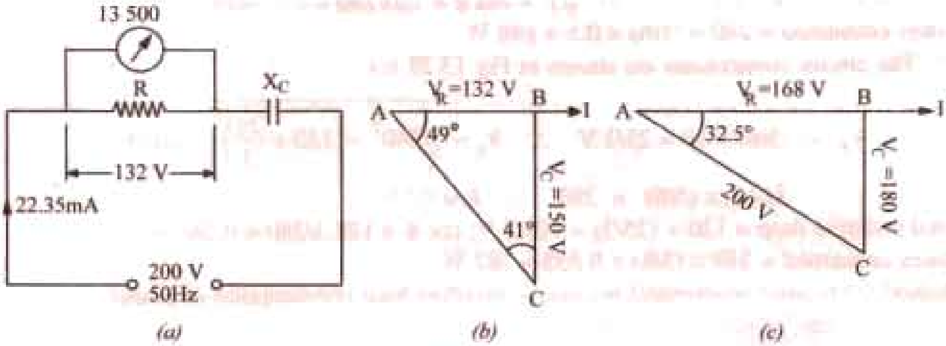


Fig. 13.28

(*b*) **When voltmeter is disconnected,** $Z = \sqrt{R^2 + X_L^2} = \sqrt{10{,}500^2 + 6{,}730^2} = 12{,}500\ \Omega$

Supply current = 200/12,461 = **16.0 mA**

In this case, $V_R = 16.0 \times 10^{-3} \times 10{,}500$ = **168 V**

$V_C = 16.0 \times 10^{-3} \times 6711$ = **107.4 V**; $\tan \phi = 107.4/168$

$\therefore$ $\phi = 32.5°$.

In this case, the supply voltage lags the circuit current by 32.5° as shown in Fig. 13.28 (*c*).

Example 13.39. *It is desired to operate a 100-W, 120-V electric lamp at its current rating from a 240-V, 50-Hz supply. Give details of the simplest manner in which this could be done using (a) a resistor (b) a capacitor and (c) an indicator having resistance of 10 Ω. What power factor would be presented to the supply in each case and which method is the most economical of power.*

(Principles of Elect. Engg.-I, Jadavpur Univ. 1985)

Solution. Rated current of the bulb is = 100/120 = 5/6 A

The bulb can be run at its correct rating by any one of the three methods shown in Fig. 13.29. (*a*) With reference to Fig. 13.29 (*a*), we have

P.D. Across R = 240 − 120 = 120 V

$\therefore$ $R = 120/(5/6)$ = **144 Ω**

Power factor of the circuit is **unity**. Power consumed = 240 × 5/6 = **200 W**

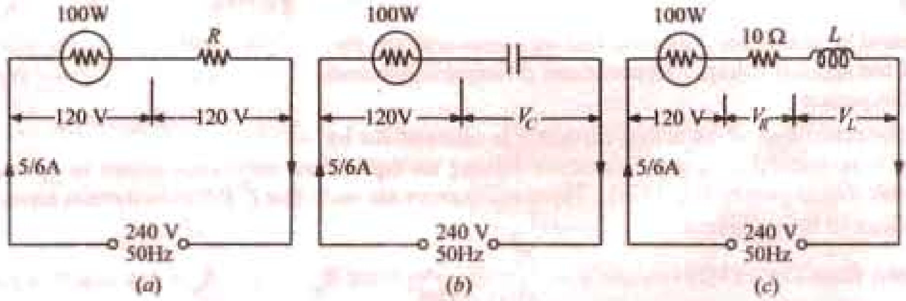


Fig. 13.29

(*b*) Referring to Fig. 13.29 (*b*), we have

$$V_C = \sqrt{240^2 - 120^2} = 207.5\,V;\ X_C = 207.5\,(5/6) = 249\,\Omega$$

$\therefore$ $1/314\,C = 249$ or $C = 12.8\ \mu F$; p.f. = $\cos\phi = 120/240 = 0.5$ **(lead)**

Power consumed = $240 \times (5/6) \times 0.5 = 100$ **W**

(*c*) The circuit connections are shown in Fig 13.29 (*c*)

$$V_R = (5/6) \times 10 = 25/3\ \text{V} \quad \therefore \quad V_L = \sqrt{240^2 - \left(120 + \frac{25}{3}\right)^2} = 203\,V$$

$$\therefore \qquad 314\,L \times (5/6) = 203 \quad \therefore \quad L = 0.775\ \text{H}$$

Total resistive drop = 120 + (25/3) = 128.3 V; $\cos\phi = 128.3/240 = 0.535$ **(lag)**

Power consumed = $240 \times (5/6) \times 0.535 = 107$ **W**

Method (*b*) is most economical because it involves least consumption of power.

Example 13.40. *A two-element series circuit consumes 700 W and has a p.f. = 0.707 leading. If applied voltage is v = 141.1 sin (314 t + 30°), find the circuit constants.*

Solution. The maximum value of voltage is 141.4 V and it leads the reference quantity by 30°. Hence, the given sinusoidal voltage can be expressed in the phase form as

$\mathbf{V} = (141.4/\sqrt{2}) \angle 30° = 100 \angle 30°$ now, $P = VI\cos\phi$ $\therefore$ $700 = 100 \times I \times 0.707$; $I = 10$ A.

Since p.f. $= 0.707$ (lead); $\phi = \cos^{-1}(0.707) = 45°$ (lead).

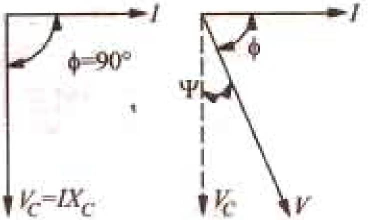
It means that current leads the given voltage by 45° for it *leads the common reference quantity by (30° + 45°) = 75°*. Hence, it can be expressed as $\mathbf{I} = 10 \angle 75°$

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{100 \angle 30°}{10 \angle 75°} = 10 \angle -45° = 7.1 - j\,7.1 \therefore\ R = \mathbf{7.1\ \Omega}$$

Since $X_C = 7.1$ $\therefore$ $1/314\,C = 7.1$; $\therefore$ $C = 450\ \mu F$

13.8. Dielectric Loss and Power Factor of a Capacitor

An ideal capacitor is one in which there are no losses and whose current leads the voltage by 90° as shown in Fig. 13.30 (*a*). In practice, it is impossible to get such a capacitor although close approximation is achieved by proper design. In every capacitor, there is always some dielectric loss and hence it absorbs some power from the circuit. Due to this loss, the phase angle is somewhat less than 90° [Fig. 13.30 (*b*)]. In the case of a capacitor with a poor dielectric, the loss can be considerable and the phase angle much less than 90°. This dielectric loss appears as heat. By *phase difference* is meant the difference between the ideal and actual phase angles. As seen from Fig. 13.30 (*b*), the phase difference ψ is given by $\psi = 90 - \phi$ where ϕ is the actual phase angle, $\sin\psi = \sin(90 - \phi) = \cos\phi$ where $\cos\phi$ is the power factor of the capacitor.



(*a*) Ideal (*b*) Actual

(*a*) (*b*)

Fig. 13.30

Since ψ is generally small, $\sin\psi = \psi$ (in radians) $\therefore$ $\tan\psi = \psi = \cos\phi$.

It should be noted that dielectric loss increases with the frequency of the applied voltage. Hence phase difference increases with the frequency *f*.

The dielectric loss of an actual capacitor is allowed for by imagining it to consist of a pure capacitor having an equivalent resistance either in series or in parallel with it as shown in Fig. 13.31. These resistances are such that I^2R loss in them is equal to the dielectric loss in the capacitor.

As seen from Fig. (13.27 (*b*), $\tan\psi = \dfrac{IR_{se}}{IX_C} = \dfrac{R_{se}}{I/\omega C} = \omega CR_{se}$ $\quad\therefore\quad$ $R_{se} = \tan\psi/\omega\,C = \text{p.f.}/\omega C$

Similarly, as seen from Fig. 13.31 (*d*), $\tan\psi = \dfrac{I_1}{I_2} = \dfrac{V/R_{sh}}{V/X_C} = \dfrac{X_C}{R_{sh}} = \dfrac{I}{\omega\,CR_{sh}}$

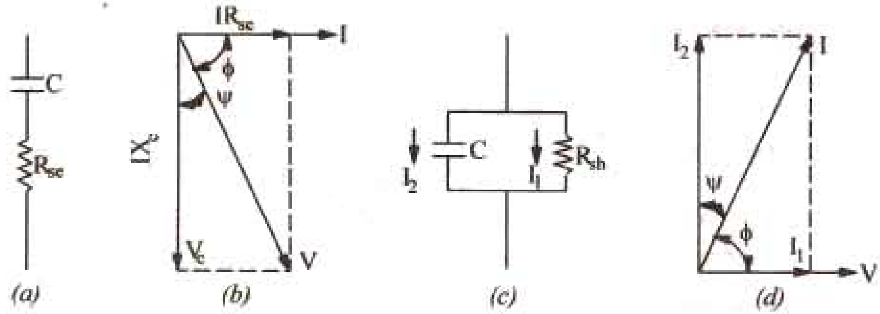


Fig. 13.31

$$R_{sh} = \frac{1}{\omega C\ .\ \tan\psi} = \frac{1}{\omega C \times \text{power factor}} = \frac{1}{\omega C \times \text{p.f.}}$$

The power loss in these resistances is $P = V^2/R_{sh} = \omega CV^2 \tan\psi = \omega\ CV^2 \times \text{p.f.}$

or $= I^2 R_{se} = (I^2 \times \text{p.f.})/\omega\ C$

where p.f. stands for the power factor of the capacitor.

Note. (*i*) In case, ψ is not small, then as seen from Fig. 13.41 (*b*) $\tan\phi = \dfrac{X_C}{R_{se}}$ (Ex. 13.41) $R_{se} = X_C/\tan\phi$

From Fig. 13.31 (*d*), we get $\tan\phi = \dfrac{I_2}{I_1} = \dfrac{V/X_C}{V/R_{sh}} = \dfrac{R_{sh}}{X_C}$ $\quad\therefore\quad R_{sh} = X_C \tan\phi = \tan\phi/\omega C$

(*ii*) It will be seen from above that both R_{se} and R_{sh} vary inversely as the frequency of the applied voltage. In other words, the resistance of a capacitor decreases in proportion to the increase in frequency.

$$\frac{R_{se1}}{R_{se2}} = \frac{f_2}{f_1}$$

Example 13.41. *A capacitor has a capacitance of 10 μF and a phase difference of 10°. It is inserted in series with a 100 Ω resistor across a 200-V, 50-Hz line. Find (i) the increase in resistance due to the insertion of this capacitor (ii) power dissipated in the capacitor and (iii) circuit power factor.*

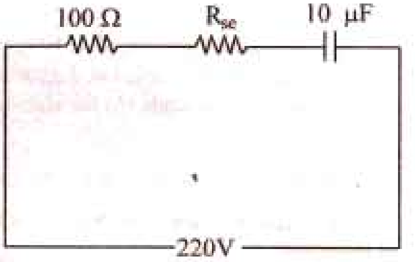


Fig. 13.32

Solution. $X_C = \dfrac{10^6}{2\pi \times 50 \times 10} = 318.3\ \Omega$

The equivalent series resistance of the capacitor in Fig. 13.32 is $R_{se} = X_C/\tan\phi$

Now $\quad \phi = 90 - \psi = 90° - 10° = 80°$

$\tan\phi = \tan 80° = 5.671$

$\therefore \quad R_{se} = 318.3/5.671 = 56.1\ \Omega$

(*i*) Hence, resistance of the circuit increases by **56.1 Ω**.

(*ii*) $\quad Z = \sqrt{(R + R_{se})^2 + X_C^2} = \sqrt{156.1^2 + 318.3^2} = 354.4\ \Omega\ ;\ I = 220/354 = 0.62\ \text{A}$

Power dissipated in the capacitor $= I^2 R_{se} = 0.62^2 \times 56.1 =$ **21.6 W**

(*iii*) Circuit power factor is $= (R + R_{se})/Z = 156.1/354/4 =$ **0.44 (lead)**

Example 13.42. *Dielectric heating is to be employed to heat a slab of insulating material 2 cm thick and 150 sq. cm in area. The power required is 200 W and a frequency of 30 MHz is to be used. The material has a relative permittivity of 5 and a power factor of 0.03. Determine the voltage necessary and the current which will flow through the material. If the voltage were to be limited to 600-V, to what would the frequency have to be raised ?*

[Elect. Engg. AMIETE (New Scheme) June 1992]

Solution. The capacitance of the parallel-plate capacitor formed by the insulating slab is

$$C = \frac{\varepsilon_0 \varepsilon_r A}{d} = \frac{8.854 \times 10^{-12} \times 5 \times 150 \times 10^{-4}}{2 \times 10^{-2}} = 33.2 \times 10^{-12}\ F$$

As shown in Art. 13.8 $\quad R_{sh} = \dfrac{1}{\omega C \times p.f.} = \dfrac{1}{(2\pi \times 30 \times 10^6) \times 33.2 \times 10^{-12} \times 0.05} = 3196\ \Omega$

Now, $P = V^2/R_{sh}$ or $V = \sqrt{P \times R_{sh}} = \sqrt{200 \times 3196} = 800\ V$

Current $I = V/X_C = \omega CV = (2\pi \times 30 \times 10^6) \times 33.2 \times 10^{-12} \times 800 = 5\ A$

Now, as seen from above $P = \dfrac{V^2}{R_{sh}} = \dfrac{V^2}{1/\omega C \times p.f.} = V^2 \omega C \times \text{p.f.}$ or $P \propto V^2 f$

$$\therefore \quad 800^2 \times 30 = 600^2 \times f \text{ or } f = \left(\frac{800}{600}\right)^2 \times 30 = \mathbf{53.3\ MHz}$$

Tutorial Problem No. 13.2

1. A capacitor having a capacitance of 20 μF is connected in series with a non-inductive resistance of 120 Ω across a 100-V, 50-Hz supply, Calculate (*a*) voltage (*b*) the phase difference between the current and the supply voltage (*c*) the power. Also draw the vector diagram. **[(*a*) 0.501A (*b*) 52.9° (*c*) 30.2 W]**
2. A capacitor and resistor are connected in series to an a.c. supply of 50 V and 50 Hz. The current is 2 A and the power dissipated in the circuit is 80 W. Calculate the resistance of the resistor and the capacitance of the capacitor. **[20 Ω ; 212 μF]**
3. A voltage of 125 *V* at 50 Hz is applied to a series combination of non-inductive resistor and a lossless capacitor of 50 μF. The current is 1.25 A. Find (*i*) the value of the resistor (*ii*) power drawn by the network (*iii*) the power factor of the network. Draw the phasor diagram for the network. **[(*i*) 77.3 Ω (*ii*) 121 W (*iii*) 0.773 (lead)]** (*Electrical Technology-I, Osmania Univ. Dec. 1979*)
4. A black box contains a two-element series circuit. A voltage (40 − *j*30) drives a current of (40 − *j*3) A in the circuit. What are the values of the elements ? Supply frequency is 50 Hz. **[*R* = 1.05 ; *C* 4750 μF]** (*Elect. Engg. and Electronics Bangalore Univ. 1986*)
5. Following readings were obtained from a series circuit containing resistance and capacitance :
$$V = 150\ V;\ I = 2.5\ A;\ P = 37.5\ W,\ f = 60\ Hz.$$
Calculate (*i*) Power factor (*ii*) effective resistance (*iii*) capacitive reactance and (*iv*) capacitance. **[(*i*) 0.1 (*ii*) 6 Ω (*iii*) 59.7 Ω (*iv*) 44.4 μF]**
6. An alternating voltage of 10 volt at a frequency of 159 kHz is applied across a capacitor of 0.01 μF. Calculate the current in the capacitor. If the power dissipated within the dielectric is 100 μW, calculate (*a*) loss angle (*b*) the equivalent series resistance (*c*) the equivalent parallel resistance. **[0.A (*a*) 10^{-4} radian (*b*) 0.01 Ω (*c*) 1 MΩ]**

13.9. Resistance, Inductance and Capacitance in Series

The three are shown in Fig. 13.33 (*a*) joined in series across an a.c. supply of r.m.s. voltage *V*.

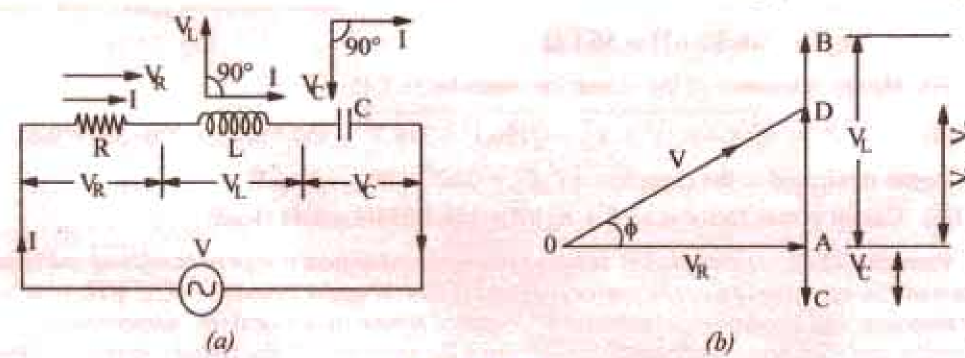


Fig. 13.33

Let $V_R = IR$ = voltage drop across R —in phase with I

$V_L = I.X_L$ = voltage drop across L —leading I by $\pi/2$

$V_C = I.X_C$ = voltage drop across C —lagging I by $\pi/2$

In voltage triangel of Fig. 13.33 (*b*), *OA* represents V_R, *AB* and *AC* represent the inductive and capacitive drops respectively. It will be seen that V_L and V_C are 180° out of phase with each other *i.e.* they are in direct opposition to each other.

Subtracting *BD* (= *AC*) from *AB*, we get the net reactive drop $AD = I\,(X_L - X_C)$

The applied voltage V is represented by *OD* and is the vector sum of *OA* and *AD*

$$\therefore \quad OD = \sqrt{OA^2 + AD^2} \text{ or } V = \sqrt{(IR)^2 + (IX_1 - IX_C)^2} = I\sqrt{R^2 + (X_L - X_C)^2}$$

$$\text{or} \quad I = \frac{V}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{V}{\sqrt{R^2 + X^2}} = \frac{V}{Z}$$

The term $\sqrt{R^2 + (X_L - X_C)^2}$ is known as the impedance of the circuit. Obviously,

$$(impedance)^2 = (resistance)^2 + (net\ reactance)^2$$

$$\text{or} \quad Z^2 = R^2 + (X_L - X_C)^2 = R^2 + X^2$$

where X is the net reactance (Fig. 13.33 and 13.34).

Phase angle ϕ is given by $\tan\phi = (X_L - X_C)/R = X/R$ = net reactance/resistance

$$\text{Power factor is } \cos\phi = \frac{R}{Z} = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{R}{\sqrt{R^2 + X^2}}$$

Hence, it is seen that if the equation of the applied voltage is $v = V_m \sin \omega t$, then equation of the resulting current in an *R-L-C* circuit is given by $i = I_m \sin(\omega t \pm \phi)$

The + ve sign is to be used when current leads *i.e.* $X_C > X_L$.

The − ve sign is to be used when current lags *i.e.* when $X_L > X_C$.

In general, the current lags or leads the supply voltage by an angle ϕ such that $\tan\phi = X/R$

Using symbolic notation, we have (Fig. 13.35), $\mathbf{Z} = R + j\,(X_L - X_C)$

Numerical value of impedance $Z = \sqrt{R^2 + (X_L - X_C)^2}$

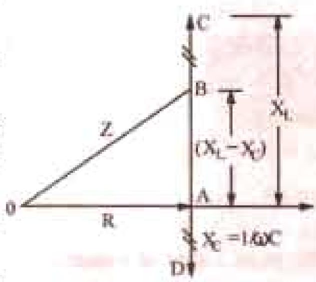


Fig. 13.34

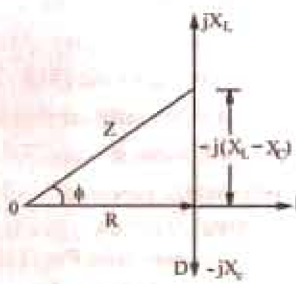


Fig. 13.35

Its phase angle is $\Phi = \tan^{-1}[X_1 - X_C/R]$

$Z = Z\angle\tan^{-1}[(X_L - X_C)/R] = Z\angle\tan^{-1}(X/R)$

If $\mathbf{V} = V\angle 0$, then, $\mathbf{I} = \mathbf{V}/\mathbf{Z}$

Summary of Results of Series AC Circuits

Type of Impedance	*Value of Impedance*	*Phase angle for current*	*Power factor*
Resistance only	R	0°	1
Inductance only	ωL	90° lag	0
Capacitance only	$1/\omega C$	90° lead	0
Resistance and Inductance	$\sqrt{[R^2 + (\omega L)^2]}$	$0 < \phi < 90°$ lag	1 > p.f. > 0 lag
Resistance and Capacitance	$\sqrt{[R^2 + (-1/\omega C)^2]}$	$0 < \phi < 90°$ lead	1 > p.f. > 0 lead
R-L-C	$\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}$	between 0° and 90° lag or lead	between 0 and unity lag or lead

Example 13.43. *A resistance of 20 Ω, an inductance of 0.2 H and a capacitance of 100 μF are connected in series across 220-V, 50-Hz mains. Determine the following (a) impedance (b) current (c) voltage across R, L and C (d) power in watts and VA (e) p.f. and angle of lag.*

(Elect. Engg. A.M.Ae S.I. 1992)

Solution. $X_C = 0.2 \times 314 = 63\ \Omega$ $C = 10\ \mu F = 100 \times 10^{-6} = 10^{-4}$ farad

$$X_C = \frac{1}{\omega C} = \frac{1}{314 \times 10^{-4}} = 32\ \Omega,\ X = 63 - 32 = 31\ \Omega\ (inductive)$$

(*a*) $Z = \sqrt{(20^2 + 31)^2} = 37\ \Omega$ (*b*) $I = 220/37 = 6$ **A (approx)**

(*c*) $V_R = I \times R = 6 \times 20 = 120$ **V**; $V_L = 6 \times 63 = 278$ **V**, $V_C = 6 \times 32 = 192$ **V**

(*d*) Power in $VA = 6 \times 220 = 1320$

Power in watts $= 6 \times 220 \times 0.54 = 713$ **W**

(*e*) p.f. $= \cos\phi = R/Z = 20/37 = 0.54$; $\phi = \cos^{-1}(0.54) = 57°18'$

Example 13.44. *A voltage e(t) = 100 sin 314 t is applied to series circuit consisting of 10 ohm resistance, 0.0318 henry inductance and a capacitor of 63.6 μF. Calculate (i) expression for i (t) (ii) phase angle between voltage and current (iii) power factor (iv) active power consumed (v) peak value of pulsating energy.* **(Elect. Technology, Indore Univ. 1985)**

Solution. Obviously, $\omega = 314$ rad/s ; $X_l = \omega L = 314 \times 0.0318 = 10\ \Omega$

$X_C = 1/\omega C = 1/314 \times 63.6 \times 10^{-6} = 50\ \Omega$; $X = X_L - X_C = (10 - 50) = -40\ \Omega$ (capacitive)

$\mathbf{Z} = 10 - j\,40 = 41.2 \angle -76°$;

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{(100/\sqrt{2}) \angle 0°}{41.2 \angle -76°} = 1.716 \angle 76°$$

$$I_m = I \times \sqrt{2} = 1.716 \times \sqrt{2} = 2.43\ A$$

(*i*) $i(t) = 2.43 \sin(314\,t + 76°)$

(*ii*) $\phi = 76°$ with current leading

(*iii*) p.f. $= \cos\phi = \cos 76° = 0.24$ **(lead)**

(*iv*) Active power, $P = VI\cos\phi$

$= (100/\sqrt{2})\,(2.43/\sqrt{2}) \times 0.24 = 29.16$ **W**

(*v*) As seen from Fig. 13.36, peak value of pulsating energy is $\dfrac{V_m I_m}{2} + \dfrac{V_m I_m}{2}\cos\phi$

$$= \frac{V_m I_m}{2}(1 + \cos\phi) = \frac{100 \times 2.43}{2}(1 + 0.24) = 151\ \text{W}$$

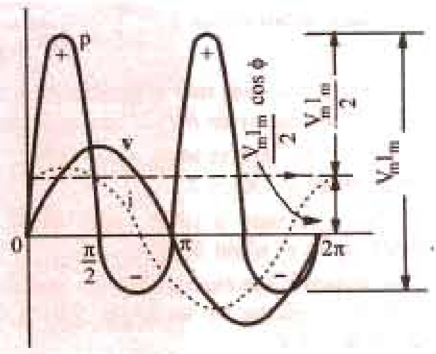


Fig. 13.36

Example 13.45. *Two impedances Z_1 and Z_2 when connected separately across a 230-V, 50-Hz suppy consumed 100 W and 60 W at power factors of 0.5 lagging and 0.6 leading respectively. If these impedances are now connected in series across the same supply, find :*

(i) total power absorbed and overall p.f. (ii) the value of the impedance to be added in series so as to raise the overall p.f. to unity. **(Elect. Circuits-I, Banglore Univ. 1987)**

Solution. Inductive Impedance $V_1 I \cos\phi_1 =$ power; $230 \times I_1 \times 0.5 = 100$; $I_1 = 0.87$ A

Now, $I_1^2 R_1$ = power or $0.87^2 R_1 = 100$; $R_1 = 132\ \Omega$; $Z_1 = 230/0.87 = 264\ \Omega$

$$X_L = \sqrt{Z_1^2 - R_1^2} = \sqrt{264^2 - 132^2} = 229\ \Omega$$

Capacitance Impedance $I_2 = 60/230 \times 0.6 = 0.434$ A ; $R_2 = 60/0.434^2 = 318\ \Omega$

$$Z_2 = 230/0.434 = 530\ \Omega\ ;\ X_C = \sqrt{530^2 - 318^2} = 424\ \Omega\ \text{(capacitive)}$$

When Z_1 and Z_2 are connected in series

$$R = R_1 + R_2 = 132 + 318 = 450\ \Omega;\ X = 229 - 424 = -195\ \Omega\ \text{(capacitive)}$$

$$Z = \sqrt{R^2 + X^2} = \sqrt{450^2 + (-195)^2} = 490\ \Omega,\ I = 230/490 = 0.47\ \text{A}$$

(*i*) Total power absorbed = $I^2 R = 0.47^2 \times 450$ = **99 W**, $\cos\phi\ R/Z = 450/490$ = **0.92 (lead)**

(*ii*) Power factor will become unity when the net capacitive reactance is neutralised by an equal inductive reactance. The reactance of the required series pure inductive coil is **195 Ω.**

Example 13.46. *A resistance R, an inductance L = 0.01 H and a capacitance C are connected in series. When a votlage v = 400 cos (300 t − 10°) votls is applied to the series combination, the current flowing is* $10\sqrt{2}\cos(3000\,t - 55°)$ *amperes. Find R and C.*

(Elect. Circuits Nagpur Univ. 1992)

Solution. The phase difference between the applied voltage and circuit current is (55° − 10°) = 45° with current lagging. The angular frequency is ω = 3000 radian/second. Since current lags, $X_L > X_C$.

∴ Net reactance $X = (X_L - X_C)$. Also $X_L = \omega L = 3000 \times 0.01 = 30\ \Omega$

$\tan\phi = X/R$ or $\tan 45° = X/R$ ∴ $X = R$ Now, $Z = \dfrac{V_m}{I_m} = \dfrac{400}{10\sqrt{2}} = 28.3\ \Omega$

$Z^2 = R^2 + X^2 = 2R^2$ ∴ $R = Z/\sqrt{2} = 28.3/\sqrt{2} = 20\ \Omega$; $X = X_L - X_C = 30 - X_C = 20$

$X_C = 10\ \Omega$ or $\dfrac{1}{\omega C} = 10$ or $\dfrac{1}{3000\,C} =$ or C = **33 μF**

Example 13.47. *A non-inductive resistor is connected in series with a coil and a capacitor. The circuit is connected to a single-phase a.c. supply. If the voltages are as indicated in Fig. 13.37 when current flowing through the circuit is 0.345 A, find the applied votlage and the power loss in coil.*

(Elect. Engg. Pune Univ. 1988)

Solution. It may be kept in mind that the coil has not only inductance *L* but also some resistance *r* which produces power loss. In the voltage vector diagram, *AB* represents drop across *R* = 25 V. Vector *BC* represents drop across coil which is due to *L* and *r*. Which value is 40 V and the vector *BC* is at any angle of φ with the current vector. *AD* represents 50 V which is the drop across *R* and coil combined. *AE* represents the drop aross the capacitor and leads the current by 90°.

It will be seen that the total horizontal drop in the circuit is *AC* and the vertical drop is *AG*. Their vector sum *AF* represents the applied votlage *V*.

From triangle *ABD*, we get $50^2 = 40^2 + 25^2 + 2 \times 25 \times 40 \times \cos\phi$ ∴ $\cos\phi = 0.1375$ and $\sin\phi = 0.99$. Considering the coil, $IZ_L = 40$ ∴ $Z_L = 40/0.345 = 115.94\ \Omega$

Now $r = Z_L \cos\phi = 115.94 \times 0.1375 = 15.94\ \Omega$

Power loss in the coil = $I^2 r = 0.345^2 \times 15.94$ = **1.9 W**

$BC = BD\cos\phi = 40 \times 0.1375 = 5.5$ V $CD = BD \sin\phi = 40 \times 0.99 = 39.6$ V

$AC = 25 + 5.5 = 30.5$ V; $AG = AE - DC = 55 - 39.6 = 15.4$ V

$$AF = \sqrt{AC^2 + CF^2} = \sqrt{30.5^2 + 15.4^2} = \textbf{34.2 V}$$

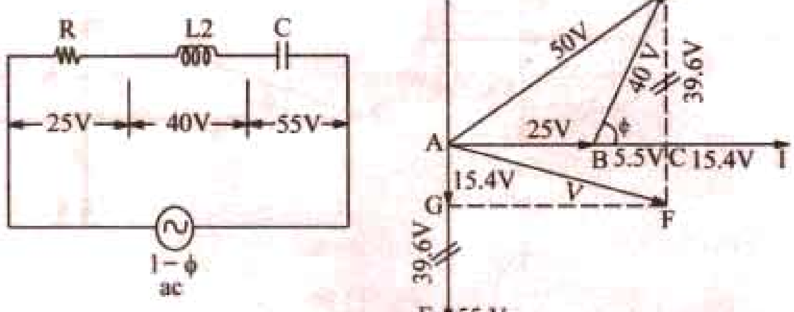


Fig. 13.37

Example 13.48. *A 4.7 H inductor which has a resistance of 20 Ω, a 4-μF capacitor and a 100-Ω non-inductive resistor are connected in series to a 100-V, 50-Hz supply. Calculate the time interval between the positive peak value of the supply voltage and the next peak value of power.*

Solution. Total resistance = 120 Ω; $X_L = 2\pi \times 50 \times 4.7 = 1477\ \Omega$

$X_C = 10^6/2\pi \times 50 \times 4 = 796\ \Omega$; $X = 1477 - 798 = 681\ \Omega$; $Z = \sqrt{120^2 + 681^2} = 691.3\ \Omega$

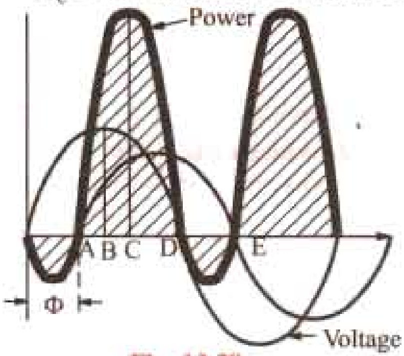


Fig. 13.38

$\cos\phi = R/Z = 120/691.3 = 0.1736$; $\phi = 80°$

Now, as seen from Fig. 13.38, the anglular displacement between the peak values of supply voltage and power cycles is $BC = \phi/2$ because $AB = 90\text{-}\phi$ and $AD = 180\text{-}\phi$.

Hence $AC = 90 - \phi/2$

$\therefore \quad BC = AC - AB = (90 - \phi/2) - (90 - \phi) = \phi/2$

Angle difference $= \phi/2 = 80°/2 = 40°$

Since a full cycle of 360° corresponds to a time interval of 1.50 second

$\therefore \quad 40°$ angular interval $= \dfrac{40}{50 \times 360} =$ **2.22 ms.**

Example 13.49. *A coil is in series with a 20 μF capacitor across a 230-V, 50-Hz supply. The current taken by the circuit is 8 A and the power consumed is 200 W. Calculate the inductance of the coil if the power factor of the circuit is (i) leading (ii) lagging.*

Sketch a vector diagram for each condition and calculate the coil power factor in each case.

(Elect. Engg.-I Nagpur Univ. 1993)

Solution. (*i*) Since power factor is leading, net reactance $X = (X_C - X_L)$ as shown in Fig. 13.39 (*a*).

$$I^2R = 200 \text{ or } 8^2 \times R = 200; \quad \therefore \quad R = 200/64 = 25/8\ \Omega = 3.125\ \Omega$$

$$Z = V/I = 230/8 = 28.75\ \Omega;\ X_C = 10^6/2\pi \times 50 \times 20 = 159.15\ \Omega$$

$R_2 + X_2 = 28.75^2 \quad \therefore \quad X = 28.58\ \Omega \quad \therefore \quad (X_C - X_L) = 28.58 \quad$ or $\quad 159.15 - X_L = 28.58$

$\therefore \quad X_L = 130.57\ \Omega$ or $2\pi \times 50 \times L = 130.57 \quad \therefore \quad L = 0.416\ H$

If θ is the p.f. angle of the coil, then $\tan\theta = R/X_L = 3.125/130.57 = 0.024$; $\theta = 1.37°$, p.f. of the coil = 0.9997

(*ii*) When power factor is lagging, net reactance is $(X_L - X_C)$ as shown in Fig. 13.39 (*b*).

$\therefore \quad X_L - 159.15 = 28.58 \quad$ or $\quad X_L = 187.73\ \Omega \quad \therefore \quad 187.73 = 2\pi \times 50 \times L$ or $L = 0.597$ H.

In this case, $\tan\theta = 3.125/187.73 = 0.0167$; $\theta = 0.954°$ $\therefore \cos\theta = 0.9998$.

The vector diagrams for the two conditions are shown in Fig. 13.35.

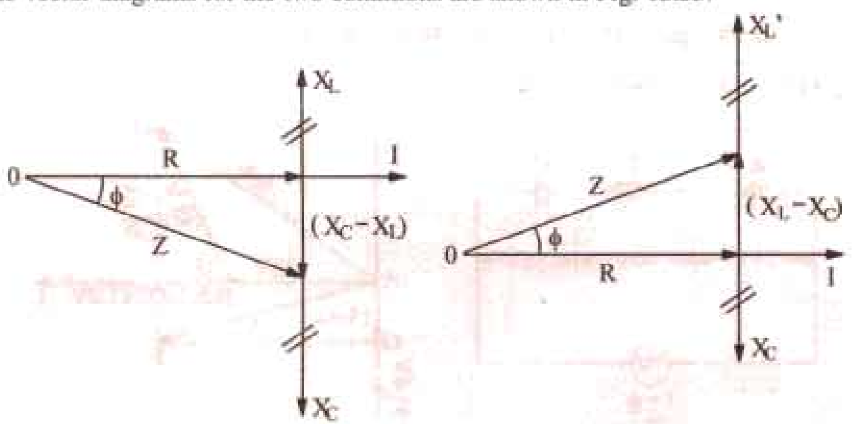


Fig. 13.39

Example 13.50. *In Fig. 13.40, calculate (i) current (ii) votage drops V_1, V_2, and V_3 and (iii) power absorbed by each impedance and total power absorbed by the circuit. Take votlage vector along the reference axis.*

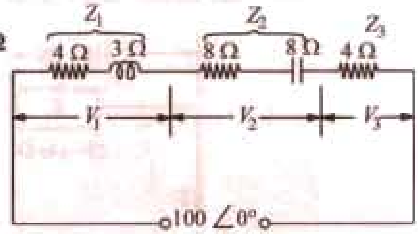


Fig. 13.40

Solution. $\mathbf{Z}_1 = (4 + j3)\ \Omega$; $\mathbf{Z}_2 = (6 - j8)\ \Omega$; $\mathbf{Z}_3 = (4 + j0)\Omega$

$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 = (4 + j3) + (6 - j8) + (4 + j0) = (14 - j5)\Omega$

Taking $\mathbf{V} = V \angle 0° = 100 \angle 0° = (100 + j0)$

$$\therefore\ \mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{100}{(14 - j5)} = \frac{100\,(14 + j5)}{(14 - j5)\,(14 + j5)} = 6.34 + j2.26$$

(*i*) Magnitude of the current

$= \sqrt{(6.34^2 + 2.26^2)} = \mathbf{6.73\ A}$

(*ii*) $\mathbf{V}_1 = \mathbf{IZ}_1 = (6.34 + j\,2.26)\,(4 + j3) = 18.58 + j\,28.06$

$\mathbf{V}_2 = \mathbf{IZ}_2 = (6.34 + j\,2.26)\,(6 - j8) = 56.12 - j\,37.16$

$\mathbf{V}_3 = \mathbf{IZ}_3 = (6.34 + j\,2.26)\,(4 + j0) = 25.36 + j\,9.04$

$\mathbf{V}\ = 100 + j\,0$ (check)

(*iii*) $P_1 = 6.73^2 \times 4 = 181.13$ W.

$P_2 = 6.73^2 \times 6 = 271.74$ W, $P_3 = 6.73^2 \times 4 = 181.13$ W,

Total = 34 W

Otherwise $\quad \mathbf{P}_{VA} = (100 + j\,0)\,(6.34 - j\,2.26)$ (using current conjugate)

$= 634 - j\,226$

real power $= 634$ W (as a check)

Example 13.51. *Draw a vector for the circuit shown in Fig. 13.41 indicating the resistance and reactance drops, the terminal votlages V_1 and V_2 and the current. Find the values of (i) the current I (ii) V_1 and V_2 and (iii) p.f.* **(Elements of Elect Engg-I, Bangalore Univ. 1988)**

Solution. $L = 0.05 + 0.1 = 0.15$ H; $X_L = 314 \times 0.5 = 47.1\ \Omega$

$X_C = 10^6/314 \times 50 = 63.7\ \Omega$; $X = 47.1 - 63.7 = -16.6\ \Omega$, $R = 30\ \Omega$,

$Z = \sqrt{30^2 + (-16.6)^2} = 34.3\ \Omega$

(*i*) $I = V/Z = 200/34.3 = \mathbf{5.83\ A,}$

from Fig. 13.41 (*a*)

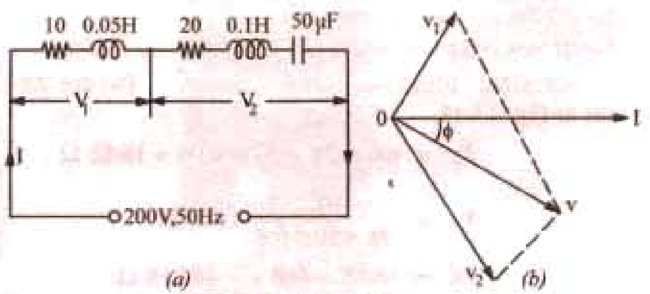


Fig. 13.41

(*ii*) $X_{L1} = 314 \times 0.05$

$= 15.7\ \Omega$

$Z_1 = \sqrt{10^2 + 15.7^2} = 18.6\ \Omega$

$V_1 = IZ_1 = 5.83 \times 18.6$

$= \mathbf{108.4\ V}$

$\phi_1 = \cos^{-1}(10/18.6) = 57.5°$ (lag)

$X_{L2} = 314 \times 0.1 = 31.4\ \Omega$, $X_c = -63.7\ \Omega$, $X = 31.4 - 63.7 = -32.2\ \Omega$, $Z_2 = \sqrt{20^2 + (-32.3)^2} = \mathbf{221\ V}$

$\phi_2 = \cos^{-2}(20/38) = 58.2$ (lead)

(*iii*) Combined p.f. = $\cos\phi = R/Z = 30/34.3 = \mathbf{0.875\ (lead)}$, from Fig. 13.41 (*b*).

Example 13.52. *In a circuit, the applied votlage is found to lag the current by 30°.*

(a) Is the power factor lagging or leading ? (b) What is the value of the power factor ? (c) Is the circuit inductive or capacitive ?

In the diagram of Fig 13.42, the voltage drop across Z_1 is (10 + j0) volts. Find out *(i) the current in the circuit (ii) the voltage drops across Z_2 and Z_3 (iii) the voltage of the generator.* **(Elect. Engg.-I. Bombay Univ. 1991)**

Solution. (*a*) Power factor is *leading* because current leads the voltage.

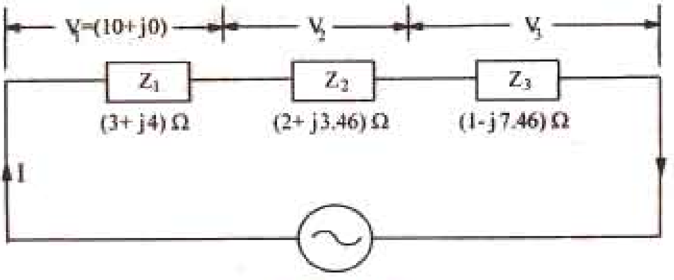


Fig. 13.42

(*b*) p.f. = cos 30° = **0.86 (lead)** (*c*) The circuit is **capacitive.**

(*i*) Circuit current can be found by dividing voltage drop $\mathbf{V}_1$ by $\mathbf{Z}_1$

$$\mathbf{I} = \frac{10+j0}{3+j4} = \frac{10\angle 0^\circ}{5\angle 53.1^\circ} = 2\angle -53.1^\circ = 2(\cos 53.1^\circ - j\sin 53.1^\circ)$$

$$= 2(0.6 - j0.8) = 1.2 - j1.6$$

$$\mathbf{Z}_2 = 2 + j3.46;\ \mathbf{V}_2 = \mathbf{IZ}_2 = (1.2 - j1.6)(2 + j3.46) = (7.936 + j0.952) \text{ volt}$$

$$\mathbf{V}_3 = (1.2 - j1.6)(1 - j7.46) = (-10.74 - j10.55) \text{ volt}$$

(*ii*)
$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 = (10 + j0) + (7.936 + j0.952) + (-10.74 - j10.55)$$

$$= (7.2 - j9.6) = 12\angle -53.1^\circ$$

Incidentally, it shows that current **I** and voltage **V** are in phase with each other.

Example 13.53. *A 230-V, 50-Hz alternating p.d. supplies a choking coil having an inductane of 0.06 henry in series with a capacitance of 6.8 µF, the effective resistance of the circuit being 2.5 Ω. Estimate the current and the angle of the phase difference between it and the applied p.d. If the p.d. has a 10% harmonic of 5 times the fundamental frequency, estimate (a) the current due to it and (b) the p.d. across the capacitance.*

(Electrical Network Analysis, Nagpur Univ. 1993)

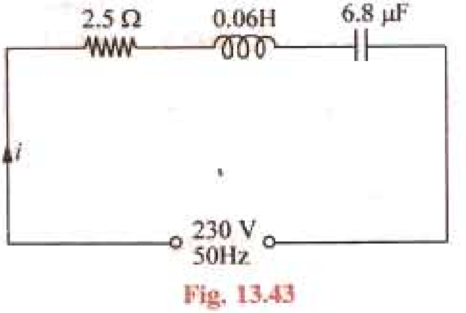


Fig. 13.43

Solution. Fundamental Frequency : For the circuit in Fig. 13.43,

$$X_L = \omega L = 2\pi \times 50 \times 0.06 = 18.85\ \Omega$$

$$X_C = \frac{10^6}{2\pi \times 50 \times 6.8} = 648\ \Omega$$

$$\therefore \quad X = 18.85 - 468 = -449.15\ \Omega$$

$$Z = \sqrt{2.5^2 + (-449.15)^2} = 449.2\ \Omega$$

Current $I_f = 230/449.2 = 0.512$ A phase angle $= \tan^{-1}\left(\frac{-449.2}{2.5}\right) = -89^\circ.42'$

$\therefore$ current leads p.d. by 89° 42′.

Fifth Harmonic Frequency $X_L = 18.85 \times 5 = 94.25\ \Omega$; $X_C = 468 + 5 = 93.6\ \Omega$

$X = 94.25 - 93.6 = 0.65\ \Omega$; $\mathbf{Z} = \sqrt{2.5^2 + 0.65^2} = 2.585\ \Omega$ Harmonic p.d. $= 230 \times 10/100 = 23$ V

$\therefore$ Harmonic current $I_h = 23/2.585 = 8.893$ A

P.D. Across cpacitor at harmonic freqency is, $V_h = 8.893 \times 93.6 = 832.6$ V

The total current flowing through the circuit, due to the complex voltage wave form, is found from the fundamental and harmonic components thus. Let,

I = the r.m.s. value of total circuit current,

I_f = r.m.s. value of fundamental current,

I_h = r.m.s. value of fifth harmonic current,

(*a*) $\therefore \quad I = \sqrt{I_f^2 + I_h^2} = \sqrt{0.512^2 + 8.893^2} = 8.9$ **A**

(*b*) The r.m.s. value of p.d. across capacitor is found in a similar way.

$$V_f = 0.512 \times 468 = 239.6 \text{ V}$$

$$\therefore \quad V = \sqrt{(V_f^2 + V_h^2)} = \sqrt{239.6^2 + 832.6^2} = 866.4 \text{ V}$$

Tutorial Problem No. 13.3

1. An e.m.f. represented by $e = 100 \sin 100\pi t$ is impressed across a circuit consisting of 40-Ω resistor in series with a 40-μ*F* capacitor and a 0.25 H indicator.
   Determine (*i*) the r.m.s. value of the current (*ii*) the power supplied (*iii*) the power factor.
   [(*i*) 1.77 A (*ii*) 125 W (*iii*) 1.0] (*London Univ.*)
2. A series circuit with a resistor of 100 Ω capacitor of 25 μ*F* and inductance of 0.15 H is connected across 220-V, 60-Hz supply. Calculate (*i*) current (*ii*) power and (*iii*) power factor in the circuit.
   [(*i*) 1.97 A; (*ii*) 390 W (*iii*) 0.9 (lead)] (*Elect. Engg. and Electronics Bangalore Univ. 1985*)
3. A series circuit with $R = 10$ Ω, $L = 50$ *mH* and $C = 100$ μ*F* is supplied with 200 V/50 Hz. Find (*i*) the impedance (*ii*) current (*iii*) power (*iv*) power factor.
   [(*i*) 18.94 Ω (*ii*) 18.55 A (*iii*) 1966 W (*iv*) 0.53 (leading)]
   (*Elect. Engg. & Electronics Bangalore Univ. 1986*)
4. A coil of resistance 10 Ω and inductance 0.1 H is connected in series with a 150-μ*F* capacitor across a 200-V, 50-Hz supply. Calculate (*a*) the inductive reactance, (*b*) the capacitive reactance, (*c*) the impedance (*d*) the current, (*e*) the power factor (*f*) the voltage across the coil and the capacitor respectively. **[(*a*) 31.4 Ω (*b*) 21.2 Ω (*c*) 14.3 Ω (*d*) 14 A (*e*) 0.7 lag (*f*) 460 V, 297 V]**
5. A circuit is made up of 10 Ω resistance, 12 mH inductance and 281.5 μ*F* capacitance is series. The supply voltage is 100 V (constant). Calculate the value of the current when the supply frequency is (*a*) 50 Hz and (*b*) 150 Hz. **[8 A leading; 8 A lagging]**
6. A coil having a resistance of 10 Ω and an inductance of 0.2 H is connected in series with a capacitor of 59.7 μ*F*. The circuit is connected across a 100-V, 50-Hz a.c. supply. Calculate (*a*) the current flowing (*b*) the voltage across the capacitor (*c*) the voltage across the coil. Draw a vector diagram to scale. **[(*a*) 10 A (*b*) 628 V (*c*) 635 V]**
7. A coil is in series with a 20 μ*F* capacitor across a 230-V , 50-Hz supply. The current taken by the circuit is 8 A and the power consumed is 200 W. Calculate the inductance of the coil if the power factor of the circuit is (*a*) leading and (*b*) lagging.
   Sketch a vector diagram for each condition and calculate the coil power factor in each case.
   [0.415 H; 0.597 H; 0.0238 ; 0.0166]
8. A circuit takes a current of 3 A at a power factor of 0.6 lagging when connected to a 115-V, 50-Hz supply. Another circuit takes a current, of 5 A at a power factor of 0.707 leading when connected to the same supply. If the two circuits are connected in series across a 230-V, 50Hz supply, calculate (*a*) the current (*b*) the power consumed and (*c*) the power factor.
   [(*a*) 5.5 A (*b*) 1.188 kW (*c*) 0.939 lag]
9. A coil of insulated wire of resistance 8 ohms and inductance 0.03 H is connected to an a.c. supply at 240 V, 50-Hz. Calculate (*a*) the current, the power and power factor (*b*) the value of a capacitance which, when connected in series with the above coil, causes no change in the values of current and power taken from the supply. **[(*a*) 19.4 A, 3012 W, 0.65 lag (*b*) 168.7 μF]** (*London Univ.*)
10. A series circuit, having a resistance of 10 Ω, an inductance of 0.025 H and a variable capacitance is connected to a 100-V, 25-Hz single-phase supply. Calculate the capacitance when the value of the current is 8 A. At this value of capacitance, also calculate (*a*) the circuit impedance (*b*) the circuit power factor and (*c*) the power consumed. **[556 μF (*a*) 1.5 Ω (*b*) 0.8 leading (*c*) 640 W]**

11. An alternating voltage is applied to a series circuit consisting of a resistor and iron-cored inductor and a capacitor. The current in the circuit is 0.5 A and the voltages measured are 30 V across the resistor, 48 V across the inductor, 60 V across the resistor and inductor and 90 V across the capacitor. Find (*a*) the combined copper and iron losses in the inductor (*b*) the applied voltage.
[(*a*) 3.3 W (*b*) 56 V] (*City & Guilds, London*)
12. When an inductive coil is connected across a 250-V, 50-Hz supply, the current is found to be 10 A and the power absorbed 1.25 kW. Calculate the impedance, the resistance and the inductance of the coil. A capacitor which has a reactance twice that of the coil, is now connected in series with the coil across the same supply. Calculate the p.d. Across the capacitor. **[25 Ω; 12.5 Ω; 68.7 mH; 433 V]**
13. A voltage of 200 V is applied to a series circuit consisting on a resistor, an inductor and a capacitor. The respective voltages across these components are 170, 150 and 100 V and the current is 4 A. Find the power factor of the inductor and of the circuit. **[0.16; 0.97]**
14. A pure resistance *R*, a choke coil and a pure capacitor of 50μ *F* are connected in series across a supply of V volts, and carry a current of 1.57 A. Voltage across *R* is 30 V, across choke coil 50 V and across capacitor 100 V. The voltage across the combination of *R* and choke coil is 60 volt. Find the supply voltage *V*, the power loss in the choke, frequency of the supply and power factor of the complete circuit. Draw the phasor diagram. **[60.7 V; 6.5 W; 0.562 lead]** (*F.E. Pune Univ. No. 1986*)

13.10. Resonance in R-L-C Circuits

We have seen from Art. 13.9 that net reactance in an *R-L-C* circuit of Fig. 13.40 (*a*) is

$$X = X_L - X_C \text{ and } Z = \sqrt{[R^2 + (X_L - X_C)^2]} = \sqrt{R^2 + X^2}$$

Let such a circuit be connected across an a.c. source of constant voltage *V* but of frequency varying from zero to infinity. There would be a certain frequency of the applied voltage which would make X_L equal to X_C in magnitude. In that case, $X = 0$ an $Z = R$ as shown in Fig. 13.40 (*c*). Under this condition, the circuit is said to be in electrical resonance.

As shown in Fig. 13.40 (*c*), $V_L = I.\ X_L$ and $V_C = I.\ X_C$ and the two are equal in magnitude but opposite in phase. Hence, they cancel each other out. The two reactances taken together act as a short-circuit since no voltage develops across them. Whole of the applied voltage drops across *R* so that $V = V_R$. The circuit impedance $Z = R$. The phasor diagram for sereis resonance is shown in Fig. 13.40 (*d*).

Calculation of Resonant Frequency

The frequency at which the net reactance of the series circuit is zero is called the resonant frequency f_0. Its value can be found as under : $X_L - X_C = 0$ or $X_L = X_C$ or $\omega_0 L = 1/\omega_0 C$

or $$\omega_0^2 = \frac{1}{LC} \text{ or } (2\pi f_0)^2 = \frac{1}{LC} \text{ or } f_0 = \frac{1}{2\pi\sqrt{LC}}$$

If *L* is in henry and *C* in farad, then f_0 is given in *Hz*.

When a sereis *R-L-C* circuit is in resonance, it possesses minimum impedance $Z = R$. Hence, circuit current is maximum, it being limited by value of *R* alone. The current $I_0 = V/R$ and is in phase with *V*.

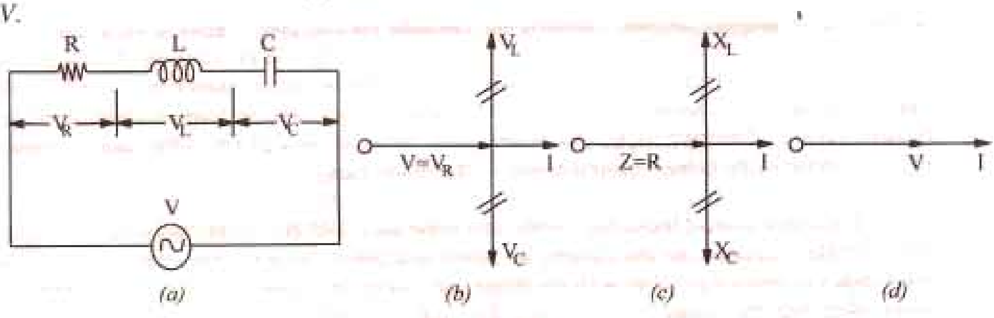


Fig. 13.44

Since circuit current is maximum, it produces large voltage drops across *L* and *C*. But these drops being equal and opposite, cancel each other out. Taken together, *L* an *C* from part of a circuit across which no voltage develops, however, large the current flowing. If it were not for the presence of *R*, such a resonant circuit would act like a short-circuit to currents of the frequency to which it resonates. Hence, a series resonant circuit is sometimes called *acceptor* circuit and the series resonance is often referred to as voltage resonance.

In fact, at resonance the series *RLC* circuit is reduced to a purely resistive circuit, as shown in Fig. 13.44.

Incidentally, it may be noted that if X_L and X_C are shown at any frequency f, that the value of the resonant frequency of such a circuit can be found the relation $f_0 = f\sqrt{X_C/X_L}$.

Summary

When an *R-L-C* circuit is in resonance

1. net reactance of the circuit is zero *i.e.* $(X_L - X_C) = 0$. or $X = 0$.
2. circuit impedance is minimum *i.e.* $Z = R$. Consequenly, circuit admittance is maximum.
3. circuit current is maximum and is given by $I_0 = V/Z_0 = V/R$.
4. power dissipated is maximum *i.e.* $P_0 = I_0^2 R = V^2/R$.
5. circuit power factor angle $\theta = 0$. Hence, power factor $\cos\theta = 1$.
6. although $V_L = V_C$ yet V_{coil} is greater than V_C because of its resistance.
7. at resonance, $\omega^2 LC = 1$
8. $Q = \tan\theta = \tan 0° = 0$*.

13.11. Graphical Representation of Resonance

Suppose an alternating voltage of constant magnitude, but of varying frequency is applied to an *R-L-C* circuit. The variations of resistace, inductive reactance X_L and capacitive reactance X_C with frequency are shown in Fig. 13.45 (*a*).

(*i*) *Resistance* : It is independent of f, hence, it is represented by a straight line.

(*ii*) *Inductive Reactance :* It is given by $X_L = \omega L = 2\pi fL$. As seen, X_L is directly proportional to f *i.e.* X_L increases linearly with f. Hence, its graph is a straight line passsing through the origin.

(*iii*) *Capacitive Reactance :* It is given by $X_C = 1/\omega C = 1/2\pi fC$. Obviously, it is inversely proportional to f. Its graph is a rectangular hyperbola which is drawn in the fourth quadrant because X_C is regarded negative. It is asymptotic to the horizontal axis at high frequencies and to the vertical axis at low frequencies.

(*iv*) *Net Reactance* : It is given by $X = X_L \sim X_C$. Its graph is a hyperbola (not rectangular) and crosses the *X*-axis at point *A* which represents resonant frequency f_0.

(*v*) *Circuit Impedance* : It is given by $Z = \sqrt{[R^2 + (X_L \sim X_C)^2]} = \sqrt{R^2 + X^2}$

At low frequencies *Z* is large because X_C is large. Since $X_C > X_L$, the net circuit reactance *X* is capacitive and the p.f. is leading [Fig. 13.45 (*b*)]. At high frequencies, *Z* is again large

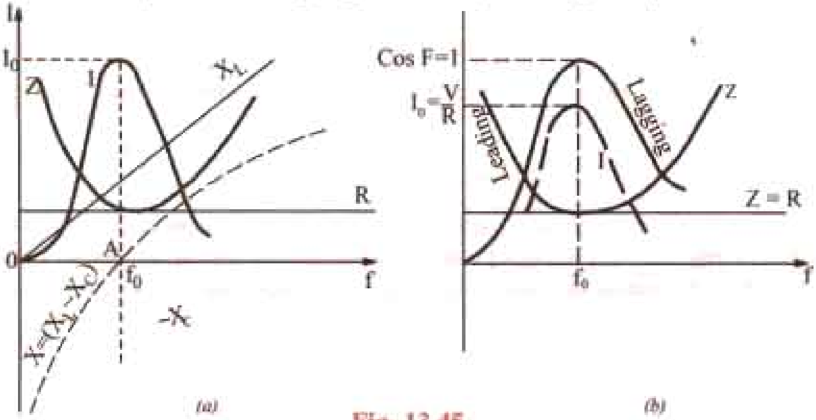


Fig. 13.45

* However, value of Q_0 is as given in Art 13.5. 13.9 and 13.17.

(because X_L is large) but is inductive because $X_L > X_C$. Circuit impedance has minimum values at f_0 give by $Z = R$ because $X = 0$.

(*vi*) *Current* I_0 : It is the reciprocal of the circuit impedance. When Z is low, I_0 is high and vice versa. As seen, I_0 has low value on both sides of f_0 (because Z is large there) but has maximum value of $I_0 = V/R$ at resonance. Hence, maximum power is dissipated by the series circuit under resonant conditions. At frequencies below and above resonance, current decerases as shown in Fig. 13.45 (*b*). Now, $I_0 = V/R$ and $I = V/Z = V/\sqrt{(R^2 + X^2)}$. Hence $I/I_0 = R/Z = V/\sqrt{(R^2 + X^2)}$ where X is the net circuit reactance at any frequency f.

(*vii*) *Power Factor*

As pointed out earlier, X is capacitive below f_0. Hence, current leads the applied voltage. However, at frequencies above f_0, X is inductive. Hence, the current lags the applied voltage as shown in Fig. 13.45. The power factor has maximum value of unity at f_0.

13.12. Resonance Curve

The curve, between circuit curent and the frequency of the applied voltage, is known as resonance curve. The shapes of such a curve, for different values of R are shown in Fig. 13.46. For smaller values of R, the resonance curve is sharply peaked and such a circuit is said to be sharply resonant or highly selective. However, for larger values of R, resonance curve is flat and is said to have poor selectivity. The ability of a resonant circuit to discriminate between one particular frequency and all others is called its selectivity. The selectivities of different resonant circuits are compared in terms of their half-power bandwidths (Art. 13.13).

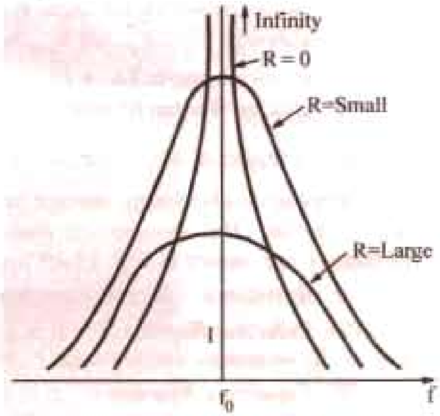


Fig. 13.46

13.13. Half-Power Bandwidth of a Resonant-Circuit

A discussed earlier, in an R-L-C circuit, the maximum current at resonance is solely determined by circuit resistance R ($\because X = 0$) but at off-resonance frequencies, the current amplitude depends on Z (where $X \neq 0$). The half-wave bandwidth of a circuit is given by the band of fequencies which lies between two points on either side of f_0 where current falls to $I_0/\sqrt{2}$. Narrower the bandwidth, higher the selectivity of the circuit and vice versa. As shown in Fig. 13.47 the half-power bandwidth AB is given by

$AB = \Delta f = f_2 - f_1$ or $AB = \Delta\omega = \omega_2 - \omega_1$ where f_1 and f_2 are the corner or edge frequencies.

As seen, $P_0 = I_0^2 R$. However, power at either of the two points A and B is

$$P_1 = P_2 = I^2 R$$

$$= (I_0/\sqrt{2})^2 R = I_0^2 R/2 = \frac{1}{2} I_0^2 R = \frac{1}{2} \times \text{power at resonance}$$

That is why the two points A and B on the resonance curve are known as half-power points* and the corresponding value of the bandwidth is called half-power bandwidth B_{hp}. It is also called – $3dB$* bandwidth. The following points regarding half-power point A and B are worth noting. At these points,

---

* The decibel power respones at these points, in terms of the maximum power at resonance, is

$$10 \log_{10} P/P_0 = 10 \log_{10} \frac{I_m^2 R/2}{I_m^2 R} = 10 \log_{10} \frac{1}{2} = -10 \log_{10} 2 = -3\, dB$$

Hence, the half-power points are also referred to as $-3\, dB$ points.

1. current is $I_0/\sqrt{2}$
2. impedance is $\sqrt{2}.R$ or $\sqrt{2}.Z_0$
3. $P - P_2 - P_0/2$
4. the circuit phase angle is $\theta = \pm 45°$
5. $Q = \tan\theta = \tan 45° = 1$
6. $B_{hp} = f_2 - f_1 = f_0/Q_0 = \sqrt{f_1 f_2}/Q_0 = R/2\pi L$

It is interesting to note that B_{hp} is independent of the circuit capacitance.

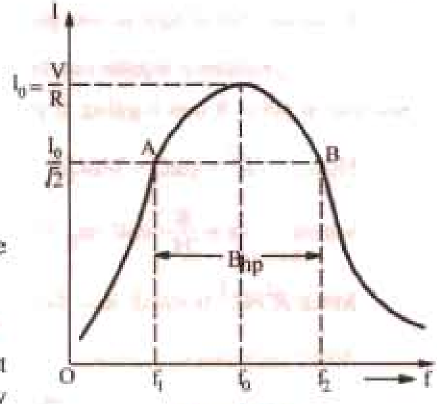


Fig. 13.47

13.14. Bandwidth B at any Off-resonance Frequency

It is found that the bandwidth of a given *R-L-C* circuit at any off-resonance frequencies f_1 and f_2 is given by

$B = f_0\, Q/Q_0 = \sqrt{f_1 f_2}\,;\; .\, Q/Q_0 = f_2 - f_1$

where f_1 and f_2 are any frequencies (not necessarily half-power frequencies) below and above f_0.

Q = tangent of the circuit phase angle at the off-resonance frequencies f_1 and f_2.

Q_0 = quality factor at resonance $= \dfrac{\omega_0 L}{R} = \dfrac{1}{R}\sqrt{\dfrac{L}{C}}$

13.15. Detemination of Upper and Lower Half-power Frequencies

As mentioned earlier, at lower half-power frequencies, $\omega_1 < \omega_0$ so that $\omega_1 L < 1/\omega_1 C$ and $\phi = -45°$

$$\therefore \quad \frac{1}{\omega_1 C} - \omega_1 L = R \quad \text{or} \quad \omega_1^2 + \frac{R}{L}\omega_1 - \frac{1}{LC} = 0$$

Putting $\dfrac{\omega_0}{Q_0} = \dfrac{R}{L}$ and $\omega_0^2 = \dfrac{1}{LC}$ in the above equation, we get $\omega_1^2 + \dfrac{\omega_0}{Q_0}\omega_1 - \omega_0^2 = 0$

The positive solution of the above equation is, $\omega_1 = \omega_0\left[\sqrt{\left(1 + \dfrac{1}{4Q_0^2}\right)} - \dfrac{1}{2Q_0}\right]$

Now at the upper half-power frequency, $\omega_2 > \omega_0$ so that $\omega_2 > 1/\omega_2 C$ and $\phi = +45°$

$$\therefore \quad \omega_2 L - \frac{1}{\omega_2 C} = R \text{ or } \omega_2^2 - \frac{\omega_0}{Q_0}\omega_2 - \omega_0^2 = 0$$

The positive solution of the above equation is $\omega_2 = \omega_0\left[\sqrt{\left(1 + \dfrac{1}{4Q_0^2}\right)} + \dfrac{1}{2Q_0}\right]$

In case $Q_0 > 10$, then the term $1/4\,Q_0^2$ is negligible as compared to 1.

Hence, in that case $\quad \omega_1 \cong \omega_0\left(1 - \dfrac{1}{2Q_0}\right)$ and $\omega_2 \cong \omega_0\left(1 + \dfrac{1}{2Q_0}\right)$

Incidentally, it may be noted from above that $\omega_2 - \omega_1 = \omega_0/Q_0$.

13.16. Values of Edge Frequencies

Let us find the values of ω_1 and ω_2. $I_0 = V/R$...at resonance

$$I = \frac{V}{[R^2 + (\omega L - 1/\omega C)^2]^{1/2}} \text{... at any frequency}$$

At points A and B, $I = \dfrac{I_0}{\sqrt{2}} = \dfrac{I}{\sqrt{2}}\cdot\dfrac{V}{R}$

$$\therefore \quad \frac{1}{\sqrt{2}}\cdot\frac{V}{R}\frac{V}{[(R^2 + (\omega L - 1/\omega C)^2]^{1/2}} \text{ or } R = \pm(\omega L - 1/\omega C) = \pm X$$

It shows that at half-power points, net reactance is equal to the resistance.

(Since resistance equals reactance, p.f. of the circuit at these points is = $1/\sqrt{2}$ *i.e.* 0.707, though leading at point *A* and lagging at point *B*).

Hence $R^2 = (\omega L - 1/\omega C)^2 \quad \therefore \quad \omega = \pm \frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} = \pm \alpha \pm \sqrt{\alpha^2 + \omega_0^2}$

where $\alpha = \frac{R}{2L}$ and $\omega_0 = \frac{1}{\sqrt{LC}}$

Since $R^2/4L^2$ is much less then $1/\sqrt{LC}$ $\therefore \omega = \pm \frac{R}{2L} \pm \frac{1}{\sqrt{LC}} = \pm \frac{R}{2L} \pm \omega_0$

Since only positive values of ω_0 are considered, $\omega = \omega_0 \pm R/2L = \omega_0 \pm \alpha$

$$\therefore \quad \omega_1 = \omega_0 - \frac{R}{2L} \text{ and } \omega_2 = \omega_0 + \frac{R}{2L}$$

$$\therefore \quad \Delta\omega = \omega_2 - \omega_1 = \frac{R}{L} \text{ rad/s and } \Delta f = f_2 - f_1 = \frac{R}{2\pi L}\, Hz = \frac{f_0}{Q_0}\, Hz$$

$$\text{Also} \quad f_1 = f_0 - \frac{R}{4\pi L}\, Hz \text{ and } f_2 = f_0 + \frac{R}{4\pi L}\, Hz$$

It is obvious that f_0 is the *centre* frequency between f_1 and f_2.

Also, $\omega_1 = \omega_0 - \frac{1}{2}\Delta\omega$ and $\omega_2 = \omega_0 + \frac{1}{2}\Delta\omega$

As stated above, bandwidth is a measure of circuits selectivity. Narrower the bandwidth, higher the selectivity and vice versa.

13.17. Q-Factor of a Resonant Series Circuit

The *Q*-factor of an *R-L-C* series circuit can be defined in the following different ways.

(*i*) it is given by the voltage magnification produced in the circuit at resonance.

We have seen that at resonance, current has maximum/value $I_0 = V/R$. Voltage across either coil or capacitor = $I_0 X_{L_0}$ or $I_0 X_{C_0}$ supply voltage $V = I_0 R$

$$\therefore \quad \text{Voltage magnification} = \frac{V_{L_0}}{V} = \frac{I_0 X_{L_0}}{I_0 R} = \frac{\text{reactive power}}{\text{active power}} = \frac{X_{L_0}}{R} = \frac{\omega_0 L}{R} = \frac{\text{reactance}}{\text{resistance}}$$

$$\text{or} \quad = \frac{V_{C0}}{V} = \frac{I_0 X_{C0}}{I_0 R} = \frac{\text{reactive power}}{\text{active power}} = \frac{X_{C0}}{R} = \frac{\text{reactance}}{\text{resistance}} = \frac{1}{\omega_0 CR}$$

$$\therefore \quad Q-\text{factor}, \quad Q_0 = \frac{\omega_0 L}{R} = \frac{2\pi f_0 L}{R} = \tan\phi \qquad ...(i)$$

where ϕ is power factor of the coil

(*ii*) The *Q*-factor may also be defined as under.

$$Q\text{-factor} = 2\pi \frac{\text{maximum stored energy}}{\text{energy dissipated per cycle}} \qquad \text{...in the circuit}$$

$$= 2\pi \frac{\frac{1}{2} L I_0^2}{I^2 R T_0} = 2\pi \frac{\frac{1}{2} L(\sqrt{2} I)^2}{I^2 R (1/f_0)} = \frac{I^2\, 2\pi f_0 L}{I^2 R} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} \qquad ...(T_0 = 1/f_0)$$

$$\text{In other words,} \quad Q_0 = \frac{\text{energy stored*}}{\text{enrergy lost}} \qquad \text{...in the circuit}$$

* The author often jokingly tells students in his class that these days the quality of a person is also measure in terms of a quality factor given by

$$Q = \frac{\text{money earned}}{\text{money spent}}$$

Obviously, a person should try to have a high a quality factor as possible by minimising the denominator and/or maximizing the numerator.

(*iii*) We have seen above that resonant frequency, $f_0 = \dfrac{1}{2\pi\sqrt{(LC)}}$ or $2\pi f_0 = \dfrac{1}{\sqrt{(LC)}}$

Substituting this value in Eq. (*i*) above, we get the Q-factor, $Q_0 = \dfrac{1}{R}\sqrt{\left(\dfrac{L}{C}\right)}$

(*iv*) In the case of series resonance, higher Q-factor means not only higher voltage magnification but also higher selectivity of the tuning coil. In fact, Q-factor of a resonant series circuit may be written as

$$Q_0 = \frac{\omega_0}{\text{bandwidth}} = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{R/L} = \frac{\omega_0 L}{R} = \frac{L}{R\sqrt{LC}} = \frac{1}{R}\sqrt{\frac{L}{C}} \qquad \text{...as before}$$

Obviously, Q-factor can be increased by having a coil of large inductance but of small ohmic resistance.

(*v*) In summary, we can say that

$$Q_0 = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R} = \frac{1}{R}\sqrt{\frac{L}{C}} = \sqrt{\frac{X_{L0}\,X_{C0}}{R}} = \frac{f_0}{B_{hp}} = \frac{\omega_0}{\omega_2 - \omega_1} = \frac{f_0}{f_2 - f_1}$$

13.18. Circuit Current at Frequencies Other Than Resonant Frequencies

At resonance, $I_0 = V/R$

At any other frequency above the resonant frequency, the current is given by I

$$\frac{V}{Z} = \frac{V}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

This current lags behind the applied voltage by a certain angle ϕ

$$\therefore \quad \frac{I}{I_0} = \frac{V}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \times \frac{R}{V} = \frac{1}{\sqrt{1 + \dfrac{1}{R^2}(\omega L - 1/\omega C)^2}} = \frac{1}{\left[1 + \left(\dfrac{\omega_0 L}{R}\right)^2 \left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)^2\right]^{1/2}}$$

Now, $\omega_0 L/R = Q_0$ and $\omega/\omega_0 = f/f_0$ hence, $\dfrac{I}{I_0} = \dfrac{1}{\left[1 + Q_0^2\left(\dfrac{f}{f_0} - \dfrac{f_0}{f}\right)^2\right]^{1/2}}$

13.19. Relation Between Resonant Power P_0 and Off-resonant Power P

In a series RLC resonant circuit, current is maximum *i.e.* I_0 at the resonant frequency f_0. The maximum power P_0 is dissipated by the circuit at this frequency where X_L equals X_C. Hence, circuit impedance $Z_0 = R$.

$$\therefore \quad P_0 = I_0^2 R = (V/R)^2 \times R = V^2/R$$

At any other frequency either above or below f_0 the power is (Fig. 13.48).

$$P = I^2 R = \left(\frac{V}{Z}\right)^2 \times R = \frac{V^2 R}{Z^2} = \frac{V^2 R}{R^2 + X^2} = \frac{V^2 R}{R^2 + X^2 R^2/R^2}$$

$$= \frac{V^2 R}{R^2 + R^2 Q^2} = \frac{V^2 R}{R^2(1+Q^2)} = \frac{V^2}{R(1+Q^2)} = \frac{P_0}{(1+Q^2)}$$

The above equation shown that any frequency other tan f_0, the circuit power P is reduced by a factor of $(1 + Q^2)$ where Q is the tangent of the circuit phase angle

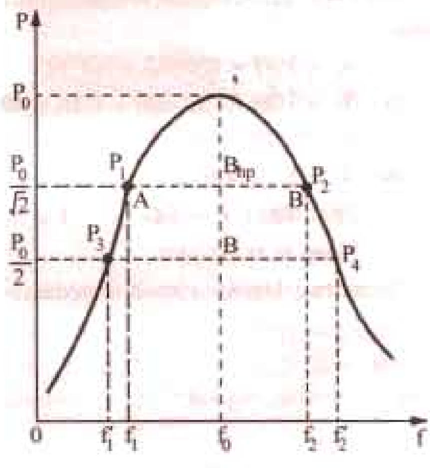


Fig. 13.48

(and not Q_0). At resonance, circuit phase angle $\theta = 0$, and $Q = \tan\theta = 0$. Hence, $P = P_0 = V^2/R$ (values of Q_0 are given in Art.)

Example 13.54. *For a series R.L.C circuit the inductor is variable. Source voltage is $200\sqrt{2}$ sin $100\pi t$. Maximum current obtainable by varying the inductance is 0.314 A and the voltage across the capacitor then is 300 V. Find the circuit element values.*

(Circuit and Field Theory, A.M.I.E. Sec B, 1993)

Solution. Under resonant conditions, $I_m = V/R$ and $V_L = V_C$.

$\therefore \quad R = V/I_m = 200/0.314 = 637\ \Omega,\ V_C = I_m \times X_{CD} = I_m/\omega_0 C$

$\therefore \quad C = I_m/\omega_0 V_C = 0.314/100\,\pi \times 300 = 3.33\ \mu F.$

$VL = I_m \times X_L = I_m\,\omega_0 L;\ L = V_1/\omega_0 I_m = 300/100\,\pi \times 0.314 =$ **3.03 H**

Example 13.55. *A coil having an inductance of 50 mH and resistance 10 Ω is connected in series with a 25 μF capacitor across a 200 V ac supply. Calculate (a) resonance frequency of the circuit (b) current flowing at resonance and (c) value of Q_0 by using different data.*

(Elect. Engg. A.M.Ae, S.I, June 1991)

Solution. (*a*)
$$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{50\times10^{-3}\times25\times10^{-6}}} = 142.3\ Hz$$

(*b*) $I_0 = V/R = 200/10 = 20$ A

(*c*)
$$Q_0 = \frac{\omega_0 L}{R} = \frac{2\pi\times142.3\times50\times10^{-3}}{10} = 4.47$$

$$Q_0 = \frac{1}{\omega_0 CR} = \frac{1}{2\pi\times142.3\times25\times10^{-6}\times10} = 4.47$$

$$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{10}\sqrt{\frac{50\times10^{-3}}{25\times10^{-6}}} = 4.47$$

Example 13.56. *A 20-Ω resistor is connected in series with an inductor, a capacitor and an ammeter across a 25-V variable frequency supply. When the frequency is 400-Hz, the current is at its maximum value of 0.5 A and the potential differnce across the capacitor is 150 V. Calculate*

(a) the capacitance of the capacitor

(b) the resistance and inductance of the inductor

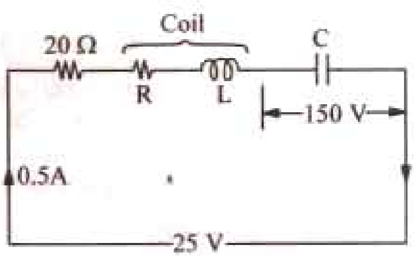


Fig. 13.49

Solution. Since current is maximum, the circuit is in resonance.

$X_C = V_C/I = 150/0.5 = 300\ \Omega$

(*a*) $X_C = 1/2\pi fC$ or $300 = 1/2\pi \times 400 \times C$

$\therefore\ C = 1.325 \times 10^{-6}\ F =$ **1.325 μF**

(*b*) $X_L = X_C = 300\ \Omega$

$\therefore\ 2\pi \times 400 \times L = 300 \quad \therefore\ L =$ **0.119 H**

(*c*) Now, at resonance,

circuit resistance = circuit impedance or $20 + R = V/I = 25/0.5 \quad \therefore\ R = 30\ \Omega$ ...Fig.13.49

Example 13.57. *An R-L-C series circuit consists of a resistance of 1000 Ω, an inductance of 100 mH an a capacitance of 10 μμ F. If a voltage of 100 V is applied across the combination, find (i) the resonance frequency (ii) Q-factor of the circuit and (iii) the half-power points.*

(Elect. Circuit Analysis, Bombay Univ. 1985)

Solution. (*i*)
$$f_0 = \frac{1}{2\pi\sqrt{10^{-1}\times10^{-11}}} = \frac{10^6}{2\pi} = \textbf{159 kHz}$$

(*ii*) $$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{1000} \times \sqrt{\frac{10^{-1}}{10^{-11}}} = \mathbf{100}$$

(*iii*) $$f_1 = f_0 - \frac{R}{4\pi L} = 159 \times 10^3 - \frac{1000}{4\pi \times 10^{-1}} = \mathbf{158.2\ kHz}$$

$$f_2 = f_0 + \frac{R}{4\pi L} = 159 \times 10^3 + \frac{1000}{4\pi \times 10^{-1}} = \mathbf{159.8\ kHz}$$

Example 13.58. *A series R-L-C circuit consists of R = 1000 Ω, L = 100 mH and C = 10 picofarads. The applied voltage across the circuit is 100 V.*

(*i*) *Find the resonant frequency of the circuit.*

(*ii*) *Find the quality factor of the circuit at the resonant frequency.*

(*iii*) *At what angular frequencies do the half power points occur ?*

(*iv*) *Calculate the bandwidth of the circuit.*

(Networks-I, Delhi Univ. Jan. 1986 & U.P. Tech. Univ. 2001)

Solution. (*i*) $$f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{100 \times 10^{-3} \times 10 \times 10^{-12}}} = 159.15 \text{ kHz}$$

(*ii*) $$Q_0 = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{1000} \times \sqrt{\frac{1000 \times 10^{-3}}{10 \times 10^{-12}}} = 100$$

(*iii*) $$B_{hp} = \frac{R}{2\pi L} = \frac{1000}{2\pi \times 100 \times 10^{-3}} = 1591.5 \text{ Hz}$$

Also. $B_{hp} = f_0/Q_0 = 159.15 \text{ kHz}/100 = 1.5915 \text{ kHz} = \mathbf{1591.5\ Hz}$ ...as above

(*iv*) $$\omega_1 = \omega_0\left(1 - \frac{1}{2Q_0}\right) = 2\pi \times 159.15\left(1 - \frac{1}{2 \times 100}\right) - 994.969 \text{ radia/sec.}$$

$$\omega_2 = \omega_0\left(1 + \frac{1}{2\,Q}\right) = 2\pi \times 159.15\left(1 + \frac{1}{2 \times 100}\right) = 1004.969 \text{ rad/sec}$$

(*v*) Band width $= (\omega_2 - \omega_1) = 1004.969 - 994.969 = 10.00$ rad/sec.

Example 13.59. *An R-L-C series resonant circuit has the following parameters :*
Resonance frequency = 5000/2π Hz; impedance at resonance = 56 Ω and Q-factor = 25.
Calculate the capacitane of the capacitor and the inductance of the inductor.
Assuming that these values are independent of the frequency, find the two frequencies at which the circuit impedance has a phase angle of π/4 radian.

Solution. Here $\omega_0 = 2\pi f_0 = 2\pi \times 5000/2\pi = 5000$ rad/s

Now, $$Q = \frac{\omega_0 L}{R} \text{ or } 25 = \frac{5000\,L}{56} \text{ or } L = 0.28 \text{ H}$$

Also at resonance $\omega_0 L = 1/\omega_0 C$ or $5000 \times 0.28 = 1/5000 \times C$ $\therefore$ $C = 0.143\ \mu F$

The circuit impedance has a phase shift of 45° and the two half-power frequencies which can be found as follows :

$$BW = \frac{f_0}{Q} = \frac{5000/2\pi}{25} = \mathbf{31.83\ Hz}$$

Therefore lower half-power frequency = $(f_0 - 31.83/2) = 5000/2\pi - 15.9 =$ **779.8 Hz.**

Upper half-power frequency = $(f_0 + 31.83/2) = 5000/2\pi + 15.9 =$ **811.7 Hz.**

Example 13.60. *An R-L-C series circuit is connected to a 20-V variable frequency supply. If R = 20 Ω, L = 20 mH and C = 0.5 μF, calculate the following :*

(*a*) *resonant frequency* f_0 (*b*) *resonant circuit* Q_0 *using L/C ratio* (*c*) *half-power bandwidth using* f_0 *and* Q_0 (*d*) *half-power bandwidth using the general formula for any bandwidth* (*e*) *half-*

power bandwidth using the given component values (f) maximum power dissipated at f_0

Solution. (*a*) $f_0 = 1/2\pi\sqrt{LC} = 1/2\pi\sqrt{(20\times10^{-3}\times 0.5\times10^{-6})} =$ **159 Hz**

(*b*) $Q_0 = \frac{1}{R}\cdot\sqrt{\frac{L}{C}} = \frac{1}{20}\cdot\sqrt{\frac{20\times10^{-3}}{0.5\times10^{-6}}} =$ **10**

(*c*) $B_{hp} = f_0/Q_0 = 1591/10 =$ **159.1 Hz**

(*d*) $B_{hp} = f_0\ Q/Q_0 = 1591 \times \tan 45°/10 =$ **159.1 Hz**

It is so because the power factor angle at half-power frequencies is ± 45°.

(*e*) $B_{hp} = R/2\pi\ L = 20/2\pi \times 20 \times 10^{-3} =$ **159.1 Hz**

(*f*) $f_0 = V^2/R = 20^2/20 =$ **20 W**

Example 13.61. *An inductor having a resistance of 25 Ω and a Q_0 of 10 at a resonant frequency of 10 kHz is fed from a 100 ∠ 0° supply. Calculate*

(a) Value of series capacitance required to prodcue resonance with the coil

(b) the inductance of the coil (c) Q_0 using the L/C ratio (d) voltage across the capacitor (e) voltage across the coil.

Solution. (*a*) $X_{L0} = Q_0 R = 10 \times 25 = 250\ \Omega$.

Now, $X_{C0} = X_{L0} = 250\ \Omega$.

Hence, $C = 1/2\pi f_0 \times X_{C0} = 1/2\pi \times 10^4 \times 250 = 63.67 \times 10^{-9}\ F =$ **63.67 μF**

(*b*) $L = X_{L0}/2\pi f_0 = 250/2\pi \times 10^4 =$ **3.98 mH**

(*c*) $Q_0 = \frac{1}{R}\cdot\sqrt{\frac{L}{C}}$

Now, $\frac{L}{C} = \frac{3.98\times10^{-3}}{63.67\times10^{-9}} = 6.25\times10^4$

∴ $Q_0 = \frac{1}{25}\times\sqrt{6.25\times10^4} =$ **10** (verification)

(*d*) $V_{CO} = -jQ_0V = -j\ 100\angle 0° \times 10 = -j\ 1000\text{V} = -$ **100 ∠ − 90° V**

(*e*) Since $V_{L0} = V_{CO}$ in magnitude, hence, $V_{L0} = +j\ 1000$ V

$= 1000\angle 90°\text{V}$; Also, $V_R = V = 100\angle 0°$

Hence, $V_{coil} = V_R = V_{L0}$

$= 100 + 1000\angle 90° = 100 + j\ 1000 =$ **1005 ∠ 84.3°**

Example 13.62. *A series L.C circuit has L = 100 pH, C = 2500 μF and Q = 70. Find (a) resonant frequency f_0 (b) half-power points and (c) bandwidth.*

Solution. (*a*) $f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{10^9}{2\pi\sqrt{100\times2500}} =$ **318.3 kHz**

(*b*) $f_2 - f_1 = \Delta f = f_0/Q = 318.3/70 =$ **4.55 kHz**

(*c*) $f_1 f_2 = f_0^2 = 318.3^2;\ f_2 - f_1 = 4.55$ kHz

Solving for f_1 and f_2, we get, $f_1 =$ **3104 kHz** and $f_2 =$ **320.59 kHz**

Note. Since Q is very high, there would be negligible error in assuming that the half-power points are equidistant from the resonant frequency.

Example 13.63. *A resistor and a capacitor are connected in series across a 150 V ac supply. When the frequency is 40 Hz, the circuit draws 5 A. When the frequency is increased to 50 Hz, it draws 6 A. Find the values of resistance and capacitance. Also find the power drawn in the second case.* **[Bombay University, 1997]**

Solution. Suffix 1 for 40 Hz and 2 for 50 Hz will be given.

$Z_1 = 150/5 = 30$ ohms, at 40 Hz

or $R^2 + X_{C2}^2 = 900$

Similarly, $R^2 + X_{C2}^2 = 625$, at 50 Hz, since $Z_2 = 25\ \Omega$

Further, capacitive reactance is inversely proportional to the frequency.

$X_{C1}/X_{C2} = 50/40$ or $X_{C1} = 1.25\ X_{C2}$

$X_{C1}^2 - X_{C2}^2 = 900 - 625 = 275$

$X_{C1}^2\ (1.25^2 - 1) = 275$ or $X_{C2}^2 = 488.9$, $X_{C2} = 22.11$

$X_{C1} = 1.25 \times 22.11 = 27.64$ ohms

$R^2 = 900 - X_{C1}^2 = 900 - 764 = 136$, $R = 11.662$ ohms

$$C = \frac{1}{2\pi \times 40 \times 27.64}\ 144\ \mu$$

Power drawn in the second case $= 6^2 \times 11.662 = 420$ watts

Example 13.64. *A constant voltage at a frequency of 1 MHz is applied to an inductor in series with a variable capacitor. When the capacitor is set 500 pF, the current has its maximum value while it is reduced to one-half when the capacitance is 600 pF. Find*

(i) the resistance, (ii) the inductance, (iii) the Q-factor of the inductor.

[Bombay University, 1996]

Solution. Resonance takes palce at 1 MHz, for $C = 500$ pF.

$$LC = \frac{1}{\omega_0\ 2} = \frac{1}{(2\pi \times 10^6)^2} = \frac{10^{-12}}{4\pi^2}$$

$$L = 10^{-12}/(4 \times \pi^2 \times 500 \times 10^{-12}) = 1/(4 \times \pi^2 \times 500)$$

$$= 50.72 \text{ mH}$$

$$Z_1 = \text{Impedance with 500 pF capacitor} = R + j\ \omega L - j\ 1/wC$$

$$= R + j\ (2p \times 10^6 \times 50.72 \times 10^{-6}) - j\ \frac{1}{2\pi \times 10^6 \times 500 \times 10^{-12}}$$

$$= R, \text{ since resonance occurs.}$$

$$Z_2 = \text{Impedance with 600 pF capacitor}$$

$$|Z|_2 = R + j\ \omega L - j \frac{1}{\omega \times 600 \times 10^{-12}} = 2\ R, \text{ since current is halved.}$$

$$\omega L - 1/\omega C = \sqrt{3}\ R$$

$$\sqrt{3}\ R = 2\pi \times 10^6 \times 50.7 \times 10^{-6} - \frac{1}{2\pi \times 10^6 \times 600 \times 10^{-12}}$$

$$= 2\pi \times 50.72 \frac{10^6}{2\pi\ 600}$$

$$= 318.52 - 265.4 = 53.12$$

$$R = 30.67 \text{ ohms}$$

$$Q \text{ Factor of coil} = (\omega_0 L)/R$$

$$= 50.72 \times 10^{-6} \times 2\pi \times 10^6/30.67 = 50.72 \times 2\pi/30.67 = 10.38$$

Example 13.65. *A large coil of inductance 1.405 H and resistance 40 ohms is connected in series with a capacitor of 20 microfarads. Calculate the frequency at which the circuit resonates.*

If a votlage of 100 V at the corresponding frequency is applied to the circuit, calculate the current drawn from the supply and the voltages across the coil and across the capacitor.

[Nagpur University Nov. 1999]

Solution.
$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1.405 \times 20 \times 10^{-6}}} = 188.65 \text{ radians/sec.}$$
$$f_0 = \frac{188.65}{2\pi} = 30.04 \text{ Hz}$$

Reactance at 30.04 Hz have to be calculated for voltages across the coil and the capacitor

$$X_L = \omega_0 L = 188.65 \times 1.405 = 265\ \Omega$$
$$X_C = \frac{1}{\omega_0 C} = \frac{1}{188.65 \times 20 \times 10^{-6}} = 265\ \Omega$$

Coil Impedance $= \sqrt{40^2 + 265^2} = 268\ \Omega$

Impedance of the total circuit

$$= 40 + j\,265 - j\,265 = 40\ \Omega$$

Supply Curmet $= \dfrac{100}{40} = 2.5$ amp, at unity p.f.

Voltage across the coil = 2.5 × 268 = 670 V

Voltage across the capacitor = 2.5 × 265 = 662.5 V

The phasor diagram is drawn below; in Fig. 13.50 (*a*) for the circuit in Fig. 13.50 (*b*)

$$OB = V,\ BC = V_C$$
$$AB = IX_L$$

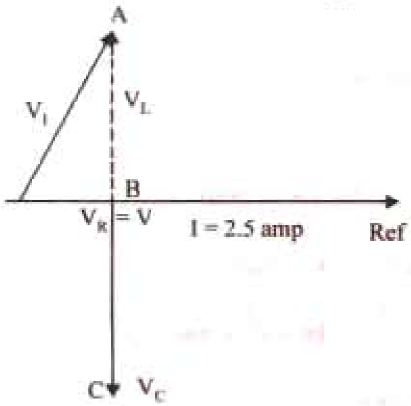


Fig. 13.50 (*a*) Phasor diagram

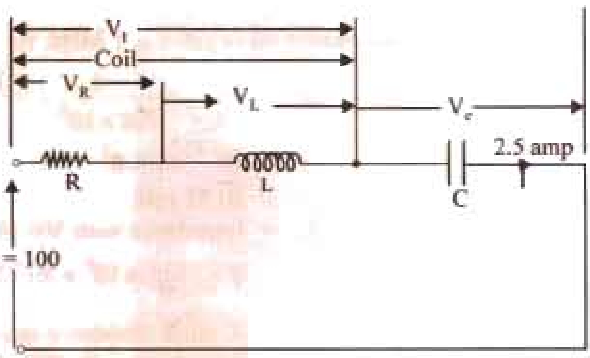


Fig. 13.50 (*b*) Series resonating circuit

Example 13.66. *A series R-L-C circuit is excited from a constant-voltage variable frequency source. The current in the circuit becomes maximum at a frequency of 600/2π Hz and falls to half the maximum value at 400/2π Hz. If the resistance in the circuit is 3 Ω, find L and C*

(Grad. I.E.T.E. Summer 1991)

Solution. Current at resonance is $I_0 = V/R$

Actual current at any other frequency is $I = \dfrac{V}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}$

$$\therefore \frac{I}{I_0} = \frac{V}{\left[R^2 + \left(\omega L - \dfrac{1}{\omega L}\right)^2\right]^{1/2}} \cdot \frac{R}{V} = \frac{1}{\left[1 + \dfrac{1}{R^2}\left(\omega L - \dfrac{1}{\omega L}\right)^2\right]^{1/2}} = \frac{1}{\left[1 + \left(\dfrac{\omega_0 L}{R}\right)^2 \left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)^2\right]^{1/2}}$$

Now $Q = \dfrac{\omega_0 L}{R}$ and $\dfrac{\omega}{\omega_0} = \dfrac{f}{f_0}$, hence $\dfrac{I}{I_0} = \dfrac{1}{\left[1 + Q^2\left(\dfrac{f}{f_0} - \dfrac{f_0}{f}\right)^2\right]^{1/2}}$

In the present case, $f_0 = 600/2\pi$ Hz, $f = 400/2\pi$ Hz and $I/I_0 = 1/2$

$$\therefore \quad \frac{1}{2} = \frac{1}{\left[1+Q^2\left(\frac{400}{600}-\frac{600}{400}\right)^2\right]^{1/2}} = \frac{1}{\left[1+Q^2\left(\frac{2}{2}-\frac{3}{2}\right)^2\right]^{1/2}}$$

$$\text{or} \quad \frac{1}{4} = \frac{1}{1+25\,Q^2/36} \quad \therefore\ Q = 2.08$$

$$\text{Now,} \quad Q = \frac{1}{\omega_0 RC} \text{ or } 2.08 = \frac{1}{600\times 3\times C} \quad \therefore\ C = 267\times 10^{-6}\ F = \mathbf{267\ mF}$$

$$\text{Also} \quad Q = \omega_0\, L/R \ \therefore\ 2.08 = \frac{600L}{R} = \frac{600L}{3} \ \therefore\ L = \mathbf{10.4\ mH}$$

Example 13.67. *Discuss briefly the phenomenon of electrical resonance in simple R-L-C circuits.*

A coil of inductance L and resistance R in series with a capacitor is supplied at constant voltage from a variable-frequency source. Call the resonance frequency ω_0 and find, in terms of L, R and ω_0, the values of that frequency at which the circuit current would be half as much as at resonance.

(Basic Electricity, Bombay Univ. 1985)

Solution. For discussion of resonance, please refer to Art. 13.10.

The current at resonance is maximum and is given by $I_0 = V/R$. Current at any other frequency is

$$I = \frac{V}{[R^2+(\omega L - 1/\omega C)^2]^{1/2}}$$

$$\therefore \quad \frac{I_0}{I} = \frac{[R^2+(\omega L - 1/\omega C)^2]^{1/2}}{R}$$

$$\text{or} \quad N = \left[1+\frac{1}{R^2}\left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}$$

$$\text{Now} \quad Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

$$\therefore \quad R = \frac{\omega_0 L}{Q} = \frac{1}{\omega_0 CQ}$$

Substituting this value in the above equation, we get

$$N = \left[1+Q^2\left(\frac{f}{f_0}-\frac{f_0}{f}\right)^2\right]^{1/2}.$$

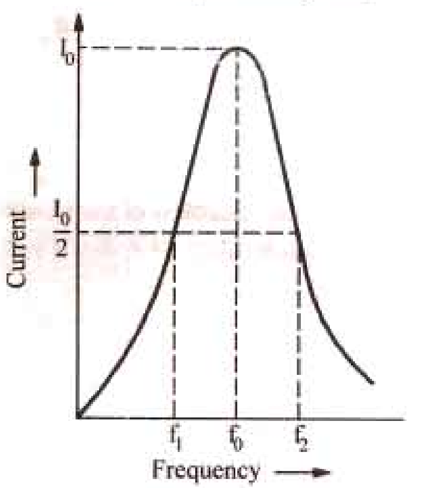


Fig. 13.51

$$\text{or} \quad \left(\frac{f}{f_0}-\frac{f_0}{f}\right)^2 = \frac{N^2-1}{Q^2} \ \therefore\ \frac{\sqrt{(N^2-1)}}{Q} = \pm\left(\frac{f}{f_0}-\frac{f_0}{f}\right)$$

$$\text{or} \quad \frac{\sqrt{(N^2-1)}}{Q} = \frac{f_2}{f_0}-\frac{f_0}{f_2} = \frac{f_0}{f_1}-\frac{f_1}{f_0}$$

where $f_2 > f_0$ and $f_1 < f_0$ are the two frequencies at which the current has fallen to 1/N of the resonant value.

In the present case, $N = 2$ (Fig. 13.50) $\therefore\ \frac{f_2}{f_0}-\frac{f_0}{f_2} = \frac{\sqrt{3}}{Q}$ and $\frac{f_0}{f_1}-\frac{f_1}{f_0} = \frac{\sqrt{3}}{Q}$

From these equations, f_1 and f_2 may be calculated.

Example 13.68. *A coil of inductance 9 H and resistance 50 Ω in series with a capacitor is supplied at constant votlage from a variable frequency source. If the maximum current of 1A occurs at 75 Hz, find the frequency when the current is 0.5 A.*

(Principles of Elect. Engg. Delhi Univ. 1987)

Solution. Here, $N = I_0/I = I/0.5 = 2$; $Q = \omega_0 L/R = 2\pi \times 75 \times 9/50 = 84.8$

Let f_1 and f_2 be the frequencies at which current falls to half its maximum value at resonance frequency. Then, as seen from above.

$$\frac{f_0}{f_1} - \frac{f_1}{f_0} = \frac{\sqrt{3}}{Q} \text{ or } \frac{75}{f_1} - \frac{f_1}{75} = \frac{\sqrt{3}}{84.4}$$

or $$(75^2 - f_1^2)/75 f_1 = 0.02 \text{ or } f_1^2 + 1.5 f_1 - 5625 = 0 \text{ or } f_1 = \mathbf{74.25\ Hz}$$

Also $$\frac{f_2}{75} - \frac{75}{f_2} = \frac{\sqrt{3}}{84.4} \text{ or } f_2^2 - 1.5 f_2 - 5625 = 0 \text{ or } f_2 = \mathbf{75.75\ Hz.}$$

Example 13.69. *Using the data given in Ex. 13.45 find the following when the power drops to 4 W on either side of the maximum power at resonance.*

(a) circuit Q (b) circuit phase angle ϕ (c) 4-W bandwidth B

(d) lower freqency f_1 (e) upper frequency f_2.

(f) resonance frequency using the value of f_1 and f_2.

Solution. (*a*) $$P = \frac{P_0}{(1+Q_0^2)} \quad \therefore\ Q = \sqrt{(P_0/P_1) - 1} = \sqrt{(20/4) - 1} = \mathbf{2}$$

(*b*) $$\tan(\pm\theta) = 2\,j\,\pm\theta/\tan^{-1} 2 = \pm\ \mathbf{63.4°}$$

(*c*) $$B_{hp} = \frac{f_0 Q}{Q_0} = \frac{1591 \times 2}{10} = 318.2 \text{ Hz}$$

(*d*) $$f_1 = f_0 - B/2 = 1591 - (318.2/2) = 1431.9 \text{ Hz}$$

(*e*) $$f_2 = f_0 + B/2 = 1591 + (318.2/2) = 1750.1 \text{ Hz}$$

(*f*) $$f_0 = \sqrt{f_1 f_2} = \sqrt{1431.9 \times 1750.2} = \mathbf{1591\ Hz.}$$

It shows that regardless of the bandwidth magnitude, f_0 is always the geometric mean of f_1 and f_2.

Example 13.70. *A constant e.m.f. source of varibale frequency is connected to a series R.L.C. circuit of Fig. 13.51.*

(a) Shown in nature of the frequency – V_R graph

(b) Calculate the following (i) frequency at which maximum power is consumed in the 2 Ω resistor

(ii) Q-factor of the circuit at the above frequency (iii) frequencies at which the power consumed in 2 Ω resistor is one-tenth of its maximum value.

(Network Analysis A.M.I.E Sec. B.W. 1989)

Solution. (*a*) The graph of angular frequency ω versus voltage drop across *R i.e.* V_R is shown in Fig. 13.52. It is seen that as frequency of the applied voltage increases, V_R increases till it reaches its maximum value when the given *RLC* circuit becomes purely reactive *i.e.* when $X_L = X_C$ (Art. 13.10). (*b*) (*i*) maximum power will be consumed in the 2 Ω resistor when maximum current flows in the circuit under resonant condition.

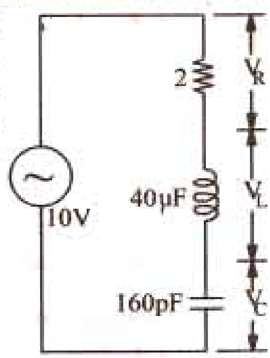


Fig. 13.52

For resonance $\omega_0 L = 1/\omega_0 X_C$ or

$$\omega_0 = 1/\sqrt{LC} = 1/\sqrt{40 \times 10^{-6} \times 160 \times 10^{-12}} = 10^9/80 \text{ rad/s}$$

$\therefore\ f_0 = \omega_0/2\pi = 10^9/2\pi \times 80 = \mathbf{1.989\ MHz}$

(*ii*) Q-factor, $Q_0 = \dfrac{\omega_0 L}{R} = \dfrac{10^9 \times 40 \times 10^{-6}}{80 \times 2} = \mathbf{250}$

(*iii*) Maximum current $I_0 = V/R$ (Art. 13.10). Current at any other frequency is $I = \dfrac{V}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$

Power at any frequency $I^2R = \dfrac{V^2}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} . R$

Maximum power $I_0^2\, R = \left(\dfrac{V}{R}\right)^2 . R$

Hence, the frequencies at which power consume would be one-tenth of the maximum power will be given by the relation. $\dfrac{1}{10} . \left(\dfrac{V}{r}\right)^2 R = \dfrac{V^2}{R^2 + (\omega L - 1/\omega C)^2} . R$

or cross multiplying, we get $R^2 + (\omega L - 1/\omega C)^2 = 10R^2$ or $(\omega L - 1/\omega)\, C^2 = 9\, R^2$

$$\therefore \qquad (\omega L - 1/\omega C) = \pm\, 3\, R \text{ and } \omega_1 L - 1/\omega_1\, C = 3R$$

$$\text{and } \omega_2\, L - 1/\omega_2\, C = -\, 3\, R$$

Adding the above two equations, we get

$$(\omega_1 + \omega_2)\, L - \frac{1}{C}\left(\frac{\omega_2 + \omega_1}{\omega_1 . \omega_2}\right) = 0 \text{ or } \omega_1\omega_2 = \frac{1}{LC} - \omega_0^2$$

Subtracting the same two equations, we have $L\,(\omega_1 - \omega_2)\,\dfrac{1}{C}\left(\dfrac{1}{\omega_2} - \dfrac{1}{\omega_1}\right) = 6R\,(\omega_1 - \omega_2) + \dfrac{1}{LC}$

$\left(\dfrac{\omega_1 - \omega_2}{\omega_1 . \omega_2}\right) = \dfrac{6R}{L}$

Substituting the value of $1/LC = \omega^2 = \omega_1\omega_2$, we get

$$(\omega_1 - \omega_2) + \omega_1\omega_2\left(\frac{\omega_1 - \omega_2}{\omega_1\omega_2}\right) = \frac{6R}{L} \text{ or } (\omega_1 - \omega_2) = \Delta\omega = \frac{3R}{L}$$

Now, $\qquad \omega_2 = \omega_0 + \dfrac{\Delta\omega}{2} = \omega_0 + \dfrac{1.5R}{L}$ and $\omega_1 = \omega_0 - \dfrac{1.5R}{L}$

$\therefore f_2 = f_0 + 1.5\, R/2\pi L = 1.989 \times 10^6 + 1.5 \times 2.2\pi \times 40 \times 10^{-6} = 1.989 \times 10^6 + 0.0119 \times 10^6 =$ **2 MHz**

$f_1 = f_0 - 1.5\, R/2\pi L = 1.989 \times 10^6 - 0.0119 \times 10^6 =$ **1.977 MHz**

Example 13.71. *Show that in R-L-C circuit, the resonant frequency ω_0 is the geometric mean of the lower and upper half-power frequencies ω_1 and ω_2 respectively.*

Solution. As stated earlier, at lower half-power radiant frequency ω_1; $X_C > X_L$ and at frequencies higher than half-power frequencies $X_L > X_C$. However, the difference between the two equals R.

$$\therefore \qquad \text{at } \omega_1,\ X_C - X_L = R \text{ or } 1/\omega_1 C - \omega_1 L = R \qquad ...(i)$$

$$\text{At } \omega_2,\ X_L - X_C = R \text{ or } \omega_2 L - 1/\omega_2\, C = R$$

Multiplying both sides of Eq. (*i*) by C and substituting $\omega_0^2 = 1\sqrt{LC}$, we get

$$\frac{1}{\omega_1} - \frac{\omega_1}{\omega_0^2} = \frac{\omega_2}{\omega_0^2} - \frac{1}{\omega_2} \text{ or } \frac{1}{\omega_1} = \frac{1}{\omega_2} = \frac{\omega_1 + \omega_2}{\omega_0^2} \text{ or } \omega_0 - = \sqrt{\omega_1\omega_2}$$

Example 13.72. *Prove that in a sereis R-L-C circuit. $Q_0 = \omega_0 L/R = f_0$/bandwidth $= f_0/BW$.*

Solution. As has been proved in Art. 14.13, at half power frequencies, net reactance equals resistance. Moreover, at ω_1, capacitive reactance exceeds inductive reactance whereas at ω_2, inductive reactance exceeds capacitive reactance.

$$\therefore \qquad \frac{1}{2\pi f_1\, C} - 2\pi\, f_1 L = R \text{ or } f_1 = \frac{-\, R + \sqrt{R^2 + 4L/C}}{4\pi L}$$

Also $\qquad 2\pi\, f_2 L - \dfrac{1}{2\pi f_2\, C} = \text{ or } f_2 = \dfrac{R + \sqrt{R^2 + 4L/C}}{4\pi L}$

Now, $\qquad BW = f_2 - f_1 = R/2\pi L$. Hence, $Q_0 = f_0/BW = 2\pi f_0 L/R = \omega_0 L/R$.

Tutorial Problem No. 13.4

1. An a.c. series circuit has a resistance of 10 Ω, an inductance of 0.2 H and a capacitance of 60 μF. Calculate
   (*a*) the resonant frequency (*b*) the current and (*c*) the power at resonance.
   Give that the applied voltage is 200 V. **[46 Hz; 20 A; 4 kW]**
2. A circuit consists of an inductor which has a resistance of 10 Ω and an inductance of 0.3 H, in series with a capacitor of 30 μ*F* capacitance. Calculate
   (*a*) the impedance of the circuit to currents of 40 Hz (*b*) the resonant frequency (*c*) the peak value of stored energy in joules when the applied voltage is 200 V at the resonant frequency.
   [58.31 Ω; 53 Hz; 120 J]
3. A resistor and a capcitor are connected in series with a variable inductor. When the circuit is connected to a 240-V, 50-Hz supply, the maximum current given by varying the inductance is 0.5 A. At this current, the voltage across the capacitor is 250 V. Calculate the values of
   (*a*) the rsistance (*b*) the capacitance (*c*) the inductance. **[480 Ω, 6.36 μF; 1.59 H]**
   Neglect the resistance of the inductor
4. A circuit consisting of a coil of resistance 12Ω and inductance 0.15 H in series with a capacitor of 12μF is connected to a variable frequency supply which has a constant voltage of 24 V.
   Calculate (*a*) the resonant frequency (*b*) the current in the circuit at resonance (*c*) the voltage across the capacitor and the coil at resonance. **[(*a*) 153 Hz (*b*) 2 A (*c*) 224 V]**
5. A resistance, a capacitor and a variable inductance are connected in series across a 200-V, 50-Hz supply. The maximum current which can be obtained by varying the inductance is 314 mA and the voltage across the capacitor is then 300 V. Calculate the capacitance of the capacitor and the values of the inductance and resistance. **[3.33 μF, 3.04 H, 637 Ω]** (*I.E.E. London*)
6. A 250-V circuit, consisting of a resistor, an inductor and a capacitor in series, resonates at 50 Hz. The current is then 1A and the p.d. across the capacitor is 500 V. Calculate (*i*) the resistance (*ii*) the inductance and (*iii*) the capacitance. Draw the vector diagram for this condition and sketch a graph showing how the current would vary in a circuit of this kind if the frequency were varied over a wide range, the applied voltage remaining constant.
   [(*i*) 250 Ω (*ii*) 0.798 H (*iii*) 12.72 μ F] (*City & Guilds, London*)
7. A resistance of 24 Ω, a capacitance of 150 μF and an inductance of 0.16 H are connected in series with each other. A supply at 240 V, 50 Hz is applied to the ends of the combination. Calculate (*a*) the current in the circuit (*b*) the potential differences across each element of the circuit (*c*) the frequency to which the supply would need to be changed so that the current would be at unity power-factor and find the current at this frequency.
   [(*a*) 6.37 A (*b*) V_R = 152.9 V, V_C = 320 V, V_C = 123.3 V (*c*) 32 Hz; 10 A] (*London Univ.*)
8. A series circuit consists of a resistance of 10 Ω, an inductance of 8 mH and a capacitance of 500 μμ*F*. A sinusoidal E.M.F. of constant amplitude 5 V is introduced into the circuit and its frequency varied over a range including the resonant frequency.
   At what frequencies will be current be (*a*) a maximum (*b*) one-half the-maximum ?
   [(*a*) 79.6 kHz (*b*) 79.872 kHz, 79.528 kHz] (*App. Elect. London Univ.*)
9. A circuit consists of a resistance of 12 ohms, a capacitance of 320 μF and an inductance of 0.08 H, all in series. A supply of 240 V, 50 Hz is applied to the ends of the curcuit. Calculate :
   (*a*) the current in the coil.
   (*b*) the potential differences across each element of the circuit.
   (*c*) the frequency at which the current would have unity power-factor.
   [(*a*) 12.4 A (*b*) 149 V, 311 V (*c*) 32 Hz] (*London Univ.*)
10. A series circuit consists of a reactor of 0.1 henry inductance and 5 ohms resistance and a capacitor of 25.5 μ F capacitance.
    Find the resonance frequency and the precentage change in the current for a divergence of 1 percent from the resonance frequency.
    [100 Hz, 1.96% at 99 Hz; 4.2% at 101 Hz] (*City and Guilds, London*)

OBJECTIVE TESTS –13

1. In a series *R-L* circuit, V_L—V_R by—degrees.
   (*a*) lags, 45 (*b*) lags, 90
   (*c*) leads, 90 (*d*) leads, 45
2. The voltage applied across an *R-L* circuit is equal to—of V_R and V_L.
   (*a*) arithmetic sum (*b*) algebraic sum
   (*c*) phasor sum (*d*) sum of the squares.
3. The power in an a.c. circuit is given by
   (*a*) $VI\cos\phi$ (*b*) $VI\sin\phi$
   (*c*) I^2Z (*d*) I^2X_L
4. The p.f. of an *R-C* circuit is
   (*a*) often zero
   (*b*) between zero and 1
   (*c*) always unity
   (*d*) between zero and – 1.0
5. Which phasor diagram of Fig. 13.53 is correct for a series *R-C* circuit ?

(*a*) (*b*) (*c*) (*d*)

Fig. 13.53

6. In an *R-L-C* circuit, $v(t) = 20\sin(314t + 5\pi/6)$ and $i(t) = 10\sin(314t + 2\pi/3)$. The p.f. of the circuit is
   (*a*) 0.5 lead (*b*) 0.866 lag
   (*c*) 0.866 lead (*d*) 0.5 lag
   and power drawn is —watt.
   (*e*) 200 (*f*) 86.6
   (*g*) 173.2 (*h*) 50
7. The input of an a.c. circuit having p.f. of 0.8 lagging is 20kVA. The power drawn by the circuit is—kW.
   (*a*) 12 (*b*) 20
   (*c*) 16 (*c*) 8
8. The power factor of an a.c. circuit is given by
   (*a*) cosine of the phase angle
   (*b*) tangent of the phase angle
   (*c*) the ratio R/X_L
   (*d*) the ratio X_L/Z
9. In a series *R-L-C* circuit, $R = 100\ \Omega$, $X_L = 300\ \Omega$ and $X_C = 200\ \Omega$. The phase angle ϕ of the circuit is—degrees.
   (*a*) 0 (*b*) 90
   (*c*) 45 (*d*) – 45
10. The phase angle of a series *R-L-C* circuit is leading if
    (*a*) $X_L = 0$ (*b*) $R = 0$
    (*c*) $X_C > X_L$ (*d*) $X_C < X_L$
11. In an a.c. circuit, the ratio of kW/kVA represents
    (*a*) power factor
    (*b*) load factor
    (*c*) form factor
    (*d*) diversity factor
12. If p.f. of a circuit is unity, its reactive power is
    (*a*) a maximum
    (*b*) equal to I^2R
    (*c*) zero
    (*d*) a negative quantity
13. In a series *R-L-C* circuit, resonance occurs when
    (*a*) $R = X_L - X_C$
    (*b*) $X_L = X_C$
    (*c*) $X_L = 10\,X_C$ or more
    (*d*) net $X > R$
14. The p.f. of a series *R-L-C* circuit at its half-power points is
    (*a*) unity (*b*) lagging
    (*c*) leading (*d*) either (*b*) or (*c*)
15. A resonanec curve for a series circuit is a plot of frequency versus—.
    (*a*) voltage (*b*) impedance
    (*c*) current (*d*) reactance
16. At half-power points of a resonance curve, the current is—times the maximum current.
    (*a*) 2 (*b*) $\sqrt{2}$
    (*c*) $1\sqrt{2}$ (*d*) 1/2
17. Higher the *Q* of a series circuit,
    (*a*) greater its bandwidth
    (*b*) sharper its resonance
    (*c*) broader its resohace curve
    (*d*) narrower its passband
18. As the *Q*-factor of a circut—, its selectivity becomes —.
    (*a*) increases, better
    (*b*) increases, worse
    (*c*) decreases, better
    (*d*) decreases, narrower
19. An *R-L-C* circuit has a resonance frequency of 160 kHz and a *Q*-factor of 100. Its bandwidth is
    (*a*) 1.6 kHz
    (*b*) 0.625 kHz
    (*c*) 16 kHz
    (*d*) none of the above

20. An R-L-C circuit has $R = 10\ \Omega$, $X_L = 20\ \Omega$ and $X_C = 20\ \Omega$. The impedance of the circuit is given by the expression
    (*a*) $\mathbf{Z} = 10 + j\,20$ (*b*) $\mathbf{Z} = 10 + j\,50$
    (*c*) $\mathbf{Z} = 10\,j\,20$ (*d*) $\mathbf{Z} = -\,10 + j\,20$
21. The power factor of an a.c. circuit is equal to
    (*a*) consine of the phase angle
    (*b*) sine of the phase angle
    (*c*) unity for a resistive circuit
    (*d*) unity for a reactive circuit
    (*Network Theory Nagpur Univ. 1993*)
22. The power factor of an ordinary electric bulb is —.
    (*a*) zero (*b*) unity
    (*c*) slightly more than zero
    (*d*) Slightly less than unity
    (*Principles of Elect. Engg. Delhi. Univ. July 1984*)
23. In a parallel R-L circuit if I_R is the current in the resistor and I_L is the current in the indcutor, then
    (*a*) I_R lags I_L by 90°
    (*b*) I_R leads I_L by 270°
    (*c*) I_L leads I_R by 270°
    (*d*) I_L lags I_R by 90°
    (*Principles of Elect. Engg. Delhi Univ. 1984*)
24. In a parallel resonant circuit there is practically no difference between the condition for unity power factor and the condition for maximum impedance so long as Q is
    (*a*) very small of the order of 5
    (*b*) small of the order of 20
    (*c*) large of the order of 1000
    (*Principles of Elect. Engg. Delhi Univ. 1988*)
25. A series R-L-C circuit will have unity power factor if operated at a frequency of
    (*a*) $1/LC$
    (*b*) $1/\omega\sqrt{LC}$
    (*c*) $1/\omega^2 LC$
    (*d*) $1/2\pi\sqrt{LC}$
    (*Principles of Elect. Engg. Delhi Univ. July 1984*)
26. A two terminal black box contains one of the R-L-C elements. The black box is connected to a 220 volts A.C. supply. The current through the source is I. When a capacitance of 0.1 F is inserted in series between the source and the box the current through the source is 2I. The element is
    (*a*) a resistance
    (*b*) an inductance
    (*c*) a capacitance
    (*d*) it is not possible to determine the element
    (*Network Theory Nagpur Univ. 1993*)

ANSWERS

1. *c* 2. *c* 3. *a* 4. *b* 5. *b* 6. *b, f* 7. *c* 8. *a* 9. *c* 10. *c* 11. *a* 12. *c* 13. *b* 14. *d* 15. *c*
16. *c* 17. *d* 18. *a* 19. *a* 20. *c* 21. *a* 22. *b* 23. *d* 24. *c* 25. *d* 26. *d*.

14 PARALLEL A.C. CIRCUITS

14.1. Solving Parallel Circuits

When impedances are joined in parallel, there are three methods available to sovle such circuits:

(*a*) *Vector or phasor Method* (*b*) *Admittance Method and* (*c*) *Vector Algebra*

14.2. Vector or Phasor Method

Consider the circuits shown in Fig. 14.1. Here, two reactors *A* and *B* have been joined in parallel across an r.m.s. supply of *V* volts. The voltage across two parallel branches *A* and *B* is the same, but currents through them are different.

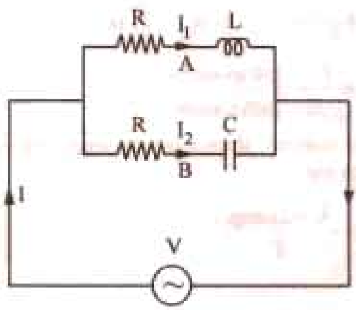


Fig. 14.1

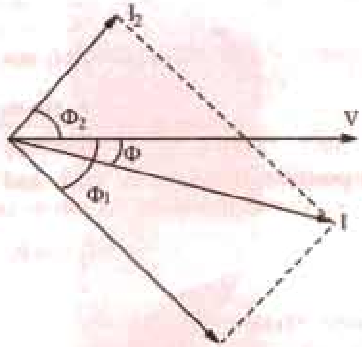


Fig. 14.2

For Branch A, $Z_1 = \sqrt{(R_1^2 + X_L^2)}$; $I_1 = V/Z_1$; $\cos \phi_1 = R_1/Z_1$ or $\phi_1 = \cos^{-1}(R_1/Z_1)$

Current I_1 lags behind the applied voltage by ϕ_1 (Fig. 14.2).

For Branch B, $Z_2 = \sqrt{(R_2^2 + X_c^2)}$; $I_2 = V/Z_2$; $\cos \phi_2 = R_2/Z_2$ or $\phi_2 = \cos^{-1}(R_2/Z_2)$

Current I_2 leads *V* by ϕ_2 (Fig. 14.2).

Resultant Current I

The resultant circuit current *I* is the vector sum of the branch currents I_1 and I_2 and can be found by (*i*) using parallelogram law of vectors, as shown in Fig. 14.2. or (*ii*) resolving I_2 into their *X*- and *Y*-components (or active and reactive components respectively) and then by combining these components, as shown in Fig. 14.3. Method (*ii*) is preferable, as it is quick and convenient.

With reference to Fig. 14.3. (*a*) we have

Sum of the active components of I_1 and I_2

$= I_1 \cos \phi_1 + I_2 \cos \phi_2$

Sum of the reactive components of I_1 and $I_2 = I_2 \sin \phi_2 - I_1 \sin \phi_1$

If I is the resultant current and ϕ its phase, then its active and reactive components must be equal to these X-and Y-components respectively [Fig. 14.3. (*b*)]

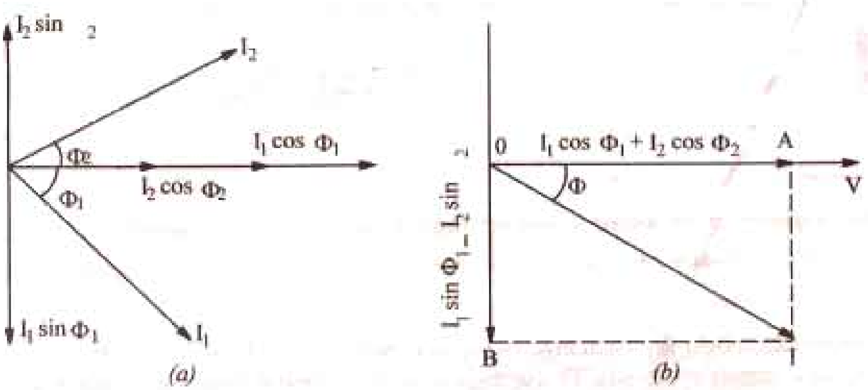


Fig. 14.3

$$I \cos \phi = I_1 \cos \phi_1 + I_2 \cos \phi_2 \text{ and } I \sin \phi = I_2 \sin \phi_2 - I_1 \sin \phi_1$$

$$\therefore \qquad I = \sqrt{[(I_1 \cos \phi_1 + I_2 \cos \phi_2)^2 + (I_2 \sin \phi_2 - I_1 \sin \phi_1)^2}$$

and
$$\tan \phi = \frac{I_2 \sin \phi_2 - I_1 \sin \phi_1}{I_1 \cos \phi_1 + I_2 \cos \phi_2} = \frac{Y - \text{component}}{X - \text{comonent}}$$

If $\tan \phi$ is positive, then current leads and if $\tan \phi$ is negative, then current lags behind the applied voltage V. Power factor for the whole circuit is given by

$$\cos \phi = \frac{I_1 \cos \phi_1 + I_2 \cos \phi_2}{1} = \frac{X - \text{comp.}}{I}$$

14.3. Admittance Method

Admittance of a circuit is defined as *the reciprocal of its impedance*. Its symbol is Y.

$$\therefore \qquad Y = \frac{1}{Z} = \frac{I}{V} \text{ or } Y = \frac{\text{r.m.s. amperes}}{\text{r.m.s. volts}}$$

Its unit is Siemens (S). A circuit having an impedance of one ohm has an admittance of one Siemens. The old unit was mho (ohm spelled backwards).

As the impedance Z of a circuit has two components X and R (Fig. 14.4.), similarly, admittance Y also has two components as shown in Fig. 14.5. The X-component is known as *conductance* and Y-component as *susceptance*.

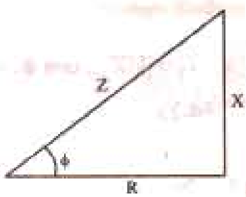


Fig. 14.4

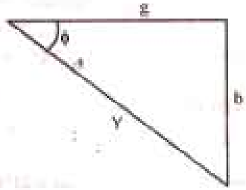


Fig. 14.5

Obviously, conductance $g = Y \cos \phi$

or $g = \frac{1}{Z} \cdot \frac{R}{Z}$ (from Fig. 14.4)

$$\therefore \qquad g^* = \frac{R}{Z^2} = \frac{R}{R^2 + X^2}$$

* In the special case when $X = 0$, then $g = 1/R$ *i.e.*, conductance becomes reciprocal of resistance, not otherwise.

Similarly, susceptance $b = Y \sin \phi = \frac{1}{2} \cdot \frac{X}{Z}$ $\quad \therefore \quad b^* = X/Z^2 = X/(R^2 + X^2)$ (from Fig. 14.5)

The admittance $\qquad Y = \sqrt{(g^2 + b^2)}$ just as $Z = \sqrt{(R^2 + X^2)}$

The unit of g, b and Y is Siemens. We will regard the *capacitive susceptance as positive and inductive susceptance as negative.*

14.4. Application of Admittance Method

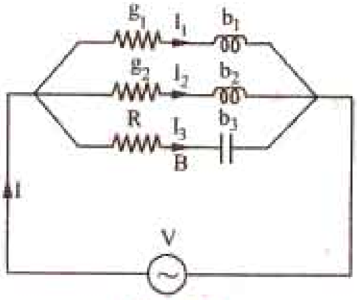


Fig. 14.6

Consider the 3-branched circuit of Fig. 14.6. Total conductance is found by merely adding the condutances of three branches. Similarly, total susceptance is found by *algebraically* adding the individual susceptances of different branches.

Total conductance $G = g_1 + g_2 + g_3$ ....

Total susceptance $B = (-b_1) + (-b_2) + b_3$ ......

(algebraic sum)

$\therefore$ Total admittance $Y = \sqrt{(G^2 + B^2)}$

Total current $I = VY$; Power factor $\cos \phi = G/Y$

14.5. Complex or Phasor Algebra

Consider the parallel circuit shown in Fig. 14.7. The two impedances, $\mathbf{Z}_1$ and $\mathbf{Z}_2$, being in parallel, have the same p.d. across them.

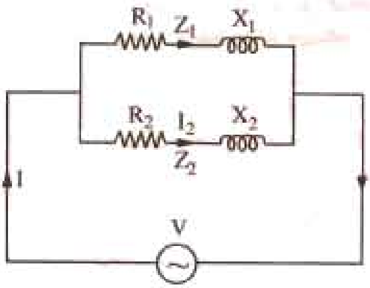


Fig. 14.7

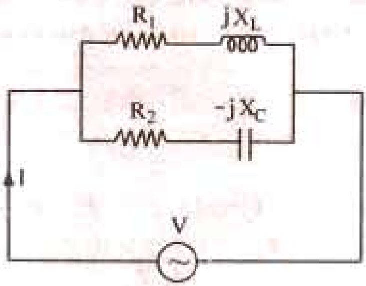


Fig. 14.8

Now $\qquad \mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1}$ and $\mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_2}$

Total current $\qquad \mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z}_1} + \frac{\mathbf{V}}{\mathbf{Z}_2} = \mathbf{V}\left(\frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_3}\right) = \mathbf{V}(\mathbf{Y}_1 + \mathbf{Y}_2) = \mathbf{VY}$

where $\qquad \mathbf{Y}$ = total admittance $= \mathbf{Y}_1 + \mathbf{Y}_2$

It should be noted that *admittances* are *added* for parallel branches, whereas for branches in series, it is the *impedances* which are *added*. *However, it is important to remember that since both admittances and impedances are complex quantities, all additions must be in complex form.* Simple arithmetic additions must not be attempted.

Considering the two parallel branches of Fig. 14.8, we have

* Similarly, in the special case when $R = 0$, $b = 1/X$ *i.e.*, susceptance becomes reciprocal of reactance, not otherwise.

$$\mathbf{Y_1} = \frac{1}{\mathbf{Z_1}} = \frac{1}{R_1 + jX_L} = \frac{(R_1 - jX_L)}{(R_1 + jX_L)(R_1 - jX_L)}$$

$$= \frac{R_1 - jX_L}{R_1^2 + X_L^2} = \frac{R_1}{R_1^2 + X_L^2} - j\frac{X_L}{R_1^2 + X_L^2} = g_1 - jb_1$$

where $g_1 = \dfrac{R_1}{R_1^2 + X_L^2}$ - conductance of upper branch,

$b_1 = -\dfrac{X_L}{R_1^2 + X_L^2}$ - susceptance of upper branch

Similarly, $\mathbf{Y_2} = \dfrac{1}{Z_2} = \dfrac{1}{R_2 - jX_C}$

$$= \frac{R_2 + jX_C}{(R_2 - jX_C)(R_2 + jX_C)} = \frac{R_2 + jX_C}{R_2^2 + X_C^2} = \frac{R_2}{R_2^2 + X_C^2} + j\frac{X_C}{R_2^2 + X_C^2} = g_2 + jb_2$$

Total admittance $\mathbf{Y} = \mathbf{Y_1} + \mathbf{Y_2} = (g_1 - jb_1) + (g_2 + jb_2) = (g_1 + g_2) - j(b_1 - b_2) = G - jB$

$$\mathbf{Y} = \sqrt{[(g_1 + g_2)^2 + (b_1 - b_2)^2]}\ ;\ \phi = \tan^{-1}\left(\frac{b_1 - b_2}{g_1 + g_3}\right)$$

The polar form for admittance is $\mathbf{Y} = \mathbf{Y} \angle \phi°$ where ϕ is as given above.

$$Y = \sqrt{G^2 + B^2} \angle \tan^{-1}(B/G)$$

Total current $\mathbf{I} = \mathbf{VY}\ ;\ \mathbf{I_1} = \mathbf{VY_1}$ and $\mathbf{I_2} = \mathbf{VY_2}$

If $\mathbf{V} = V \angle 0°$ and $\mathbf{Y} = Y \angle \phi$ then $\mathbf{I} = \mathbf{VY} = V \angle 0° \times Y \angle \phi = VY \angle \phi$

In general, if $\mathbf{V} = V \angle \alpha$ and $\mathbf{Y} = Y \angle \beta$, then $\mathbf{I} = \mathbf{VY} = V \angle \alpha \times Y \angle \beta = VY \angle \alpha + \beta$

Hence, it should be noted that when vector voltage is multiplied by admittance either in complex (rectangular) or polar form, the result is vector current in its proper phase relationship with respect to the voltage, *regardless of the axis to which the voltage may have been referred to.*

Example 14.1. *Two circuits, the impedance of which are given by $Z_1 = 10 + j\,15$ and $Z_2 = 6 - j8$ ohm are connected in parallel. If the total current supplied is 15 A, what is the power taken by each branch ? Find also the p.f. of individual circuits and of combination. Draw vector diagram.*

(Elect. Technology, Vikram Univ, Ujjain 1989)

Solution. Let $\mathbf{I} = 15 \angle 0°\ ;\ \mathbf{Z_1} = 10 + j15 = 18 \angle 57°$

$\mathbf{Z_2} = 6 - j8 = 10 \angle -53.1°$

Total impedance, $\mathbf{Z} = \dfrac{\mathbf{Z_1 Z_2}}{\mathbf{Z_1 + Z_2}} = \dfrac{(10 + j\,15)(6 - j\,8)}{16 + j\,7}$

$= 9.67 - j\,3.6 = 10.3 \angle -20.4°$

Applied voltage is given by

$\mathbf{V} = \mathbf{IZ} = 15 \angle 0° \times 10.3 \angle -20.4° = 154.4 \angle -20.4°$

$\mathbf{I_1} = \mathbf{V/Z_1} = 154.5 \angle -20.4°/18 \angle 57° = 8.58 \angle -77.4°$

$\mathbf{I_2} = \mathbf{V/Z_2} = 154.5 \angle -20.4°/10 \angle -53.1°$

$= 15.45 \angle 32.7°$

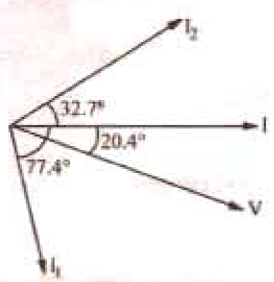


Fig. 14.9

We could also find branch currents as under :

$\mathbf{I_1} = \mathbf{I.\,Z_2/(Z_1 + Z_2)}$ and $\mathbf{I_2} = \mathbf{I.Z_1/(Z_1 + Z_2)}$

It is seen from phasor diagram of Fig. 14.9 that $\mathbf{I_1}$ lags behind $\mathbf{V}$ by $(77.4° - 20.4°) = 57°$ and $\mathbf{I_2}$ leads it by $(32.7° + 20.4°) = 53.1°$.

$\therefore$ $P_1 = I_1^2 R_1 = 8.58^2 \times 10 =$ **736 W** p.f. $= \cos 57° =$ **0.544 (lag)**

$P_2 = I_2^2 R_2 = 15.45^2 \times 6 =$ **1432 W** ; p.f. $= \cos 53.1° =$ **0.6**

Combined p.f. $= \cos 20.4° =$ **0.937 (lead)**

Example 14.2. *Two impedance $Z_1 = (8 + j6)$ and $Z_2 = (3 - j4)$ are in parallel. If the total current of the combination is 25 A, find the current taken and power consumed by each impedance.*

(F.Y. Engg. Pune Univ. May 1988)

Solution. $Z_1 = (8 + j6) = 10 \angle 36.87^\circ$; $Z_2 = (3 - j4) = 5\angle -53.1^\circ$

$$Z = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(10 \angle 36.87^\circ)(5 \angle -53.1^\circ)}{(8 + j6) + (3 - j4)} = \frac{50 \angle -16.23^\circ}{11 + j2} = \frac{50 \angle -16.23^\circ}{11.18 \angle 10.3^\circ} = 4.47 \angle -26.53^\circ$$

Let $I = 25 \angle 0^\circ$; $V = I Z = 25 \angle 0^\circ \times 4.47 \angle -26.53^\circ = 111.75 \angle -26.53^\circ$

$I_1 = V/Z_1 = 111.75 \angle -26.53^\circ / 10 \angle 36.87^\circ = 11.175 \angle -63.4^\circ$

$I_2 = 111.75 \angle -26.53/5 \angle -53.1^\circ = 22.35 \angle 26.57^\circ$

Now, the phase difference between V and I_1 is $63.4^\circ - 26.53^\circ = 36.87^\circ$ with current lagging. Hence, $\cos \phi_1 = \cos 36.87^\circ = 0.8$.

Power consumed in $\mathbf{Z_1} = VI_1 \cos \phi = 11.175 \times 111.75 \times 0.8 =$ **990 W**

Similarly, $\phi_2 = 26.57 - (-26.53) = 53.1^\circ$; $\cos 53.1^\circ = 0.6$

Power consumed in $\mathbf{Z_2} = VI_2 \cos \phi_2 = 111.75 \times 22.35 \times 0.6 =$ **1499 W**

Example 14.3. *Refer to the circuit of Fig. 14.10 (a) and determine the resistance and reactance of the lagging coil load and the power factor of the combination when the currents are as indicated.*

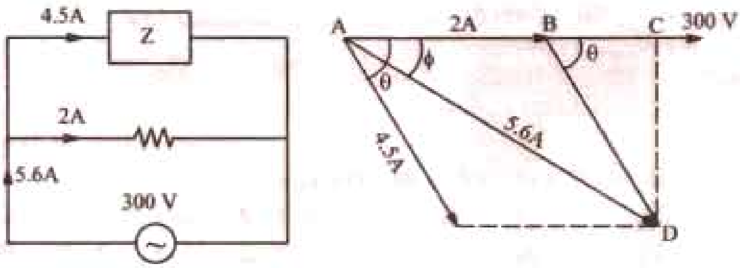
(Elect. Engg. A.M.Ae. S.I. Dec. 1989)

Solution. As seen from the $\Delta\, ABC$ of Fig. 14.10 (*b*).

$5.6^2 = 2^2 + 4.5^2 + 2 \times 2 \times 4.5 \times \cos \theta$, $\therefore \cos \theta = 0.395$, $\sin \theta = 0.919$. $Z = 300/4.5 = 66.67\ \Omega$

$R = Z \cos \theta = 66.67 \times 0.919 = 61.3\ \Omega$

p.f. $= \cos \phi = AC/AD = (2 + 4.5 \times 0.395)/5.6 =$ **0.67 (lag)**



(*a*) (*b*)

Fig. 14.10

Example 14.4. *A mercury vapour lamp unit consists of a 25μ f condenser in parallel with a series circuit containing the resistive lamp and a reactor of negligible resistance. The whole unit takes 400 W at 240 V, 50 Hz at unity p.f. What is the voltage across the lamp ?*

(F.Y. Engg. Pune Univ. Nov. 1987)

Solution. $$X_C = \frac{1}{2\pi fC} = \frac{1}{2\pi \times 50 \times (25 \times 10^{-6})} = 127.3\ \Omega, \ \therefore\ I_C = \frac{240}{127.3} = 1.885 \text{ A}$$

$$W = VI \cos \phi = VI \quad \therefore \quad I = W/V = 400/240 = 1.667 \text{ A}$$

In the vector diagram of Fig. 14.10 (*b*) I_C leads V by 90° and current I_1 in the series circuit lags V by ϕ_1 where ϕ_1 is the power factor angle of the series circuit. The vector sum of I_C and ϕ_1 gives the total current I. As seen $\tan \phi_1 = I_C/I = 1.885/1.667 = 1.13077$. Hence, $\phi_1 = 48.5^\circ$ lag. The applied voltage V is the vector sum of the drop across the resistive lamp which is in phase with I_1 and drop across the coil which leads I_1 by 90°.

Voltage across the lamp $= V \cos \Phi_1 = 340 \times \cos 48.5 = 240 \times 0.662 =$ **159 V.**

Example 14.5. *The currents in each branch of a two-branched parallel circuit are given by the expression $i_a = 7.07 \sin (314t - \pi/4)$ and $i_b = 21.2 \sin (314t + \pi/3)$*

The supply voltage is given by the expression v = 354 sin 314t. Derive a similar expression for the supply current and calculate the ohmic value of the component, assuming two pure components in each branch. State whether the reactive components are inductive or capacitive.

(Elect. Engineering., Calcutta Univ. 1991)

Solution. By inspection, we find that i_a lags the voltage by $\pi/4$ radian or 45° and i_b leads it by $\pi/3$ radian or 60°. Hence, branch *A* consists of a resistance in series with a pure inductive reactance. Branch *B* consists of a resistance in series with pure capacitive reactance as shown in Fig. 14.11 (*a*).

Maximum value of current in branch A is 7.07 A and in branch *B* is 21.2 A. The resultant current can be found vectorially. As seen from vector diagram.

X-comp = 21.2 cos 60° + 7.07 cos 45° = 15.6 A

Y-comp = 21.2 sin 60° − 7.07 sin 45° = 13.36 A

Maximum value of the resultant current is = $\sqrt{15.6^2 + 13.36^2}$ = 20.55 A

$\phi = \tan^{-1}(13.36/15.6) = \tan^{-1}(0.856) = 40.5°$ (lead)

Hence, the expression for the supply current is **i = 20.55 sin (314 t + 40.5°)**

$Z_A = 354/7.07 = 50\ \Omega$; $\cos\phi_A = \cos 45°$

$= 1/\sqrt{2}$. $\sin\phi_A = \sin 45° = 1/\sqrt{2}$

$R_A = Z_A \cos\phi_A = 50 \times 1/\sqrt{2} =$ **35.4 Ω**

$X_L = Z_A \sin\phi_A = 50 \times 1/\sqrt{2} =$ **35.4 Ω**

$Z_B = 354/20.2 =$ **17.5 Ω**

$R_B = 17.5 \times \cos 60° =$ **8.75 Ω**

$X_C = 17.5 \times \sin 60° =$ **15.16 Ω**

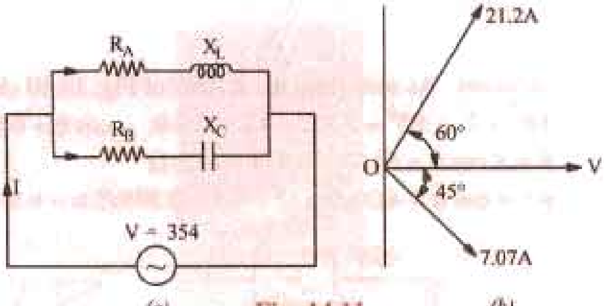


Fig. 14.11

Example 14.5 (*a*). *A total current of 10 A flows through the parallel combination of three impedance : (2 − j5) Ω, (6 + j3) Ω and (3 + j4) Ω. Calculate the current flowing through each branch. Find also the p.f. of the combination*

(Elect. Engg., -I Delhi Univ. 1987)

Solution. Let $\mathbf{Z_1} = (2 - j5)$, $\mathbf{Z_2} = (6 + j3)$, $\mathbf{Z_3} = (3 + j4)$

$\mathbf{Z_1Z_2} = (2 - j5)(6 + j3) = 27 - j24$. $\mathbf{Z_2 Z_3} = (6 + j3)(3 + j4) = 6 + j33$

$\mathbf{Z_3Z_1} = (3 + j4)(2 - j5) = 26 - j7$; $\mathbf{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1} = 59 + j2$

With reference to Art, 1.25

$$\mathbf{I_1} = \mathbf{I}.\frac{\mathbf{Z_2 Z_3}}{\mathbf{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}} = (10 + j0) \times \frac{6 + j33}{59 + j2} = 1.21 + j5.55$$

$$\mathbf{I_2} = \mathbf{I}.\frac{\mathbf{Z_3 Z_1}}{\mathbf{\Sigma Z_1 Z_2}} = (10 + j0) \times \frac{26 + j7}{59 + j2} = 4.36 - j1.33$$

$$\mathbf{I_3} = \mathbf{I}.\frac{\mathbf{Z_1 Z_2}}{\mathbf{\Sigma Z_1 Z_2}} = (10 + j0) \times \frac{27 - j24}{59 + j2} = 4.43 - j4.22$$

Now,

$$\mathbf{Z} = \frac{\mathbf{Z_1Z_2Z_3}}{\mathbf{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}} = \frac{(2 - j5)(6 + j33)}{59 + j2} = 3.01 + j0.51$$

$$\mathbf{V} = 10\angle 0° \times 3.05\angle 9.6° = 30.5\angle 9.6°$$

Combination p.f. = cos 9.6° = **0.986 (lag)**

Example 14.6. *Two impedances given by Z_B = (10 + j 5) and Z_2 = (8 + j 6) are joined in parallel and connected across a voltage of v = 200 + j0. Calculate the circuit current, its phase and the branch currents. Draw the vector diagram.* **(Electrotechnics-I, M.S. Univ. Baroda 1985)**

Solution. The circuit is shown in Fig. 14.12

Branch A, $\mathbf{Y_1} = \dfrac{\mathbf{1}}{\mathbf{Z_1}} = \dfrac{1}{(10 + j5)}$

$$= \frac{10 - j5}{(10 + j5)(10 - j5)} = \frac{10 - j5}{100 + 25}$$

$$= 0.08 - j0.04 \text{ Siemens}$$

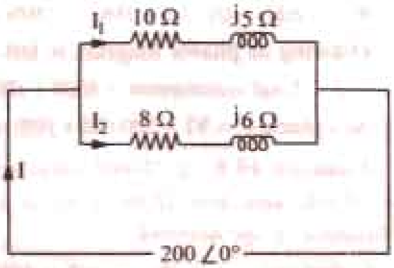


Fig. 14.12

Branch B, $\mathbf{Y_2} = \dfrac{1}{\mathbf{Z_2}} = \dfrac{1}{(8 + j6)}$

$$= \frac{8 - j6}{(8 + j6)(8 - j6)} = \frac{8 - j6}{64 + 36} = 0.08 - j0.06 \text{ Siemens}$$

$$\mathbf{Y} = (0.08 - j\,0.04) + (0.08 - j\,0.06) = 0.16 - j\,0.1 \text{ Siemens}$$

Direct Method

We could have found total impedance straightway like this : $\dfrac{1}{\mathbf{Z}} = \dfrac{1}{\mathbf{Z_1}} + \dfrac{1}{\mathbf{Z_2}} = \dfrac{\mathbf{Z_1 + Z_2}}{\mathbf{Z_1 Z_2}}$

$$\therefore \qquad \mathbf{Y} = \frac{\mathbf{Z_1 + Z_2}}{\mathbf{Z_1 Z_2}} = \frac{(10 + j5) + (8 + j6)}{(10 + j5)(8 + j6)} = \frac{18 + j11}{50 + j100}$$

Rationalizing the above, we get

$$\mathbf{Y} = \frac{(18 + j\,11)(50 - j\,100)}{(50 + j\,100)(50 - j\,100)} = \frac{200 - j\,1250}{12{,}500} = 0.16 - j\,0.1 \text{ (same as before)}$$

Now $\mathbf{Y} = 200\angle 0^\circ = 200 + j0$

$\therefore \qquad \mathbf{I} = \mathbf{VY} = (200 + j0)(0.16 - j0.1)$

$= 32 - j20 = 37.74\angle -32^\circ$....polar form

Power factor = cos 32° = 0.848

$\mathbf{I_1} = \mathbf{VY_1} = (200 + j0)(0.08 - j0.04)$

$= 16 - j8 = 17.88\angle -26^\circ 32'$

It lags behind the applied voltage by 26°32′.

$\mathbf{I_2} = \mathbf{VY_2} = (200 + j0)(0.08 - j0.06)$

$= 16 - j12 = 20\angle -36^\circ 46'$

It lags behind the applied voltage by 36°46′. The vector diagram is shown in Fig. 14.13.

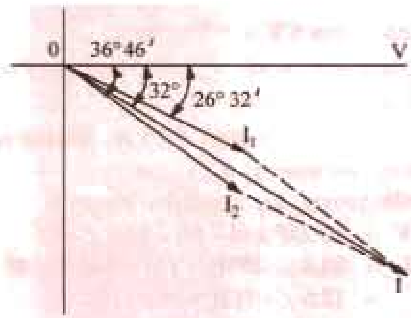


Fig. 14.13

Example 14.7. *Explain the term admittance. Two impedance $Z_1 = (6 - j\,8)$ ohm and $Z_2 = (16 + j12)$ ohm are connected in parallel. If the total current of the combination is $(20 + j10)$ amperes, find the complexor for power taken by each impedance. Draw and explain the complete phasor diagram.* **(Basic Electricity, Bombay Univ 1986)**

Solution. Let us first find out the applied voltage, $\mathbf{Y} = \mathbf{Y_1} + \mathbf{Y_2} = \dfrac{1}{6 - j8} + \dfrac{1}{16 + j12}$

$$= (0.06 + j0.08) + (0.04 - j0.03) = 0.1 + j0.05 = 0.1118\angle 26^\circ 34'$$

$$\mathbf{I} = 20 + j10 = 22.36\angle 26^\circ 34'$$

Now $\qquad \mathbf{I} = \mathbf{VY} \quad \therefore \quad \mathbf{V} = \dfrac{I}{Y} = \dfrac{22.36\angle 26^\circ 34'}{0.1118\angle 26^\circ 34'} = 200\angle 0^\circ$

$\mathbf{I_1} = \mathbf{VY_1} = (200 + j0)(0.06 + j0.008) = 12j\,16$ A, $\mathbf{I_2} = 200\,(0.04 - j0.03) = 8 - j6$ A

Using the method of conjugates and taking voltage conjugate, the complexor power taken by each branch can be foudn as under :

$\mathbf{P}_1 = (200 - j0)\ (12 + j16) = 2400 + j3200$; $P_2 = (200 - j0)\ (8 - j6) = 100 - j1200$

Drawing of phasor diagram is left to the reader.

Note. Total voltamperes = 4000 + j2000

As a check, $\mathbf{P} = \mathbf{VI} = 200\ (20 + j10) = 4000 + j2000$

Example 14.8. *A 15-mH inductor is in series with a parallel combination of an 80 Ω resistor and 20 μR capacitor. If the angular frequency of the applied voltage is ω = 1000 rad/s, find the admittance of the network.* **(Basic Circuit Analysis Osmania Univ. Jan/Feb 1992)**

Solution. $X_L = \omega L = 100 \times 15 \times 10^{-3} = 15\Omega$; $X_C = 1/\omega C = 10^6/1000 \times 20 = 50\ \Omega$

Impedance of the parallel combination is given by

$$Z_p = 80 \| -J\ 50 = -j4000/(80 - j50) = 22.5 - j36.$$

$$\text{Total impedance} = j15 + 22.5 - j36 = 22.5 - j21$$

$$\text{Admittance} \quad Y = \frac{1}{Z} = \frac{1}{22.5 - j21} + 0.0238 - j0.022 \text{ siemens}$$

Example 14.9. *An impedance (6 + j 8) is connected across 200-V, 50-Hz mains in parallel with another circuit having an impedance of (8 − j6) Ω. Calculate (a) the admittance, the condutance, the susceptance of the combined circuit (b) the total current taken from the mains and its p.f.* **(Elect. Engg-AMAE, S.I. 1992)**

Solution. $\mathbf{Y}_1 = \dfrac{1}{6 + j8} = \dfrac{6 - j8}{6^2 + 8^2} = 0.06 - j0.08$ Siemens, $\mathbf{Y} = \dfrac{1}{8 - j6} = \dfrac{8 + j6}{100} = 0.08 + j0.06$ Siemens

(*a*) Combined admittance is $\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 = 0.14 - j0.02 =$ **0.1414 ∠− 8.8°′ Siemens**

Conductance, G = **0.14 Siemens** ; Susceptance, B = **−0.02 Siemens** (inductive)

(*b*) Let $\mathbf{V} = 200\ \angle 0°$; $\mathbf{I} = \mathbf{VY} = 200 \times 0.1414\ \angle -8°8\ V = 28.3\ \angle -8°8'$

p.f. = cos 8°8′ = **0.99 (lag)**

Example 14.10. *If the voltmeter in Fig. 14.14 reads 60 V, find the reading of the ammeter.*

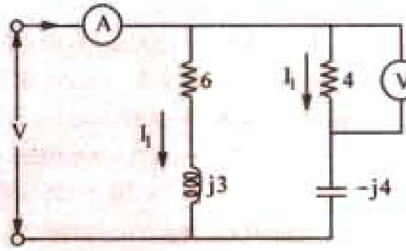


Fig. 14.14

Solution. $I_2 = 60/A = 15$ A. Taking it as reference quantity, we have $\mathbf{I}_2 = 15 \angle 0$.

Obviously, the applied voltages is

$$\mathbf{V} = 15\ \angle 0° \times (4 - j4) = 84.8\ \angle -45°$$

$$\mathbf{I}_1 = 84.8\ \angle -45°/(6 + j\,3) = 84.8\ \angle -45 + 6.7\ \angle 26.6$$

$$= 12.6\ \angle -71.6° = (4 - j\,12)$$

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = (15 + j\,0) + (4 - j\,12)$$

$$= 19 - j\,12 = 22.47\ \angle -32.3°$$

Hence, ammeter **reads 22.47.**

Example 14.11. *Find the reading of the ammeter when the voltmeter across the 3 ohm resistor in the circuit of Fig. 14.15 reads 45 V* **(Elect. Engg. & Electronics Bangalore Univ. 1988)**

Solution. Obviously $I_1 = 45/3 = 15\ A$. If we take it as reference quantity, $I_1 = 3\ \angle 0°$

Now, $Z_1 = 3 - j3 = 4.24\ \angle -45°$.

Hence, $V = I_1\ Z_1 = 15\ \angle 0° \times 4.24\ \angle -45° = 63.6\ \angle -45°$

$$I_2 = \frac{V}{Z_2} = \frac{63.6 \angle -45°}{5 + j2} = \frac{63.6 \angle -45°}{5.4 \angle 21.8°}$$

$$= 11.77 \angle -66.8° = 4.64 - j\,10.8$$

$$I = I_1 + I_2 = 19.64 - j10.8 = \mathbf{22.4\ \angle 28.8°}$$

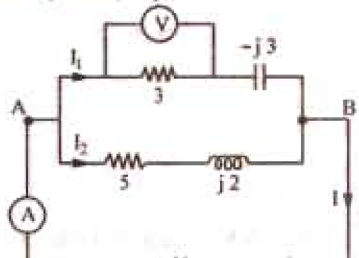


Fig. 14.15

Example 14.12. *A coil having a resistance of 5 Ω and an inductance of 0.02 H is arranged in parallel with another coil having a resistance of 1 Ω and an inductance of 0.08 H. Calculate* *the current through the combination and the power absorbed when a voltage of 100 V at 50 Hz is applied. Estimate the resistance of a single coil which will take the same current at the same power factor.*

Solution. The circuit and its phasor diagram are shown in Fig. 14.16.

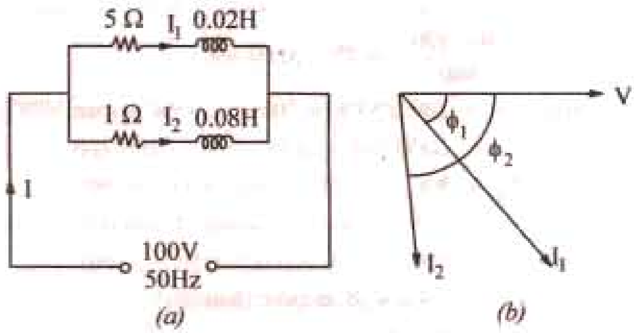


Fig. 14.16.

Branch No. 1

$X_1 = 314 \times 0.02 = 6.28\ \Omega$

$Z_1 = \sqrt{5^2 + 6.28^2} = 8\ \Omega$

$I_1 = 100/8 = 12.5$ A

$\cos\phi_1 = R_1/Z_1 = 5/8$

$\sin\phi_1 = 6.28/8$

Branch No. 2

$X_2 = 314 \times 0.08 = 25.12\ \Omega,\ Z_2 = \sqrt{1^2 + 25.12^2} = 25.14\ \Omega,\ I_2 = 100/25.14 = 4$ A

$\cos\phi_2 = 1/25.14$ and $\sin\phi_2 = 25.12/25.14$

X - components of I_1 and $I_2 = I_1\cos\phi_1 + I_2\cos\phi_2 = (12.5 \times 5/8) + (4 \times 1/25.14) = 7.97$ A

Y - components of I_1 and $I_2 = I_1\sin\phi_1 + I_2\sin\phi_2 = (12.5 \times 6.28/8) + (4 \times 25.12/25.14) = 13.8$ A

$I = \sqrt{7.97^2 + 13.8^2} =$ **15.94 A**

$\cos\phi = 7.97/15.94 = 0.5$ (lag)

$\phi = \cos^{-1}(0.5) = 60^\circ$

Power absorbed

$= 100 \times 15.94 \times 0.5 =$ **797 W**

The equivalent series circuit is shown in Fig. 14.17 (*a*).

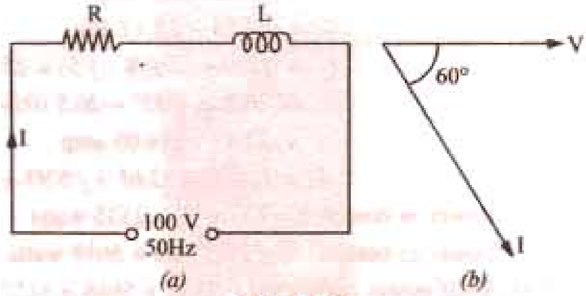


Fig. 14.17

$V = 100$ V ; $I = 15.94$ A ; $\phi = 60^\circ$

$Z = 100/15.94 = 6.27\ \Omega$;

$R = Z\cos\phi = 6.27 \times \cos 60^\circ =$ **3.14 Ω**

$X = Z\sin\phi = 6.27 \times \sin 60^\circ =$ **5.43 Ω**

Admittance Method For Finding Equivalent Circuit

$$\mathbf{Y}_1 = \frac{1}{5 + j6.28} = \frac{5 - j6.28}{5^2 + 6.28^2} = 0.078 - j\,0.098 \text{ S},$$

$$\mathbf{Y}_2 = \frac{1}{1 + j25.12} = \frac{1 - j25.12}{1^2 + j25.12^2} = 0.00158 - j\,0.0397 \text{ S},$$

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 = 0.0796 - j0.138 = 0.159\ \angle{-60^\circ}$$

Here $\quad G = 0.0796$ S, $B = -0.138$ S, $Y = 0.159\ \Omega$

$\therefore \quad R_{eq} = G/Y^2 = 0.0796/0.159^2 =$ **3.14 Ω**, $X_{eq} = B/Y^2 = 0.138/0.159^2 =$ **5.56 Ω**

Example. 14.13. *A voltage of 200 ∠53°8′ is applied across two impedances in parallel. The values of impedances are (12 + j 16) and (10 − j 20). Determine the kVA, kVAR and kW in each branch and the power factor of the whole circuit.* **(Elect. Technology, Indore Univ, 1986)**

Solution. The circuit is shown in Fig. 14.18.

$\mathbf{Y_A} = 1/(12 + j16) = (12 - j16)/[(12 + j16)(12 - j16)]$

$$= (12 - j16)/400 = 0.03 - j0.04 \text{ mho}$$

$$\mathbf{Y_B} = 1/(10 - j20) = 10 + j20/[(10 - j20)(10 + j20)]$$

$$= \frac{10 + j20}{500} + 0.02 + j0.04 \text{ mho}$$

Now $V = 200 \angle 53°8' = 200 (\cos 53°8' + j \sin 53°8')$

$$= 2000 (0.6 + j0.8) = 120 + j160 \text{ volt}$$

$$\mathbf{I_A} = \mathbf{VY_A} = (120 + j160)(0.03 - j0.04)$$

$$= (10 + j0) \text{ ampere (along the reference axis)}$$

$$\therefore \quad \mathbf{I_B} = \mathbf{VY_B} = (120 + j160)(0.02 + j0.04)$$

$$= -4.0 + j8 \text{ ampere (leading)}$$

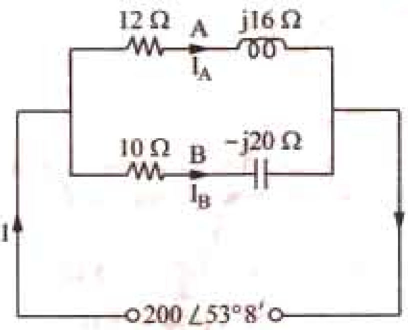


Fig. 14.18

Example 14.14. *Two circuits, the impedances of which are given by $Z_1 = 15 + j12$ ohms and $Z_2 = 8 - j5$ ohms are connected in parallel. If the potential difference across one of the impedance is $250 + j0$ V, calculate*

(i) total current and branch currents

(ii) total power and power consumed in each branch

and (iii) overall power-factor and power-factor of each branch.

(Nagpur University, November 1998)

Solution. (*i*) $I_1 = (250 + j\,0)/(15 + j\,12) = 250 \angle 0°/19.21 \angle 38.6°$

$$= 13 \angle -38.6° \text{ amp} = 13 (0.78 - j\,0.6247)$$

$$= 10.14 - j\,8.12 \text{ amp}$$

$$I_2 = (250 + j\,0)/(8 - j\,5) = 250 \angle 0°/9.434 \angle -32°$$

$$= 26.5 \angle +32° = 26.5 (0.848 + j\,0.530)$$

$$= 22.47 + j14.05 \text{ amp}$$

$$I = I_1 + I_2 = 32.61 + j\,5.93 = 33.15 \angle +10.36°$$

(*ii*) Power in branch 1 = $13^2 \times 15 = 2535$ watts

Power in branch 2 = $26.5^2 \times 8 = 5618$ watts

Total power consumed = 2535 + 5618 = 8153 watts

(*iii*) Power factor of branch 1 = cos 38.60° = 0.78 lag

Power factor of branch 2 = cos 32° = 0.848 lead.

Oveall power factor = cos 10.36° = 0.984 lead.

Additional hint : Drawn phasor-diagram for these currents, in fig. 14.19, for the expressions written above.

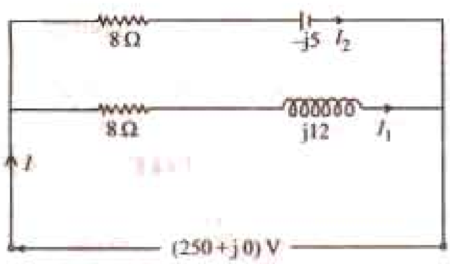


Fig. 14.19

Example 14.15. *An inductive circuit, in parallel with a resistive circuit of 20 ohms is connected across a 50 Hz supply. The inductive current is 4.3 A and the resistive current is 2.7 A. The total current is 5.8 A Find : (a) Power factor of the combined circuit. Also draw the phasor diagram.*

(Nagpur University, November 1997)

Solution. I_2 (= 2.7 A) is in phase with V which is 54 V in magnitude. The triangle for currents is drawn in the phasor diagram in fig. 14.20 (*b*)

Solving the triangle, $\phi_1 = 180° - \cos^{-1} [(2.7^2 = 4.3^2 - 5.8^2)/(2 \times 4.3 \times 2.7)] = 70.2°$

Further, $5.8 \sin \phi = 4.3 \text{ Sin } \phi_1$, giving $\phi = 44.2°$.

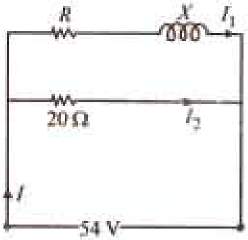


Fig. 14.20 (*a*)

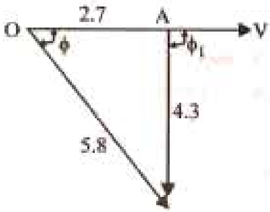


Fig. 14.20 (*b*)

OA = I_2 = 2.7 in phase with V
AB = I_1 = 4.3 lagging behind V by ϕ_1
OB = I = 5.8 lagging behind V by φ

$$|Z_1| = 54/4.3 = 12.56 \text{ ohms}$$
$$R = Z_1 \cos \phi_1 = 4.25 \text{ ohms}$$
$$X = Z_1 \sin \phi_1 = 11.82 \text{ ohms, since } \phi_1 \text{ is the lagging angle}$$

(*a*) Power absorbed by the Inductive branch

$$= 4.3^2 \times 4.25 = 78.6 \text{ watts}$$

(*b*) $L = 11.82/314 = 37.64$ mH

(*c*) P.f. of the combined circuit = cos φ = 0.717 lag

Check : Power consumed by 20 ohms resistor = $2.7^2 \times 20 = 145.8$ W

Total Power consumed in two branches = 78.6 + 145.8 = 224.4 W

This figure must be obtained by input power = VI cos φ

= 54 × 5.8 × cos 44.2° = 224.5 W. Hence checked.

Example 14.16. *In a particular A.C. circuit, three impedances are connected in parallel, currents a shown in fig. 14.21 are flowing through its parallel branches.*

(i) Write the equations for the currents in terms of sinusoidal variations and draw the wave forms.

Find the total current supplied by the source. **[Nagpur University, April 1998]**

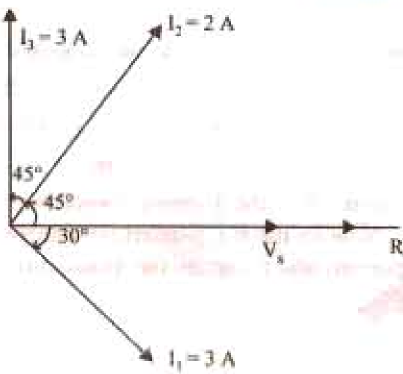


Fig. 14.21

Solution. In Fig. 14.21, V is taken as reference, and is very convenient for phasor diagrams for parallel circuits.

(*i*) I_1 lags behind by 30°. Branch no. 1must, therefore, have an R-L series combination. With 10-volt source, a current of 3 A in branch 1 means that its impedance Z_1 is given by

$Z_1 = 10/3 = 3.333$ ohms

The phase-angle for I_1 is 30° lagging

$R_1 = 3.333 \cos 30° = 2.887$ ohms

$X_{L1} = 3.333 \sin 30° = 1.6665$ ohms

(*ii*) I_2 is 2 amp and it leads the voltage by 45°. Branch 2 must, therefore, have R-C series combination.

$$Z_2 = 10/2 = 5 \text{ ohms}$$
$$R_2 = 5 \cos 45° = 3.5355 \text{ ohms}$$
$$X_{c2} = 5 \sin 45° = 3.5355 \text{ ohms}$$

(*iii*) Third branch draws a current of 3 amp which leads the voltage by 90°. Hence, it can only have a capacitive reactance.

$$|Z_3| = X_{c3} = 10/3 = 3.333 \text{ ohms}$$

Total current supplied by the source = I amp

$$I = I_1 + I_2 + I_3$$
$$= 3 [\cos 30° - j \sin 30°] + 2 [\cos 45° + j \sin 45°] + 3 [0 + j\, 1]$$
$$= 4.0123 + j\, 2.9142$$
$$|I| = 4.96 \text{ amp, leading } V_s \text{, by } 36°.$$

Expressions for currents : Frequency is assumed to be 50 Hz

$$v_s = 10\sqrt{2} \sin (314\, t)$$
$$i_1 = 3\sqrt{2} \sin (314\, t - 30°)$$
$$i_2 = 2\sqrt{2} \sin (314\, t + 45°)$$
$$i_3 = 3\sqrt{2} \sin (314\, t + 90°)$$

Total current, $i\,(t) = 4.96\sqrt{2} \sin (314\, t + 36°)$

$$\text{Total power consumed} = \text{Voltage} \times \text{active (or in phase-) component of current}$$
$$= 10 \times 4.012 = 40.12 \text{ watts}$$

Example 14.17. *A resistor of 12 ohms and an inductance of 0.025 H are connected in series across a 50 Hz supply. What values of resistance and inductance when connected in parallel will have the same resultant impedance and p.f. Find the current in each case when the supply voltage is 230 V.* **(Nagpur University, Nov. 1996)**

Solution. At 50 Hz, the series R-L circuit has an impedance of Z_s given by

$$Z_s = 12 + j\,(314 \times 0.025) = 12 + j\, 7.85 = 14.34 + \angle 33.2°$$
$$I_s = (230 + j0) / (12 + j\, 7.85) = 16.04 - \angle 33.2°$$
$$= 13.42 - j\, 8.8 \text{ amp}$$

Out of these two components of I_s, the in-phase components is 13.42 amp and quadrature component (lagging) is 8.8 amp. Now let the *R-L* parallel combination be considered. In Fig. 14.22 (*b*), *R* carries the in-phase component, and *L* carries the quadrature-component (lagging). For the two systems to be equivalent,

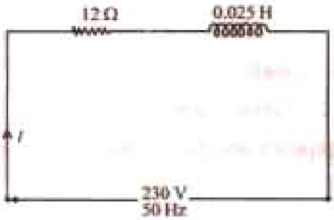


Fig. 14.22 (*a*)

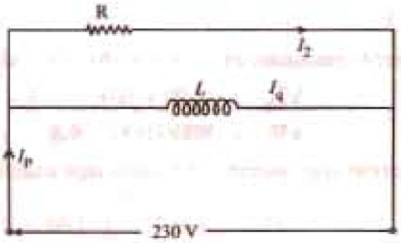


Fig. 14.22 (*b*)

$$I_s = I_p$$

It means $\quad I_s = 13.42$ amp

$$I_q = 8.8 \text{ amp}$$

Thus, $\quad R = 230 / 13.42 = 17.14$ ohms

$$X_L = 230 / 8.8 = 26.14 \text{ ohms}$$

$$L = 26.14/314 = 83.2 \text{ mH}$$

Example 14.18. *An inductive coil of resistance 15 ohms and inductive reactance 42 ohms is connected in parallel with a capacitor of capacitive reactance 47.6 ohms. The combination is energized from a 200 V, 33.5 Hz a.c. supply. Find the total current drawn by the circuit and its power factor. Draw to the scale the phasor diagram of the circuit.* **(Bombay University, 2000)**

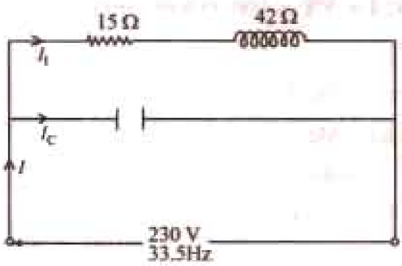


Fig. 14.23 (*a*)

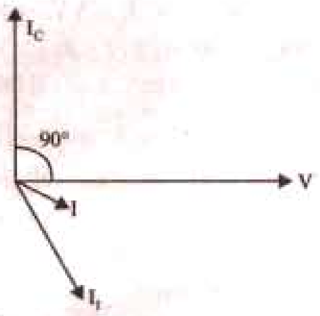


Fig. 14.23 (*b*)

Solution. $\quad Z_1 = 15 + j42,\ Z_1 = 44.6$ ohms, $\cos \phi_1 = 15.44.6 = 0.3363$

$$\phi_1 = 70.40 \text{ Lagging},\ I_1 = 200/44.6 = 4.484 \text{ amp}$$

$$I_c = 200/47.6 = 4.2 \text{ amp}$$

$$I = 4.484\,(0.3355 - j0.942) + j4.2 = 1.50 - j0.025 = 1.5002 - \angle 1°$$

For the circuit in Fig. 14.23 (*a*), the phasor diagram is drawn in Fig. 14.23 (*b*).

Power Calculation

Power etc. can be calculated by the method of conjugates as explained in Ex. 14.3

Branch A

The current conjugate of $(10 + j\,0)$ is $(10 - j\,0)$

$\therefore \quad$ VIA $= (120 + j\,160)\,(10 - j\,0) = 1200 + j\,1600 \quad \therefore \quad$ kW $= 1200/1000 = 1.2$

$\therefore \quad$ kVAR $= 1600/1000 = 1.6$. The fact that it is positive merely shows the reactive volt-amperes are due to a lagging current* kVA $= \sqrt{(1.2^2 + 1.6^2)} = 2$

* If voltage conjugate is used, then capacitive VARs are positive and inductive VARs negative.

Branch B

The-current conjugate of $(-4.0 + j8)$ is $(-4.0 - j8)$

$\therefore \quad \mathbf{VI_B} = (120 + j160)(-4 - j8) = 800 - j1600$

$\therefore \quad kW = 800/1000 = \mathbf{0.8} \quad \therefore \quad kVAR = -1600/1000 = -1.6$

The negative sign merely indicates that reactive volt-amperes are due to the leading current

$$\therefore \quad kVA = \sqrt{[0.8^2 + (-1.6)^2]} = 1.788$$

$$\mathbf{Y} = \mathbf{Y_A} + \mathbf{Y_B} = (0.03 - j0.04) + (0.02 + j0.04) = 0.05 + j0$$

$$\mathbf{I} = \mathbf{VY} = (120 + j160)(0.05 + j0) = 6 + j8 = 10 \angle 53°8'$$

or $\quad \mathbf{I} = \mathbf{I_A} + \mathbf{I_B} = (10 + j0) + (-4 + j8) = 6 + j8$ (same as above)

Circuit p.f. $= \cos 0° = 1$ ($\because$ current is in phase with voltage)

Example 14.19. *An impedance $Z_1 = (8 - j5)\,\Omega$ is in parallel with an impedance $Z_2 = (3 + j7)\,\Omega$. If 100 V are impressed on the parallel combination, find the branch currents I_1, I_2 and the resultant current. Draw the corresponding phasor diagram showing each current and the voltage drop across each parameter. Calculate also the equivalent resistance, reactance and impedance of the whole circuit.* **(Elect. Techology-I, Gwalior Univ. 1998)**

Solution. Admittance Method

$\mathbf{Y_1} = 1/(8 - j5) = (0.0899 + j0.0562)\,S$

$\mathbf{Y_2} = 1/(3 + j7) = (0.0517 - j0.121)\,S$, $\mathbf{Y} = \mathbf{Y_1} + \mathbf{Y_2} = (0.1416 - j0.065)S$

Let $\mathbf{V} = (100 + j0)$; $\mathbf{I_1} = \mathbf{VY_1} = 100\,(0.0899 + j0.0562) = 8.99 + j5.62$

$\mathbf{I_2} = \mathbf{VY_2} = 100\,(0.0517 - j0.121) = 5.17 - j12.1$; $\mathbf{I} = \mathbf{VY} = 100\,(0.416 - j0.056) = 14.16 - j6.5$

Now, $\quad G = 0.1416S$, $B = -0.065\,S$ (inductive;)

$$Y = \sqrt{G^2 + B^2} = \sqrt{0.1416^2 + 0.065^2} = 0.1558\ \text{S}$$

Equivalent series resistance, $R_{eq} = G/Y^2 = 0.1416/0.1558^2 = 5.38\,\Omega$

Equivalent series inductive reactance $X_{eq} = B/Y^2 = 0.065/0.1558^2 = 2.68\,\Omega$

Equivalent series impedance $\mathbf{Z} = 1/\mathbf{Y} = 1/0.1558 = 6.42\,\Omega$

Impedance Method

$$\mathbf{I_1} = \mathbf{V/Z_1} = (100 + j0)/(8 - j5) = 8.99 + j5.62$$

$$\mathbf{I_2} = \mathbf{V/Z_2} = 100/(3 + j7) = 5.17 - j12.1$$

$$\mathbf{Z} = \frac{\mathbf{Z_1 Z_2}}{\mathbf{Z_1 + Z_2}} = \frac{(8 - j5)(3 + j7)}{(11 + j2)} = \frac{59 + j41}{(11 + j2)} = 5.848 + j2.664 = 6.426 = \angle 24.5°,$$

$$\mathbf{I} = 100/6.426 \angle 24.5° = 15.56 \angle -24.5° = 14.16 - j\,6.54$$

As seen from the expression for Z, equivalent series resistance is 5.848 Ω and inductive reactance is 2.664 ohm.

Example 14.20. *The impedances $Z_1 = 6 + j\,8$, $Z_2 = 8 - j\,6$ and $Z3 = 10 + j\,0$ ohms measured at 50 Hz, form three branches of a parallel circuit. This circuit is fed from a 100 volt, 50-Hz supply. A purely reactive (inductive or capacitive) circuit is added as the fourth parallel branch to the above three-branched parallel circuit so as to draw minimum current from the source. Determine the value of L or C to be used in the fourth branch and also find the minimum current.*

(Electrical Circuits, South Gujarat Univ. 1986)

Solution. Total admittance of the 3-branched parallel circuit is

$$\mathbf{Y} = \frac{1}{6 + j8} + \frac{1}{8 - j6} + \frac{1}{10 + j0} = 0.06 - j0.08 + 0.08 + j0.06 + 0.1 = 0.24 - j0.02$$

Current taken would be minimum when net susceptance is zero. Since combined susceptance is inductive, it means that we must add capacitive susceptance to neutralize it. Hence, we must connect a pure capacitor in parallel with the above circuit such that its susceptance equals $+ j0.02$ S

$\therefore \quad 1/X_C = 0.02$ or $2\pi fC = 0.02$; $C = 0.2/314 =$ **63.7 μF**

Admittance of four parallel branches $= (0.24 - j0.02) + j0.02 = 0.24$ S

$\therefore$ Minimum current drawn by the circuit $= 100 \times 0.24 =$ **24 A**

Example 14.21. *The total effective current drawn by parallel circuit of Fig. 14.24 (a) is 20 A. Calculate (i) VA (ii) VAR and (iii) watts drawn by the circuit.*

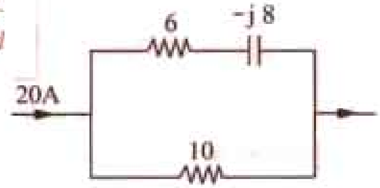


Fig. 14.24. (a)

Solution. The combined impedance of the circuit is

$$\mathbf{Z} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} = \frac{10\,(6 - j8)}{(16 - j\,8)} = (5 - j2.5) \text{ ohm}$$

(*iii*) Power $= I^2R = 20^2 \times 5 =$ **2000 W** (*ii*) $Q = I^2X = 20^2 \times 2.5$ = **1000 VAR (leading)** (*i*) $S = P + j\,Q = 2000 + j1000 = 2236 \angle 27°$; $S =$ **2236 VA**

Example 14.22. *Calculate (i) total current and (ii) equivalent impedance for the four-branched circuit of Fig. 14.24 (b).*

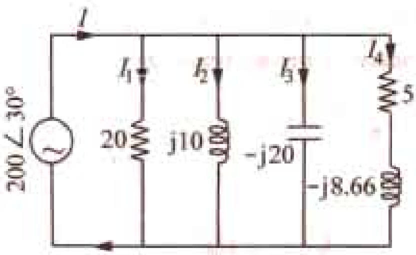


Fig. 14.24 (b)

Solution. $\mathbf{Y}_1 = 1/20 = 0.05$ S, $\mathbf{Y}_2 = 1/j10 = -j0.1$ S;

$\mathbf{Y}_3 = 1/-j20 = j0.05$ S ; $\mathbf{Y}_4 = 1/5 - j8.66 = 1/10 \angle 60°$

$= 0.1 \angle -60° = (0.05 - j0.0866)$ S

$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \mathbf{Y}_4 = (0.1 - j\,0.1366)$ S

$= 0.169 \angle -53.8°$ S

(*i*) $\mathbf{I} = \mathbf{VY} = 200 \angle 30° \times 0.169 \angle -53.8° = 33.8 \angle -23.8°$ A

(*ii*) $\mathbf{Z} = 1/\mathbf{Y} = 1/0.169 \angle -53.8° = 5.9 \angle 53.8°\ \Omega$

Example 14.23. *The power consumed by both branches of the circuit shown in Fig. 14.23 is 2200 W. Calculate power of each branch and the reading of the ammeter.*

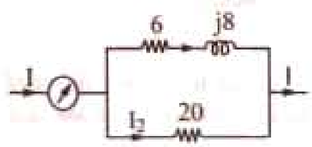


Fig. 14.25

Solution. $\mathbf{I}_1 = \mathbf{V}/\mathbf{Z}_1$

$= \mathbf{V}/(6 + j\,8) = \mathbf{V}/10 \angle 53.1°$ $\mathbf{I}_2 = \mathbf{V}/\mathbf{Z}_2 = \mathbf{V}/20$

$\therefore \quad I_1/I_2 = 20/10 = 2$, $P_1 = I_1^2 R_1$ and $P_2 = I_2^2 R_2$

$$\therefore \quad \frac{P_1}{P_2} = \frac{I_1^2 R_1}{I_2^2 R_2} = 2^2 \times \left(\frac{6}{20}\right) = \frac{6}{5}$$

Now, $\quad P = P_1 + P_2$ or $\dfrac{P}{P_2} = \dfrac{P_1}{P_2} + 1 = \dfrac{6}{5} + 1 = \dfrac{11}{5}$

or $\quad P_2 = 2200 \times \dfrac{5}{11} = 1000\,W \quad \therefore \quad P_1 = 2200 - 1000 = 1200$ W

Since $\quad P_1 = I_1^2 R_1$ or $1200 = I_1^2 \times 6$; $I_1 = 14.14$ A

If $\quad \mathbf{V} = V \angle 0°$, then $\mathbf{I}_1 = 14.14 \angle -53.1° = 8.48 - j11.31$

Similarly, $\quad P_2 = I_2^2 R_2$ or $1000 = I_2^2 \times 20$; $\mathbf{I}_2 = 7.07$ A or $\mathbf{I}_2 =$ **7.07 ∠0°**

Total current $\quad \mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 = (8.48 - j11.31) + 7.07 = 15.55 - j11.31 =$ **19.3 ∠–36°**

Hence, ammeter reads **19.3 A**

Example 14.24. *Consider an electric circuit shown in Figure 14.25 (a)*

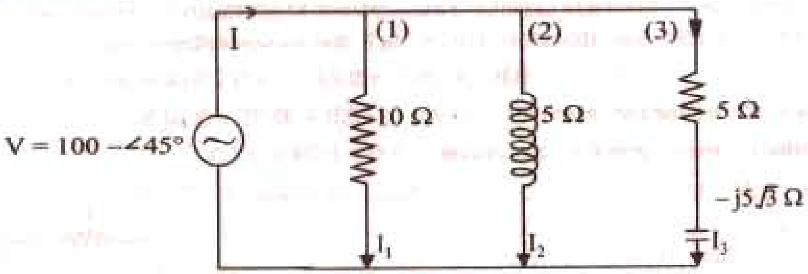


Fig. 14.25. (a)

Determine : *(i) the current and power consumed in each branch.*

(ii) the supply current and power factor. **(U.P. Technical University, 2001)**

Solution. Indicating branch numbers 1, 2, 3 as marked on the figure, and representing the source voltage by 100 ∠45°,

$$Z_1 = 10 + j\,0 = 10\angle 0^\circ,\ I_1 = 100\angle 45^\circ/10\angle 0^\circ = 10\angle 45^\circ \text{ amp}$$

$$Z_2 = 5 + j5\sqrt{3} = 10\angle 60^\circ,\ I_2 = 100\angle 45^\circ/10\angle 60^\circ = 10\ \angle -15^\circ \text{ amp}$$

$$Z_3 = 5 - j5\sqrt{3} = 10\ \angle -60^\circ,\ I_3 = 100\angle 45^\circ/10\angle -60^\circ = 10\angle 105^\circ \text{ amp}$$

Phasor addition of these three currents gives the supply current, I which comes out to be $I = 20\ \angle\ 45^\circ$ amp.

This is in phase with the supply voltage.

(*i*) Power consumed by the branches :

Branch 1 : $10^2 \times 10 = 1000$ watts

Branch 2 : $10^2 \times 5 = 500$ watts

Branch 3 : $10^2 \times 5 = 500$ watts

Total power consumed = 2000 watts

(*ii*) Power factor = 1.0 since V and I are in phase

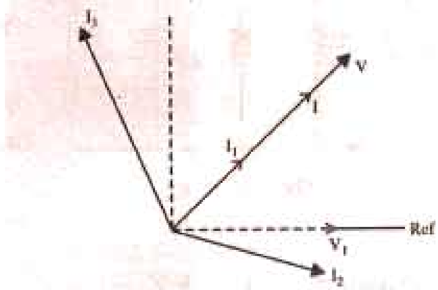


Fig. 14.25 (*b*)

14.6. Series-parallel Circuits

(*i*) *By Admittance Method*

In such circuits, the parallel circuit is first reduced to an equivalent series circuit and then, as usual, combined with the rest of the circuit. For a parallel circuit,

Equivalent series resistance $R_{eq} = Z \cos = \dfrac{1}{Y} \cdot \dfrac{G}{Y} = \dfrac{G}{Y^2}$

- See Ex. 14.14

Equivalent series reactance $X_{eq} = Z \sin\phi = \dfrac{1}{Y} \cdot \dfrac{B}{Y} = \dfrac{B}{Y^2}$

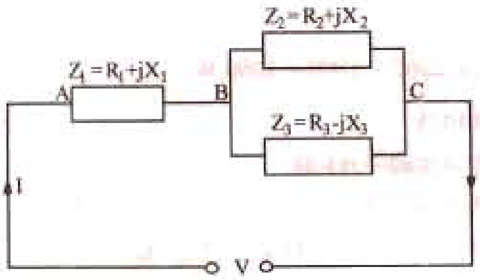


Fig. 14.26

(*ii*) *By Symbolic Method*

Consider the circuit of Fig. 14.26. First, equivalent impedance of parallel branches is calculated and it is then added to the series impedance to get the total circuit impedance. The circuit current can be easily found.

$$\mathbf{Y}_2 = \frac{1}{R_2 + jX_2}\ ;\ \mathbf{Y}_3 = \frac{1}{R_3 - jX_3}$$

$$\therefore \quad \mathbf{Y}_{23} = \frac{1}{R_2 + jX_2} + \frac{1}{R_3 - jX_3}$$

$$\therefore \qquad \mathbf{Z}_{23} = \frac{1}{\mathbf{Y}_{23}} \; ; \mathbf{Z}_1 = R_1 + jX_1 \; ; \mathbf{Z} = \mathbf{Z}_{23} + \mathbf{Z}_1$$

$$\therefore \qquad \mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} \qquad \text{(See Ex. 14.21)}$$

14.7. Series Equivalent of a Parallel Circuit

Consider the parallel circuit of Fig. 14.27 (*a*). As discussed in Art. 14.5

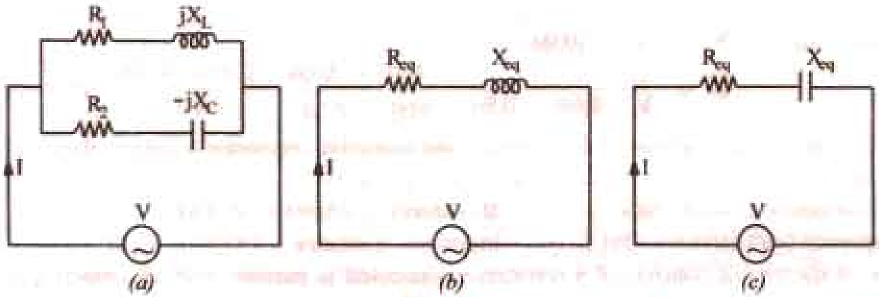


Fig. 14.27

$$\mathbf{Y}_1 = \frac{R_1}{R_1^2 + X_L^2} - j\,\frac{X_L}{R_1^2 + X_L^2} = g_1 - jb_1 \; ; \mathbf{Y}_2 = \frac{R_2}{R_2^2 + X_c^2} + j\,\frac{X_2}{R_2^2 + X_c^2} = g_2 + jb_2$$

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 = g_1 - jb_1 + g_2 + jb_2 = (g_1 + g_2) + j(b_2 - b_1) = G + jB = \sqrt{G^2 + B^2)} \angle \tan^{-1} (B/G)$$

As seen from Fig. 14.28.

$$R_{eq} = Z\cos\phi = \frac{1}{Y}\cdot\frac{G}{Y} = \frac{G}{Y^2}$$

$$X_{eq} = Z\sin\phi = \frac{1}{Y}\cdot\frac{B}{Y} = \frac{B}{Y^2}$$

Hence, equivalent series circuit is as shown in Fig. 14.27 (*b*) or (*c*) depending on whether net susceptance *B* is negative (inductive) or positive (capacitive). If *B* is negative, then it is an *R-L* circuit of Fig. 14.27 (*b*) and if *B* is positive, then it is an *R-C* circuit of Fig. 14.27 (*c*).

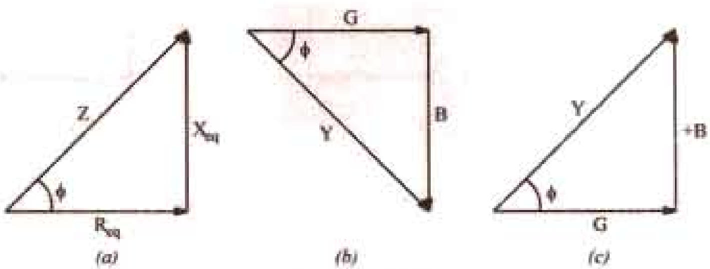


Fig. 14.28

14.8. Parallel Equivalent of a Series Circuit

The two circuits will be equivalent if **Y** of Fig. 14.29 (*a*) is equal to the **Y** of the circuit of Fig. 14.29. (*b*).

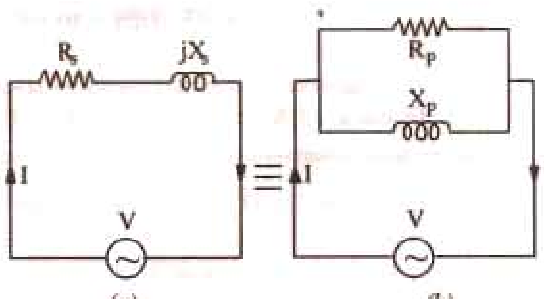


Fig. 14.29

Series Circuit

$$\mathbf{Y}_s = \frac{1}{R_s + jX_s}$$

$$= \frac{R_s - jX_s}{(R_s + jX_s)(R_s - jX_s)}$$

$$= \frac{R_s - jX_s}{R_2^2 + X_s^2} = \frac{R_s}{R_s^2 + X_s^2} - j\,\frac{X_s}{R_s^2 + X_s^2}$$

Parallel Circuit

$$\mathbf{Y}_p = \frac{1}{R_p + j0} + \frac{1}{0 + jX_p} = \frac{1}{R_p} + \frac{1}{jX_p} = \frac{1}{R_p} - \frac{j}{X_p}$$

$$\therefore \quad \frac{R_s}{R_s^2+X_s^2} - j\frac{X_s}{R_s^2+X_s^2} = \frac{1}{R_P} - \frac{j}{X_p} \quad \therefore \quad \frac{1}{R_p} = \frac{R_s}{R_s^2+X_s^2} \text{ or } R_p = R_s + \frac{X_s^2}{R_s} = R_s\left(1+\frac{X_s^2}{R_s^2}\right)$$

Similarly $X_p = X_s + \frac{R_s^2}{X_s} = X_s\left(1+\frac{R_s^2}{X_s^2}\right)$

Example 14.25. *The admittance of a circuit is (0.03 − j 0.04) Siemens. Find the values of the resistance and inductive reactance of the circuit if they are joined (a) in series and (b) in parallel.*

Solution. (*a*) $\mathbf{Y} = 0.03 - j0.04$

$$\therefore \quad \mathbf{Z} = \frac{1}{\mathbf{Y}} = \frac{1}{0.03 - j0.04} = \frac{j0.03 + j0.04}{0.03^2+0.04^2} = \frac{0.03+j0.04}{0.0025} = 12 + j16$$

Hence, if the circuit consists of a resistance and inductive reactance in series, then resistance is **12 Ω** and inductive reactance is 16 Ω as shown in Fig. 14.30.

(*b*) Conductance = 0.03 mho ∴ Resistance = 1/0.03 = 33.3 Ω

Susceptance (inductive) = 0.04 S ∴ Inductive reactance = 1/0.04 = 25 Ω

Hence, if the circuit consists of a resistance connected in parallel with an inductive reactance, then resistance is **33.3 Ω** and inductive reactance is **25 Ω** as shown in Fig. 14.31.

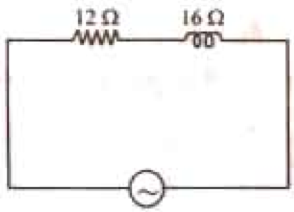


Fig. 14.30

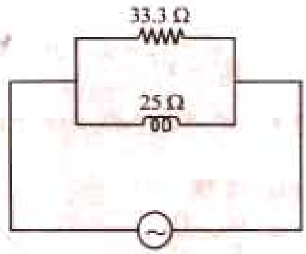


Fig. 14.31

Example 14.26. *A circuit connected to a 115-V, 50-Hz supply takes 0.8 A at a power factor of 0.3 lagging. Calculate the resistance and inductance of the circuit assuming (a) the circuit consists of a resistance and inductance in series and (b) the circuit consists of a resistance and inductance in parallel* **(Elect. Engg.-I, Sardar Patel Univ. 1988)**

Solution. Series Combination

$$Z = 115/0.8 = 143.7\ \Omega\ ;\ \cos\phi = R/Z = 0.3 \quad \therefore \quad R = 0.3 \times 143.7 = \mathbf{43.1\ \Omega}$$

Now $X_L = \sqrt{Z^2 - R^2} = \sqrt{143.7^2 - 43.1^2} = 137.1\ \Omega$

$\therefore \quad L = 137.1/2\pi \times 50 = \mathbf{0.436\ H}$

Parallel Combination

Active component of current (drawn by resistance)

$= 0.8 \cos\phi = 0.8 \times 0.3 = 0.24$ A $R = 115/0.24 = \mathbf{479\ \Omega}$

Quadrature component of current (drawn by inductance) $= 0.8 \sin\phi = 0.8\sqrt{1-0.3^2} = 0.763$ A

$\therefore \quad X_L = 115/0.763\ \Omega \quad \therefore \quad L = 115/0.763 \times 2\pi \times 50 = \mathbf{0.48\ H}$

Example 14.27. *The active and lagging reactive components of the current taken by an a.c circuit from a 250-V supply are 50 A and 25 A respectively. Calculate the conductance, susceptance, admittance and power factor of the circuit. What resistance and reactance would an inductive coil have if it took the same current from the same mains at the same factor ?*

(Elect. Technology, Sumbal Univ. 1987)

Solution. The circuit is shown in Fig. 14.32.

Resistance = 250/50 = 5 Ω ; Reactance = 250/25 = 10 Ω

$\therefore$ Conductance $g = 1/5 = \mathbf{0.2\ S}$. Susceptance $b = -1/10 = \mathbf{-0.1\ S}$

Admittance $Y = \sqrt{g^2 + b^2} = \sqrt{0.2^2 + (-0.1)^2} = \sqrt{0.05} = \mathbf{0.224\ S}$

$\mathbf{Y} = 0.2 - j\,0.1 = 0.224\ \angle -26°34'$. Obviously, the total current lags the supply voltage by 26°34′, p.f. = cos 26°34′ = **0.894 (lag)**

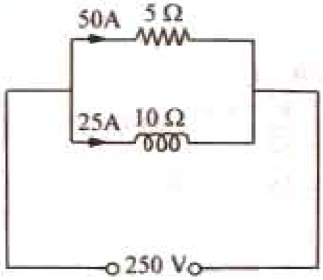


Fig. 14.32

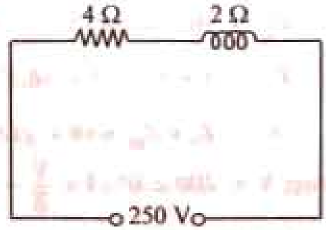


Fig. 14.33

Now $\mathbf{Z} = \dfrac{1}{\mathbf{Y}} = \dfrac{1}{0.2 - j0.1} = \dfrac{0.2 + j0.1}{0.05} = 4 + j2$

Hence, resistance of the coil = **4 Ω** Reactance of the coil = **2 Ω** (Fig. 14.33)

Example 14.28. *The series and parallel circuits shown in Fig. 14.34 have the same impedance and the same power factor. If $R = 3\ \Omega$ and $X = 4\ \Omega$, find the values of R_1 and X_1. Also, find the impedance and power factor.* **(Elect. Engg., Bombay Univ, 1980)**

Solution. Series Circuit [Fig. 14.34 (*a*)]

$$\mathbf{Y}_S = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2} = \frac{R}{R^2 + X^2} - j\frac{X}{R^2 + X^2}$$

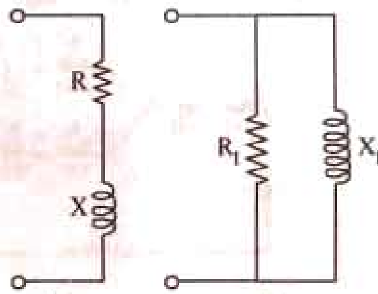
Parallel Circuit [Fig. 14.34 (*b*)]

$$\mathbf{Y}_P = \frac{1}{R_1 + j0} = \frac{1}{0 + jX_1} = \frac{1}{R_1} + \frac{1}{jX_1} = \frac{1}{R_1} - \frac{j}{X_1}$$

$$\therefore \quad \frac{R}{R^2 + X^2} - j\frac{X}{R^2 + X^2} = \frac{1}{R_1} - \frac{j}{X_1}$$

$\therefore \quad R_1 = R + X^2/R$ and $X_1 = X + R^2/X$

$\therefore \quad R_1 = 3 + (16/3) = \mathbf{8.33\ \Omega}$; $X_1 = 4 + (9/4) = \mathbf{6.25\ \Omega}$



(*a*) (*b*)

Fig. 14.34

Impedance = $3 + j\,4 = 5\ \angle 53.1°$; Power factor = cos 53.1° = **0.6 (lag)**

Example 14.29. *Find the value of the resistance R and inductance L which when connected in parallel will take the same current at the same power factor from 400-V, 50-Hz mains as a coil of resistance $R_1 = 8\ \Omega$ and an induction $L_1 = 0.2$ H from the same source of supply.*

Show that when the resistance R_1 of the coil is small as compared to its inductance L_1, then R and L are respectively equal to $\omega^2 L_1^2/R_1$ and L_1. **(Elect. Technology, Utkal Univ. 1985)**

Solution. As seen from Art. 14.8 in Fig. 14.35.

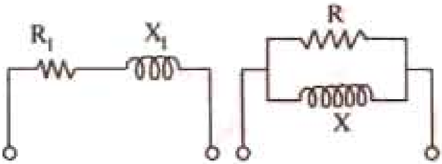
$$R = R_1 + X_1^2/R_1 \qquad \text{...(i)}$$

$$X = X_1 + R_1^2/X_1 \qquad \text{...(ii)}$$

Now $R_1 = 8\ \Omega \quad X_1 = 2\pi \times 50 \times 0.2 = 62.8\ \Omega$

$\therefore \quad R = 8 + (62.8^2/8) = \mathbf{508\ \Omega}$

$X = 62.8 + (64/62.8) = \mathbf{63.82\ \Omega}$



(*a*) (*b*)

Fig. 14.35

From (*i*), it is seen that if R_1 is negligible, then $R = X_1^2/R_1 = \omega^2 L_1^2/R_1$

Similarly, from (*ii*) we find that the term R_1^2/X_1 is negligible as compared to X_1,

$\therefore \quad X = X_1$ or $L = L_1$

Example 14.30. *Determine the current drawn by the following circuit [Fkig. 14.36 (a)] when a voltage of 200 V is applied across the same. Draw the phasor diagram.*

Solution. As seen from the figurre

$$\mathbf{Z_2} = 10 - j12 = 15.6 \angle -50.2°; \mathbf{Z_3} = 6 + j10 = 11.7 \angle 58°$$

$$\mathbf{Z_1} = 4 + j6 = 7.2 \angle 56.3° \ \mathbf{Z}_{BC} = \frac{(10 - j12)(6 - j10)}{16 - j2} = 10.9 + j3.1 = 11.3 \angle 15.9°$$

$$\mathbf{Z} = \mathbf{Z_1} + Z_{BC} = (4 + j6) + (10.9 + j3.1) = 14.9 + j9.1 = 17.5 \angle 31.4°$$

Assuming $\mathbf{V} = 200 \angle 0°$; $\mathbf{I} = \dfrac{\mathbf{V}}{\mathbf{Z}} = \dfrac{200 \angle 0°}{17.5 \angle 31.4°} = 11.4 \angle -31.4°$

For drawing the phasor diagram, let us find the following quantities :

(*i*) $\quad \mathbf{V}_{AB} = \mathbf{IZ}_1 \ 11.4 \angle -31.4° \times 7.2 \angle 56.3° = 82.2 \angle 24.9°$

$$\mathbf{V_{BC}} = \mathbf{I.Z}_{BC} = 11.4 \angle -31.4° \times 11.3 \angle 15.9 = 128.8 \angle -15.5°$$

$$\mathbf{I_2} = \frac{\mathbf{V_{AB}}}{\mathbf{Z_2}} \frac{128. \angle -15.5°}{15.6 \angle -50.2°} = 8.25 \angle 34.7$$

$$\mathbf{I_3} = \frac{128.8 \angle -15.5°}{11.7 \angle 59°} = 15.1 \angle -74.5°$$

Various currents and voltages are shown in their phase relationship in Fig. 14.36 (*b*).

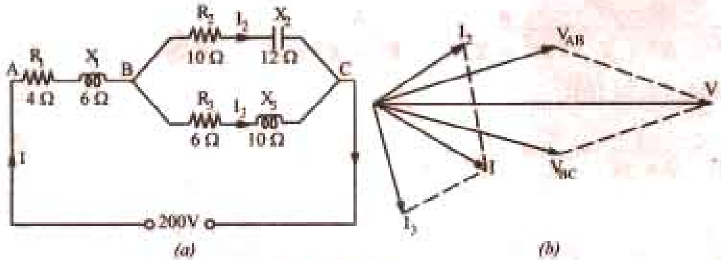


Fig. 14.36

Fig. 14.37 (*a*)

Example 14.31. *For the circuit shown in Fig. 14.37 (a), find (i) total impedance (ii) total current (iii) total power absorbed and power-factor. Draw a vector diagram.*

(Elect. Tech. Osmania Univ. Jan/Feb 1992)

Solution. $Z_{BC} = (4 + j8) \parallel (5 - j8) = 9.33 + j0.89$

(*i*) $\ Z_{AC} = 3 + j6 + 9.33 + j0.89 = 12.33 + j6.89$
$= 14.13 \angle 29.2°$

(*ii*) $I = 100/14.13 \angle 29.2°$, as drawn in Fig. 14.37 (*b*)
$= 7.08 \angle -29.2°$

(*iii*) $\phi = 29.2°$; $\cos\phi = 0.873$; $P = VI \cos\phi$
$= 100 \times 7.08 \times 0.873 = 618$ W

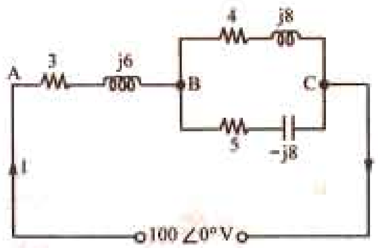


Fig. 14. 37 (*b*)

Example 14.32. *In a series-parallel circuit, the parallel branches A and B are in series with C. The impednces are : $Z_A = (4 + j3)$; $Z_B = (4 - j16/3)$; $Z_C = (2 + j8)$ ohm.*

If the current $I_C = (25 + j0)$, draw the complete phasor diagram determining the branch currents and voltages and the total voltage. Hence, calculate the complex power (the active and reactance powers) for each branch and the whole circuits. **(Basic Electricity. Bombay Univ. 1986)**

Solution. The Circuit is shown in Fig. 14.38 (*a*)

$\mathbf{Z_A} = (4 + j3) = 5 \angle 36°52'$; $\mathbf{Z_B} = (4 - j16/3) = 20/3 \angle -53°8'$; $\mathbf{Z_C} = (2 + j8) = 8.25 \angle 76°$

$\mathbf{I_C} = (25 + j0) = 25 \angle 0°$; $\mathbf{V_C} = \mathbf{I_C Z_C} = 206 \angle 76°$

$$\mathbf{Z_{AB}} = \frac{(4+j3)(4-j16/3)}{(8-j7/3)} = \frac{(32-j28/3)}{(8-j7/3)} = 4 + j\,0 = 4 \angle 0°$$

$$\mathbf{V_{AB}} = \mathbf{I_C\, Z_{AB}} = 25 \angle 0° \times 4 \angle 0° = 100 \angle 0°$$

$\mathbf{Z} = \mathbf{Z_C} + \mathbf{Z_{AB}} = (2 + j8) + (4 + j0) = (6 + j8) = 10 \angle 53°8'$ $\mathbf{V} = \mathbf{I_C Z} = 25\angle 0° \times 10 \angle 53°8'$
$= 250 \angle 53°8'$

$$\mathbf{I_A} = \frac{\mathbf{V_{AB}}}{\mathbf{Z_A}} = \frac{100 \angle 0°}{5 \angle 36°52'} = 20 \angle -36°52' ; \mathbf{I_B} = \frac{\mathbf{V_{AB}}}{\mathbf{Z_B}} = \frac{100 \angle 0°}{(20/3) \angle -53°8'} = 15 \angle 53°8'$$

Various voltages and currents are shown in Fig. 14.38 (*b*). Powes would be calculated by using voltage conjugates.

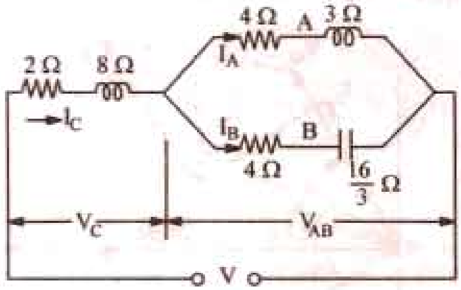
Power for whole ciruict is $\mathbf{P} = \mathbf{VI_C} = 250 \angle -53°8' \times 25 \angle 0° = 6{,}250 \angle -53°8'$

$= 6250 (\cos 53°8' - j \sin 53°8') = 3750 - j5000$

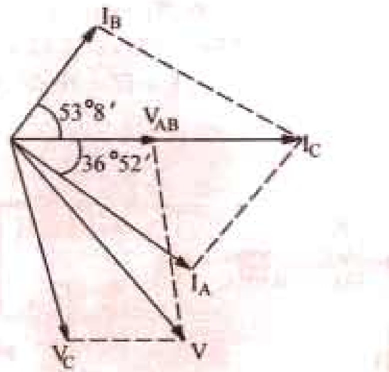
$\mathbf{P_C} = 25 \times 206 \angle -76° = 5150 (\cos 76° - j\sin 76°) = 1250 - j5000$

$\mathbf{P_A} = 100 \times 20 \angle -36°52' = 2000 \angle -36°52' = 1600 - j1200$

$\mathbf{P_B} = 100 \times 15 \angle 53°8' = (900 + j1200)$; Total $= 3{,}750 - j5000°$ (as a check)



(*a*)



(*b*)

Fig. 14.38

Example 14.33. *Find the value of the power developed in each arm of the series-paralell circuit shown in Fig. 14.39.*

Solution. In order to find the circuit current, we must first find the equivalent impedance of the whole circuit.

$Z_{AB} = (5 + j12) \parallel (-j\,20)$

$$= \frac{(5+j12)(-j20)}{5+j12-j20} = \frac{13 \angle 67.4° \times 20 \angle -90°}{9.43 \angle -58°}$$

$= 27.57 \angle 35.4° = (22.47 + j\,15.97)$

$Z_{AC} = (10 + j0) + (22.47 + j15.97) = (32.47 + j14.97)$

$= 36.2 \angle 26.2°$

$$I = \frac{V}{Z} = \frac{50 \angle 0°}{36.2 \angle 26.2°} = 1.38 \angle -26.2° A$$

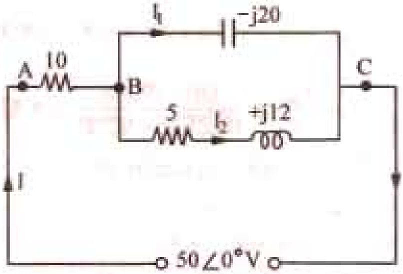


Fig. 14.39

Power developed in 10Ω resistor = $I^2R = 1.38^2 \times 10 =$ **19 W.**

Potential difference across 10 Ω resistor is

$$IR = 1.38 \angle -26.2° \times 10 = 13.8 \angle -26.2° = (12.38 - j\,6.1)$$

$$V_{BC} = \text{supply voltage} - \text{drop across } 10\ \Omega \text{ resistor}$$

$$= (50 + j0) - (12.38 - j6.1) = (37.62 + j6.1) = 38.1 \angle 9.21°$$

$$I_2 = \frac{V_{BC}}{(5 + j12)} = \frac{38.1 \angle 9.21°}{13 \angle 67.4°} = 2.93 \angle -58.2°$$

Power developed = $I_2^2 \times 5 = 2.93^2 \times 5 =$ **43 W**

No power is developed in the capacitor branch because it has no resistance.

Example 14.34. *In the circuit shown in Fig. 14.40 determine the voltage at a frequency of 50 Hz to be applied across AB in order that the current in the circuit is 10 A. Draw the phasor diagram.*
(Elect. Engg. & Electronics Bangalore Univ. 1988)

Solution. $X_{L1} = 2\,\pi \times 50 \times 0.05 = 15.71\ \Omega$; $X_{L2} = -\,2\,\pi \times 50 \times 0.02 = 6.28\ \Omega$,

$X_C = 1/2\pi \times 50 \times 400 \times 10^{-6} = 7.95\ \Omega$

$\mathbf{Z}_1 = R_1 + jX_{L1} = 10 + j15.71 = 18.6 \angle 57°33'$

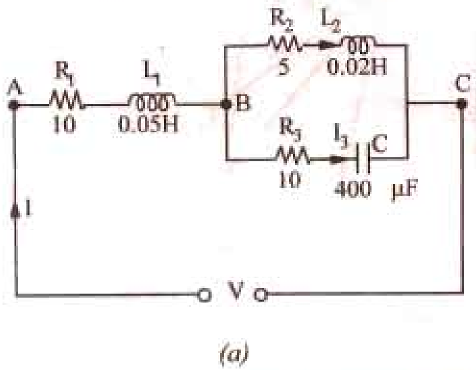
$\mathbf{Z}_2 = R_2 + jX_{L2} = 5 + j6.28 = 8 \angle 51°30'$

$\mathbf{Z}_3 = R_3 - jX_C = 10 - j7.95 = 12.77 \angle 38°30'$

$\mathbf{Z}_{BC} = Z_2 \,||\, Z_3\ (5 + j6.28) \,||\, (10 - j7.95) = 6.42 + j2.25 = 6.8 \angle 19°18'$

$Z = Z_1 + Z_{BC} = (10 + j15.71) + (6.42 + j2.25) = 16.42 + j17.96 = 24.36 \angle 47°36'$

Let I = $10\angle 0°$; $\therefore$ $\mathbf{V} = IZ = 10\angle 0° \times 24.36\angle 47°36 = 243.6\angle 47°36'$



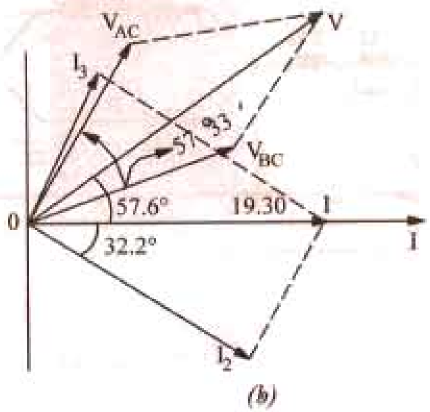


Fig. 14.40

$$V_{BC} = IZ_{BC} = 10 \angle 0° \times 6.8 \angle 19°18' = 68 \angle 19°18' \; ; I_2 = \frac{V_{BC}}{Z_2} = \frac{68 \angle 19°18'}{8 \angle 51°31'} = 8.5 \angle -32°12'$$

$$I_3 = \frac{V_{BC}}{Z_3} = \frac{68 \angle 19°18'}{12.77 \angle -38°30'} = 5.32 \angle 57°48' \; ; V_{AC} = IZ_1 = 10 \angle 0° \times 18.6 \angle 57°33' = 186 \angle 57°33'$$

The phasor diagram is shown in Fig. 14.36 (*b*).

Example 14.35. *Determine the average power delivered to each of the three boxed networks in the circuit of Fig. 14.41.* **(Basic Circuit Analysis Osmania Univ. Jan/Feb 1992)**

Solution. $Z_1 = 6 - j8 = 10 \angle -53°13°$; $Z_2 = 2 + j14 = 14.14 \angle 81.87°$; $Z_3 = 6 - j8 = 10 \angle -53.13°$

$$Z_{23} = \frac{Z_2 Z_3}{Z_2 + Z_3} = 14.14 \angle -8.13° = 14 - j2$$

Drop across two parallel impedances is given by

$$V_{23} = 100 \frac{14 - j2}{(6 - j8) + (14 - j2)} = 63.2 \angle 18.43° = 60 + j20$$

$$V_1 = 100 \frac{10 \angle -53.13°}{6 - j8 + (14 - j2)} = 47.7 \angle -26.57° = 40 - j20$$

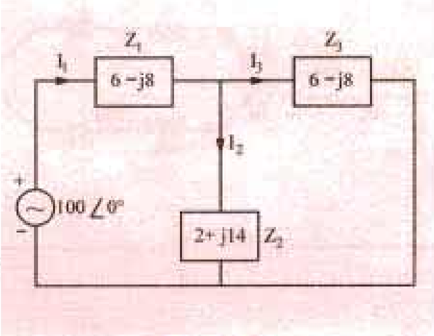


Fig. 14.41

$$I_1 = \frac{44.7 \angle -26.57°}{10 \angle -53.13°} = 4.47 \angle 26.56°$$

$$I_2 = \frac{63.2 \angle 18.43°}{14.14 \angle 81.87°} = 4.47 \angle -63.44°$$

$$I_3 = \frac{63.2 \angle 18.43°}{10 \angle -53.13°} = 6.32 \angle 71.56°$$

$$P_1 = V_1 I_1 \cos \phi_1 = 44.7 \times 4.47 \times \cos 53.13° = 120 \text{ W}$$
$$P_2 = V_2 I_2 \cos \phi_2 = 63.2 \times 4.47 \times \cos 81.87° = 40 \text{ W};$$
$$P_3 = V_3 I_3 \cos \phi_3 = 63.2 \times 6.32 \times \cos 53.13° = \mathbf{240 \text{ W}}, \text{Total} = 400 \text{ W}$$

As a check, power delivered by the 100-V source is,

$$P = VI_1 \cos \phi = 100 \times 4.47 \times \cos 26.56° = 400 \text{ W}$$

Example 14.36. *In a series-parallel circuit of Fig. 14.42 (a), the parallel branches A and B are in series with C. The impedances are $Z_A = (4 + j3)$, $Z_B = (10 - j7)$ and $Z_C = (6 + j5)\ \Omega$.*

If the voltage applied to the circuit is 200 V at 50 Hz, calculate : (a) current I_A, I_B and I_C; (b) the total power factor for the whole circuit.

Draw and explain complete vector diagram.

Solution. $\mathbf{Z_A} = 4 + j3 = 5 \angle 36.9°$; $\mathbf{Z_B} = 10 - j7 = 12.2 \angle -35°$; $\mathbf{Z_C} = 6 + j5 = 7.8 \angle 39.8°$

$$\mathbf{Z_{AB}} = \frac{\mathbf{Z_A Z_B}}{\mathbf{Z_A + Z_B}} = \frac{5 \angle 36.9° \times 12.2 \angle -35°}{14 - j4} = \frac{61 \angle 1.9°}{14.56 \angle -16°} = 4.19 \angle 17.9° = 4 + j1.3$$

$$\mathbf{Z} = \mathbf{Z_C} + \mathbf{Z_{AB}} = (6 + j5) + (4 + j1.3) = 10 + j6.3 = 11.8 \angle 32.2°$$

Let $\mathbf{V} = 200 \angle 0°$; $\mathbf{I_C}$ $(\mathbf{V/Z}) = (200/11.8) \angle 32.2° = 16.35 \angle 32.2°$

$$\mathbf{I_A} = \mathbf{I_C} \cdot \frac{\mathbf{Z_B}}{\mathbf{Z_A + Z_B}} = 16.95° \angle -32.2° \times \frac{12.2 \angle -35°}{14.56 \angle -16°} = 14.0 \angle -51.2°$$

$$\mathbf{I_B} = \mathbf{I_C} \cdot \frac{\mathbf{Z_B}}{\mathbf{Z_A + Z_B}} = 14.95 \angle -32.2° \times \frac{5 \angle 36.9°}{14.56 \angle -16°} = 5.82 \angle 20.7°$$

The Phase angle between V and total circuit current I_C is 32.2°. Hence p.f. for the whole circuit is = cos 32.2° = **0.846 (lag)**

For drawing the phasor diagram of Fig. 14.42 (*b*) following quantities have to be calculated :

$$\mathbf{V_{CA}} = \mathbf{I_C Z_C} = 17.85 \angle -32.2° \times 7.8 \angle 39.8° = 132.2 \angle 7.6°$$
$$\mathbf{V_{AB}} = \mathbf{I_C Z_{AB}} = 17.85 \angle -32.2° \times 4.19 \angle 17.9° = 71 \angle -14.3°$$

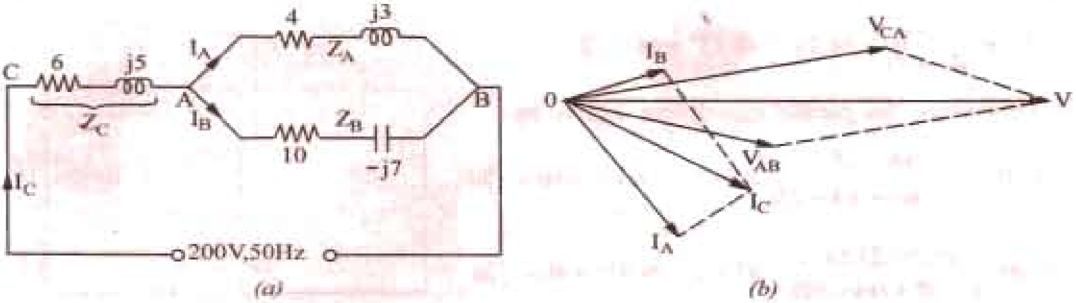


Fig. 14.42

The circuit and phasor diagrams are shown in Fig. 143.38.

Example 14.37. *A fluorescent lamp taking 80 W at 0.7 power factor lagging from a 230-V 50-Hz supply is to be corrected to unity power factor. Determine the value of the correcting apparatus required.*

Solution. Power taken by the 80-W lamp circuit can be found from the following equation,

$230 \times I \times 0.7 = 80 \quad \therefore \quad I = 80/230 \times 0.7 = 0.5$ A

Reactive component of the lamp current is $= I \sin \phi = 0.5\sqrt{1-0.7^2} = 0.357$ A

The power factor of the lamp circuit may be raised to unity by connecting a suitable capacitor across the lamp circuit. The leading reactive current drawn by it should be just equal to 0.357 A. In that case, the two will cancel out leaving only the in-phase component of the lamp current.

$I_C = 0.357$ A $\quad X_C = 230/0.357 = $ **645 Ω**

Now $\quad X_C = 1/\omega C \quad \therefore \; 645 = 1/2\pi \times 50 \times C \quad C = $ **4.95 μF**

Example 14.38. *For the circuit shown in Fig. 14.43, calculate I_1, I_2 and I_3. The values marked on the inductance and capacitance give their reactances.* **(Elect. Science-I Allahabad Univ. 1992)**

Solution. $Z_{BC} = Z_2 \| Z_3 = \dfrac{(4+j2)(1-j5)}{(3+j2)+(1-j5)} = \dfrac{14-j18}{5-j3} = \dfrac{(14-j18)(5+j3)}{5^2+3^2} = 3.65 - j1.41 = 3.9\angle 21.2°$

$\mathbf{Z = Z_1 + Z_{BC}} = (2+j3) + (3.65 - j1.41) = 5.65 + j1.59 = 5.82 \angle 74.3°$

Let $V = 10\angle 0°$; $I_1 = V/Z = 10\angle 0°/5.82\angle 74.3° = \mathbf{1.72 \angle -74.3°}$

$V_{BC} = I_1 Z_{BC} = 1.72 \angle -74.3° \times 3.9 \angle 21.2° = 6.7 \angle -53.1$

Now, $Z_2 = 4 + j2 = 4.47 \angle 63.4°$;

$Z_3 = 1 - j5 = 5.1 \angle -11.3°$

$I_2 = V_{BC}/Z_2 = 6.7\angle -53.1°/4.47\angle 63.4° = \mathbf{1.5 \angle 10.3°}$

$I_3 = V_{BC}/Z_3 = 6.7\angle -53.1°/5.1\angle -11.3° = \mathbf{1.3 \angle -41.8°}$

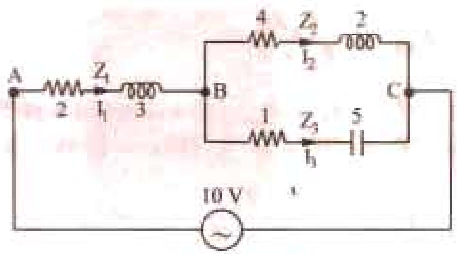


Fig. 14.43

Example 14.39. *A workshop has four 240-V, 50-Hz single-phase motors each developing 3.73 kW having 85% efficiency and operating at 0.8 power factor. Calculate the values 0.9 lagging and (b) 0.9 leading. For each case, sketch a vector diagram and find the value of the supply current.*

Solution. Total motor power input $= 4 \times 3730/0.85 = 17{,}550$ W

Motor current $\quad I_m = 17{,}550/240 \times 0.8 = 91.3$A

Motor p.f. $= \cos \phi_m = 0.8 \quad \therefore \quad \phi_m = \cos^{-1}(0.8) = 36°52'$

(*a*) Since capacitor does not consume any power, the power taken from the supply remains unchanged after connecting the capacitor. If I_s is current drawn from the supply, then $240 \times I_s \times 0.9 = 17{,}550$

$\therefore \quad I_s = 81.2$ A. $\cos \phi_s = 0.9$; $\phi_s = \cos^{-1}(0.9) = 25°50'$

As seen from vector diagram of Fig. 14.44 (*a*), I_s the vector sum of I_m and capacitor current I_C . $I_C = I_m \sin \phi_m - I_s \sin \phi_s = 91.3 \sin 36°52' - 81.2 \sin 25°50' = 54.8 - 35.4 = 19.4$ A

Now $I_C = \omega VC$

or $19.4 = 240 \times 2\pi \times 50 \times C$

$\therefore \quad C = 257 \times 10^{-6} \text{F} = \mathbf{257 \mu F}$

(*b*) In this case, I_s leads the supply voltage as shown in Fig. 14.44 (*b*)

$$I_C = I_m \sin \phi_m + I_s \sin \phi_s = 54.8 + 35.4 = 90.2 \text{ A}$$

Now $I_C = \omega VC$

$\therefore \quad 90.2 = 240 \times 2\pi \times 50 \times C$

$\therefore \quad C = 1196 \times 10^{-6}$ F = **1196 μF**

The line or supply current is, as before, 81.2 A (leading)

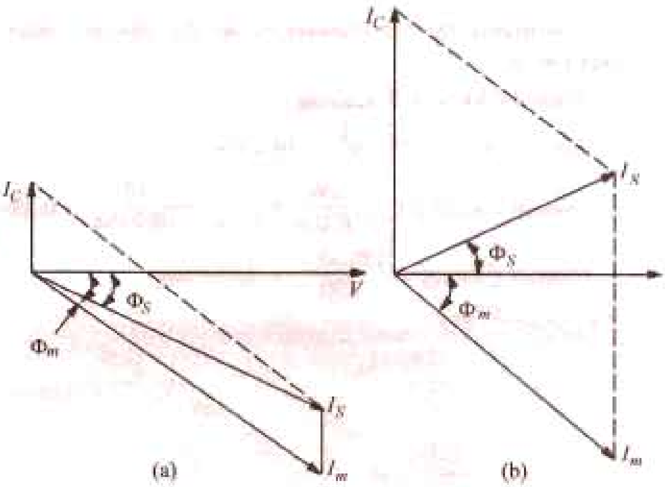


Fig. 14.44

Example 14.40. *The load taken from a supply consists of (a) lamp load 10 kW a unity power factor (b) motor load of 80 kVA at 0.8 power factor (lag) and (c) motor load of 40 kVA at 0.7 power factor leading. Calculate the total load taken from the supply in kW and in kVA and the power factor of the combined load.*

Solution. Since it is more convenient to adopt the tabular method for such questions, we will use the same as illustrated below. We will tabulate the kW, kVA and kVAR (whether leading or lagging) of each load. The lagging kVAR will be taken as negative and leading kVAR as positive.

Load	kVA	cos φ	sin φ	kW	kVAR
(*a*)	10	1	0	10	0
(*b*)	80	0.8	0.6 –	64	– 48
(*c*)	40	0.7	0.714	28	28/6
			Total	102	– 19.2

Total kW = **102** ; Total kVAR = – 19.4 (lagging) ; kVA taken = $\sqrt{102^2 + (-19.4)^2}$ = **103.9**

Power factor = kW/kVA = 102/103.9 = **0.9822 (lag)**

Example 14.41. *A 23-V, 50 Hz, 1-ph supply is feeding following loads which are connected across it.*

(i) A motor load of 4 kW, 0.8 lagging p.f. *(ii) A rectifier of 3 kW at 0.6 leading p.f.*

(iii) A lighter-load of 10 kVA at unity p.f. *(iv) A pure capacitive load of 8 kVA*

Determine : Total kW, Total kVAR, Total kVA **(I BE Nagpur University Nov. 1999)**

Solution.

S. No.	Item	kW	P.f	kVA	kVAR	I	I_s	I_r
1	Motor	4	0.8 lag	5	3 – ve, Lag	21.74	17.4	13.04 Lag (–)
2	Rectifier	3	0.6 Lead	5	4 + ve, Lead	21.74	13.04	17.4 Lead (+)
3	Light-Load	10	1.0	10	zero	43.48	43.48	zero
4	Capacitive Load	Zero	0.0 Lead	8	8 + ve Lead	34.8	zero	34.8 Lead (+)
	Toal	17	-	Phasor Addition required	+ 9 + ve Lead	Phasor addition required	73.92	39.16 Lead (+)

Performing the calculations as per the tabular entries above, following answers are obtained Total kW = 17

Total kVAR = + 9, leading

Total $kVA = \sqrt{17^2 + 9^2} = 19.2354$

Overall circuit p.f. $= \frac{\text{kW}}{\text{KVA}} = \frac{17}{\text{kVA}} = \frac{17}{19.2354} = 0.884$ leading

Overall Current $= \frac{19235.4}{230} = 83.63$ amp

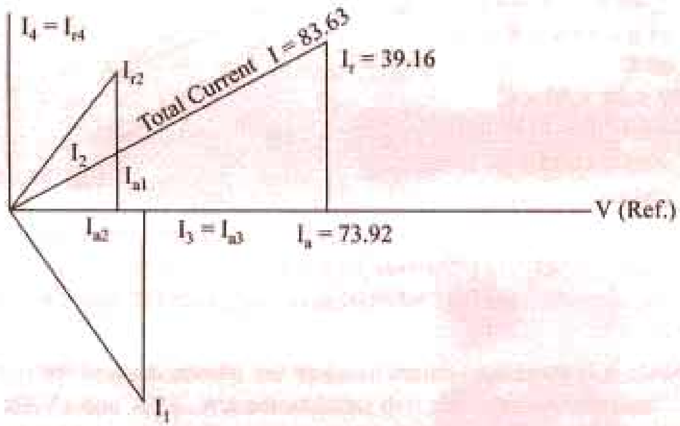


Fig. 14.45 Phasor diagram for currents corresponding to load

Example 14.42. *A three phase induction Motor delivers an output of 15 h.p. at 83 % efficiency. The motor is d (delta) connected and is supplied by 440 V, three phase, 50 Hz supply. Line current drawn by motor is 22.36 Amp. What is motor power factor ?*

It is now decided to improve the power factor to 0.95 lag by connecting three similar capacitors in delta across the supply terminals. Determine the value of the capacitance of each capacitor.

[Note : 1 h.p. = 745 watts] **(Bombay University, 2000)**

Solution. Power factor $= \frac{15 \times 745}{0.83 \times 1.732 \times 440 \times 22.36} = 0.79$, Lagging

$\phi = \cos^{-1} 0.79 = 37.8°$

$I_1 = I_{ph} = 22.36/1.732 = 12.91$ amp

Active Current $I_a = I_1 \cos \phi_1 = 12.91 \times 0.79 = 10.2$ amp

New Power-factor $\cos \phi_2 = 0.95$, $\phi_2 = 18.2°$

$I_2 = 10.2/0.95 = 10.74$ amp

Capacitive current per phase $= I_1 \sin \phi_1 - \phi_2 \sin \phi_2$

$= 4.563$

Capacitive reactance per phase = 440/4.563 = 96.43 ohms

Capacitance per phase = 33 μf

These have to be delta-connected

Example 14.43. *Draw admittance triangle between the terminals AB of Fig 14.46 (a) labeling its sides with appropriate values and units in case of :*

(i) $X_L = 4$ *and* $X_C = 8$ *(ii)* $X_L = 10$ *and* $X_C = 5$ **[Bombay University 1999]**

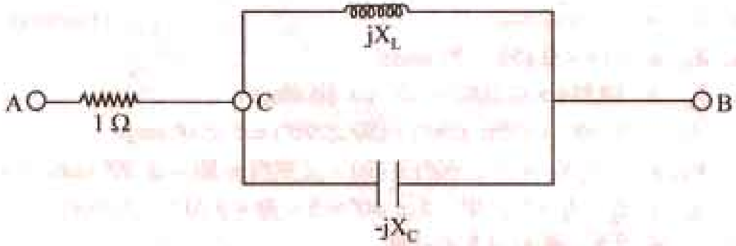


Fig. 14.46 (*a*)

Solution. (*i*)

$$X_L = 4\ \Omega,\ X_C = 8\ \Omega$$

$$Z_{CB} = \frac{jX_L\,(-\,jX_C)}{j\,(X_L - X_C)} = j8$$

$$Z_{AB} = 1 + j\,8 \text{ ohms}$$

$$Y_{AB} = 1/Z_{AB} = (1/65) - j\,(8/65) \text{ mho}$$

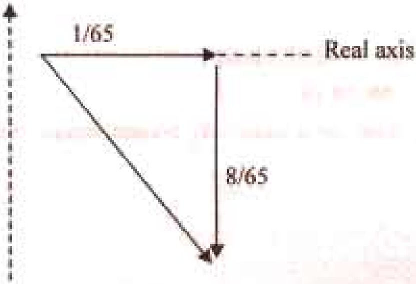
(*ii*)

$$X_L = 10\ \Omega,\ X_C = 5\ \Omega$$

$$Z_{CB} = \frac{j10 \times (-\,j5)}{j5} = -\,j10$$

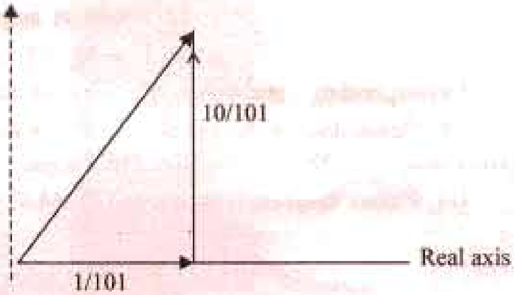
$$Z_{AB} = 1 - j\,10 \text{ ohms}$$

$$Y_{AB} = (1/101) + j\,(10/101) \text{ mho}$$



(*i*) Admittance triangle for first case

Fig. 14.46 (*b*)



(*ii*) Admittance triangle for second case

Fig. 14.46 (*c*)

Example 14.44. *For the circuit in Fig. 14.47 (a), given that L = 0.159 H*

$$C = 0.3183 \text{ mf}$$

$$I_2 = 5 \angle 60^\circ \text{ A}$$

$$V_1 = 250 \angle 90^\circ \text{ volts.}$$

Find :-

(i) Impedance Z_1 with its components.

(ii) Source voltage in the form of $V_m \cos(\omega t + \phi)$

(iii) Impedance Z_2 with its components so that source p.f. is unity, without adding to the circuit power loss.

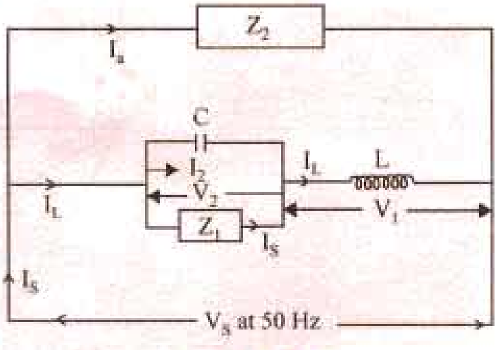


Fig. 14.47 (*a*)

(iv) Power loss in the circuit

(v) Draw the phasor diagram. **(Bombay University 1997)**

Solution. $X_L = 314 \times 0.159 = 50$ ohms

$X_C = 1/(314 \times 0.3183 \times 10^{-3}) = 10$ ohms

$I_L = V_1/jX_L = (250 \angle 90°) / (50 \angle 90°) = 5 \angle 0°$ amps

$V_2 = -jI_2 X_C = (5 \angle 60°) \times (10 - \angle 90°) = 50 - \angle 30°$ volts $= 43.3 - j\,25$ volts

$I_L = I_L - I_2 = 5 \angle 0° - 5 \angle 60° = 5 + j0 - 5\,(0.5 + j0.866)$

$= 2.5 - j4.33 = 5 \angle -60°$

(*a*) $Z_1 = V_2/I_1 = (50 \angle -30°) / 5 \angle -60° = 10 \angle +30°$

$= 10\,(\cos 30° + j \sin 30°) = 8.66 + j5$

(*b*) $V_s = V_1 + V_2 = 0\,j250 + 43.3 - j25 = 43.3 + j225$

$= 229.1 \angle 79.1°$ volts

V_s has a peak value of $(229.1 \times \sqrt{2} =)$ 324 volts

$V_s = 324 \cos (314\,t - 10.9°)$, taking V_1 as reference

or $V_s = 325 \cos (314\,t - 79\,.1°)$, taking I_L as reference.

(*c*) Source Current must be at unity P.f., with V_s

Component of I_L in phase with $V_s = 5 \cos 79.1° = 0.9455$ amp

Component of I_L in quadrature with V_S (and is lagging by 90°)

$= 5.00 \times \sin 79.1° = 4.91$ amp

Z_2 must carry I_a such that no power loss is there and I_S is at unity P.f. with V_s.

I_a has to be capacitive, to compensate, in magnitude, the quatrature component of I_L

$|I_a| = 4.91$ amp

$|Z_2| = V_s / |I_a| = 229.1/4.91 = 46.66$ ohms

Corresponding capacitance, $C_2 = 1 / (46.66 \times 314) = 68.34\ \mu F$

(*d*) Power-loss in the circuit $= I^2_1 \times 8.66 = 216.5$ watts or power $= V_s \times$ component of I_L in phase with $V_s = 299.1 \times 0.9455 = 216.5$ watts

(*e*) Phasor diagram is drawn in Fig. 14.47 (*b*)

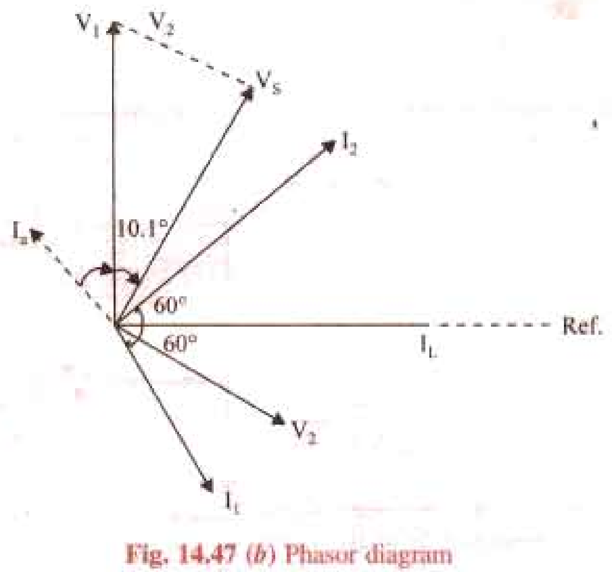


Fig. 14.47 (*b*) Phasor diagram

Tutorial Problem No. 14.1

1. A capacitor of 50 μF capacitance is connected in parallel with a reactor of 22 Ω resistance and 0.07 henry inductance across 200-V, 50-Hz mains. Calculate the total current taken. Draw the vector diagram in explanation. **[4.76 A lagging, 17°12′]** (*City & Guilds, London*)
2. A non-inductive resistor is connected in series with a capacitor of 100 μF capacitance across 200-V, 50-Hz mains. The p.d. measured across the resistor is 150 V. Find the value of resistance and the value of current taken from the mains if the resistor were connected in parallel-with the capacitor instead of in series. **[R = 36.1 Ω; 8.37 Ω]** (*City & Guilds, London*)
3. An impedance of $(10 + j15)$ Ω is connected in parallel with an impedance of $(6 - j8)$Ω. The total current is 15 A. Calculate the total power. **[2036 W]** (*City & Guilds, London*)
4. The load on a 250-V supply system is ; 12 A at 0.8 power factor lagging ; 10 A at 0.5 power factor lagging ; 15 A at unity power factor ; 20 A at 0.6 power factor leading. Find (*i*) the total lead in kVA and (*ii*) its power factor. **[(*i*) 10.4 kVA (*ii*) 1.0]** (*City & Guilds, London*)
5. A voltage having frequency of 50 Hz and expressed by $\mathbf{V} = 200 + j100$ is applied to a circuit consisting of an impedance of $50\angle 30°$ Ω in parallel with a capacitance of 10 μF. Find (*a*) the reading on a ammeter connected in the supply circuit (*b*) the phase difference between the current and the voltage. **[(*a*) 4.52 (*b*) 26.6° lag]** (*London University*)
6. A voltage of $200° \angle 30°$ V is applied to two circuits *A* and *B* connected in parallel. The current is *A* is $20\angle 60°$ A and that in *B* is $40\angle -30°$ A. Find the kVA and kW in each branch circuit and the main circuit. Express the current in the main circuit in the form $A + jB$. (*City & Guilds, London*)
[$kVA_A = 4$, $kVA_B = 8$, $kV_A = 12$, $kW_A = 3.46$, $kW_B = 4$, $kW = 7.46$, $I = 44.64 - j\,2.68$] (*City & Guilds, London*)
7. A coil having an impedance of $(8 + j6)$ Ω is connected across a 200-V supply. Express the current in the coil in (*i*) polar and (*ii*) rectangular co-ordinate forms.
If a capacitor having a susceptance of 0.1 S is placed in parallel with the coil, find (*iii*) the magnitude of the current taken from the supply. **[(*i*) $20\angle 36.8°$A (*ii*) 16 − j12 A (*iii*) 17.9 A]** (*City & Guilds, London*)
8. A coil-A of inductance 80 mH and resistance 120 Ω, is connected to a 230-V, 50 Hz single-phase supply. In parallel with it is a 16 μF capacitor is series with a 40 Ω non-inductive resistor *B*. Determine (*i*) the power factor of the combined circuit and (*ii*) the total power taken from the supply. **[(*i*) 0.945 lead (*ii*) 473 W]** (*London University*)
9. A choking coil of inductance 0.08 H and resistance 12 ohm, is connected in parallel with a capacitor of 120 μF. The combination is connected to a supply at 240 V, 50 Hz. Determine the total current from the supply and its power factor. Illustrate your answers with a phasor diagram. **[3.94 A, 0.943 lag]** (*London University*)
10. A choking coil having a resistance of 20 Ω and an inductance of 0.07 henry is connected with a capacitor of 60 μF capacitance which is in series with a resistor of 50 Ω. Calculate the total current and the phase angle when this arrangement is connected to 200-V, 50 Hz mains. **[7.15 A, 24°39′ lag]** (*City & Guilds, London*)
11. A coil of resistance 15 Ω and inductance 0.05 H is connected in parallel with a non-inductive resistance of 20 Ω. Find (*a*) the current in each branch (*b*) the total current (*c*) the phase angle of whole arrangement for an applied voltage of 200 V at 50 Hz. **[9.22 A ; 10A ; 22.1°]**
12. A sinusoidal 50-Hz voltage of 200 V (r.m.s) supplies the following three circuits which are in parallel : (*a*) a coil of inductance 0.03 H and resistance 3 Ω (*b*) a capacitor of 400 μF in series with a resistance of 100 Ω (*c*) a coil of inductance 0.02 H and resistance 7 Ω in series with a 300 μF capacitor. Find the total current supplied and draw a complete vector diagram. **[29.4 A]** (*Sheffield Univ. U.K.*)
13. A 50-Hz, 250-V single-phase power line has the following loads placed across it in parallel : 4 kW at a p.f. of 0.8 lagging ; 6 kVA at a p.f. of 0.6 lagging; 5 kVA which includes 1.2 kVAR leading. Determine the overall p.f. of the system and the capacitance of the capacitor which, if connected across the mains would restore the power factor to unity. **[0.844 lag ; 336 μF]**
14. Define the terms admittance, conductance and susceptance with reference to alternating current circuits. Calculate their respective values for a circuit consisting of resistance of 20Ω, in series with an inductance of 0.07 H when the frequency is 50 Hz. **[0.336 S, 0.0226 S, 0.0248 S]** (*City & Guilds, London*)
15. Explain the terms admittance, conductance, susceptance as applied to a.c. circuits. One branch *A*, of a parallel circuit consists of a coil, the resistance and inductance of which are 30 Ω and 0.1 *H* respectively. The other branch *B*, consists of a 100 μF capacitor in series with a 20 Ω resistor. If the combination is connected 240-V, Hz mains, calculate (*i*) the line current and (*ii*) the power. Draw to scale a vector diagram of the supply current and the branch-circuit currents. **[(*i*) 7.38 A (*ii*) 1740 W]** (*City & Guilds, London*)

16. Find the value of capacitance which when placed in parallel with a coil of resistance 22 Ω and inductance of 0.07 H, will make it resonate on a 50-Hz circuit. **[72.33 μF]** (*City & Guilds, London*)
17. A parallel circuit has two branches. Branch A consists of a coil of inductance 0.2 H and a resistance of 15 Ω ; branch B consists of a 30 mF capacitor in series with a 10 Ω resistor. The circuit so formed is connected to a 230-V, 50-Hz supply. Calculate (*a*) current in each branch (*b*) line current and its power factor (*c*) the constants of the simplest series circuit which will take the same current at the same power factor as taken by the two branches in parallel. **[3.57 A, 2.16 A ; 1.67 A, 0.616 lag, 8.48 Ω, 0.345 H]**
18. A 3.73 kW, 1-phase, 200-V motor runs at an efficiency of 75% with a power factor of 0.7 lagging. Find (*a*) the real input power (*b*) the kVA taken (*c*) the reactive power and (*d*) the current. With the aid of a vector diagram, calculate the capacitance required in parallel with the motor to improve the power factor to 0.9 lagging. The frequency is 50 Hz. **[4.97 kW ; 7.1 kVA ; 5.07 kVAR ; 35.5 A ; 212μF]**
19. The impedances of two parallel circuits can be represented by $(20 + j15)$ and $(1 - j60)$ Ω respectively. If the supply frequency is 50 Hz, find the resistance and the inductance or capacitance of each circuit. Also derive a symbolic expression for the admittance of the combined circuit and then find the phase angle between the applied voltage and the resultant current. State whether this current is leading or lagging relatively to the voltage. **[20 Ω; 0.0478 H; 10 Ω; 53 μF; (0.0347 – j 0.00778)S; 12°38′ lag]**
20. One branch *A* of a parallel circuit consists of a 60-μF capacitor. The other branch *B* consists of a 30 Ω resistor in series with a coil of inductance 0.2 *H* and negligible resistance. A 140 Ω resistor is connected in parallel with the coil. Sketch the circuit diagram and calculate (*i*) the current in the 30 Ω resistor and (*ii*) the line current if supply voltage is 230-V and the frequency 50 Hz. **[(*i*) 3.1 ∠– 44° (*ii*) 3.1 ∠ 45° A]**
21. A coil having a resistance of 45 Ω and an inductance of 0.4 *H* is connected in parallel with a capacitor having a capacitance of 20 μ*F* across a 230-V, 50-Hz system. Calculate (*a*) the current taken from the supply (*b*) the power factor of the combination and (*c*) the total energy absorbed in 3 hours. **[(*a*) 0.615 (*b*) 0.951 (*c*) 0.402 kWh]** (*London University*)
22. A series circuit consists of a resistance of 10 Ω and reactance of 5 Ω. Find the equivalent value of conductance and susceptance in parallel. **[0.08 S, 0.04 S]**
23. An alternating current passes through a non-inductive resistance *R* and an inductance *L* in series. Find the value of the non-inductive resistance which can be shunted across the inductance without altering the value of the main current. **[$\omega^2L^2/2R$]** (*Elec. Meas. London Univ.*)
24. A p.d. of 200 V at 50 Hz is maintained across the terminals of a series-parallel circuit, of which the series branch consists of an inductor having an inductance of 0.15 H and a resistance of 30 Ω, one the parallel branches consists of 100-μF capacitor and the other consists of a 40-Ω resistor.
Calculate (*a*) the current taken by the capacitor (*b*) the p.d. across the inductor and (*c*) the phase difference of each of these quantities relative to the supply voltage. Draw a vector diagram representing the various voltage and currents. **[(*a*) 29.5 A (*b*) 210 V (*c*) 7.25°, 26.25°]** (*City & Guilds, London*)
25. A coil (*A*) having an inductance of 0.2 *H* and resistance of 3.5 Ω is connected in parallel with another coil (*B*) having an inductance of 0.01 H and a resistance of 5 Ω. Calculate (*i*) the current and (*ii*) the power which these coils would take from a 100-V supply system having a frequency of 50-Hz. Calculate also (*iii*) the resistance and (*iv*) the inductance of a single coil which would take the same current and power. **[(*i*) 29.9 A (*ii*) 2116 W (*ii*) 2.365 Ω (*iv*) 0.00752 H]** (*London Univ.*)
26. Two coils, one (*A*) having $R = 5$ Ω, $L = 0.031$ *H* and the other (*B*) having $R = 3$ Ω ; $L = 0.023$ *H*, are connected in parallel to an a.c. supply at 200 *V*, 50 Hz. Determine (*i*) the current taken by each coil and also (*ii*) the resistance and (*iii*) the inductance of a single coil which will take the same total current at the same power factor as the two coils in parallel. **[(*i*) I_A = 18.28 A, I_B = 19.9 A (*ii*) 3.12 Ω (*iii*) 0.0137 H]** (*London Univ.*)
27. Two coils are connected in parallel across 200-V, 50-Hz mains. One coil takes 0.8 kW and 1.5 kVA and the other coil takes 1.0 kW and 0.6 kVAR. Calculate (*i*) the resistance and (*ii*) the reactance of a single coil which would take the same current and power as the original circuit. **[(*i*) 10.65 Ω (*ii*) 11.08 Ω]** (*City & Guilds, London*)
28. An a.c. circutis consists of two parallel branches, one (*A*) consisting of a coil, for which $R = 20$ Ω and $L = 0.1$ H and the other (*B*) consisting of a 40-Ω non-inductive resistor in series with 60-μF capacitor. Calculate (*i*) the current in each branch (*ii*) the line current (*iii*) the power, when the circuit is connected to 230-V mains having a frequency of 50 Hz. Calculate also (*iv*) the resistance and (*b*) the inductance of a single coil which will take the same current and power from the supply. **[(*i*) 6.15 A, 3.46 A (*ii*) 5.89 (*iii*) 1235 W (*iv*) 35.7 Ω (*b*) 0.0509 H]** (*London Univ.*)
29. One branch (*A*) of a parallel circuit, connected to 230-V, 50-Hz mains consists of an inductive coil ($L = 0.15$ H, $R = 40$ Ω) and the other branch (*B*) consists of a capacitor ($C = 50$ μF) in series with a 45

Ω resistor. Determine (*i*) the power taken (*ii*) the resistance and (*iii*) the reactance of the equivalent series circuit. **[(*i*) 946 W (*ii*) 55.4 Ω (*iii*) 4.6 Ω]** (*London Univ.*)

14.9. Resonance in Parallel Circuits

We will consider the practical case of a coil in parallel with a capacitor, as shown in Fig. 14.48. Such a circuit is said to be in electrical resonance when the reactive (or wattless) component of line current becomes zero. The frequency at which this happens is known as *resonant* frequency.

The vector diagram for this circuit is shown in Fig. 14.48 (*b*).

Net reactive or wattless component $= I_C - I_L \sin \phi_L$

As at resonance, its value is zero, hence

$I_C - I_L \sin \phi_L = 0$ or $I_L \sin \phi_L = I_C$

Now, $I_L = V/Z$; $\sin \phi_L = X_L$ and $I_C = V/X_C$

Hence, condition for resonance becomes

$$\frac{V}{Z} \times \frac{X_L}{Z} = \frac{V}{X_C} \quad \text{or} \quad X_L \times X_C = Z^2$$

Now, $X_L = \omega L$, $X_C = \dfrac{1}{\omega C}$

$$\therefore \quad \frac{\omega L}{\omega C} = Z^2 \quad \text{or} \quad \frac{L}{C} = Z^2 \; \ldots (i)$$

$$\text{or} \quad \frac{L}{C} = R^2 + X_L^{\,2} = R^2 + (2\pi f L)^2$$

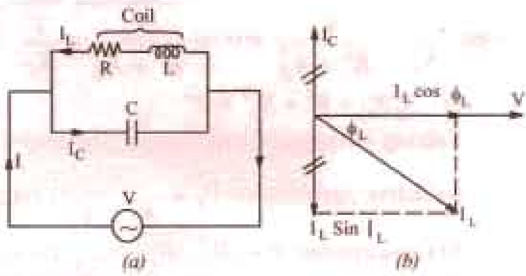


Fig. 14.48

$$\text{or} \qquad (2\pi f_0 L)^2 = \frac{L}{C} - R^2 \quad \text{or} \quad 2\pi f_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} \quad \text{or} \quad f_0 = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

This is the resonant frequency and is given in Hz in R is in ohm, L is the henry and C is the farad.

If R is the negligible, then $f_0 = \dfrac{1}{2\pi\sqrt{(LC)}}$

... same as for series resonance

Current at Resonance

As shown in Fig. 14.41 (*b*), since wattless component of the current is zero, the circuit current is

$$I = I_L \cos \phi_L = \frac{V}{Z} - \frac{R}{Z} \text{ or } I = \frac{VR}{Z^2}.$$

Putting the value of $Z^2 = L/C$ from (*i*) above, we get $I = \dfrac{VR}{L/C} = \dfrac{V}{L/CR}$

The denominator L/CR is known as the *equivalent* or *dynamic impedance* of the parallel circuit at resonance. It should be noted that impedance is 'resistive' only. Since current is minimum at resonance, L/CR must, therefore, represent the maximum impedance of the circuit. In fact, parallel resonance is a condition of maximum impedance or minimum admittance.

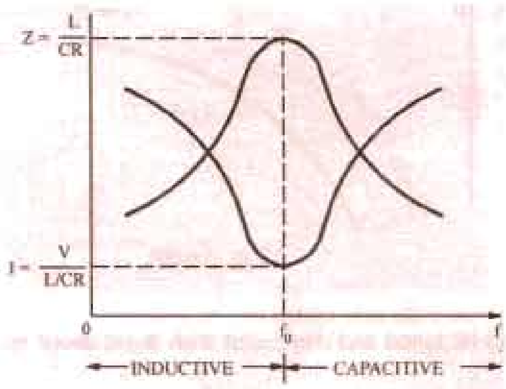


Fig. 14.49

Current at resonance is minimum, hence such a circuit (when used in radio work) is sometimes known as *rejector* circuit because it rejects (or takes minimum current of) that frequency to which it resonates. This resonance is often referred to as current resonance also because the current circulating *between* the two branches is many times greater than the line current taken from the supply.

The phenomenon of parallel resonance is of great practical importance because it forms the basis of tuned circutis in Electronics.

The variations of impedance and current with frequency are shown in Fig. 14.49. As seen, at resonant frequency, impedance is maximum and equals L/CR. Consequently, current at resonance is minimum and is $= V / (L/CR)$. At off-resonance frequencies, impedance decreases and, as a result, current increases as shown.

Alternative Treatment

$$\mathbf{Y}_1 = \frac{1}{R + jX_L} = \frac{R}{R^2 + X_L^2} - j\frac{X_L}{R^2 + X_L^2}\ ;\ \mathbf{Y}_2 = \frac{1}{-jX_C} = \frac{j}{X_C}$$

$$\mathbf{Y} = \frac{R}{R^2 + X_L^2} + j\left(\frac{1}{X_C} - \frac{X_L}{R^2 + X_L^2}\right)$$

Now, circuit would be in resonance when j-component of the complex admittance is zero *i.e.* when $\dfrac{1}{X_C} - \dfrac{X_L}{R^2 + X_L^2} = 0$ or $\dfrac{X_L}{R^2 + X_L^2} = \dfrac{1}{X_C}$

or $X_L X_C = R^2 + X_L{}^2 = Z^2$ –as before

Talking in terms of susceptance, the above relations can be put as under :

Inductive susceptance $B_L = \dfrac{X_L^2}{R^2 + X_L^2}$; capacitive susceptance $B_C = \dfrac{1}{X_C}$

Net susceptance $B = (B_C - B_L)$ $\therefore$ $Y = G + j(B_C - B_L) = G + jB$.

The parallel circuit is said to be in resonance when $B = 0$.

$$\therefore \qquad B_C - B_L = 0 \quad \text{or} \quad \frac{1}{X_C} = \frac{X_L}{R^2 + X_L^2}$$

The rest procedure is the same as above. It may be noted that at resonance, the admittance equals the conductance.

14.10. Graphic Representation of Parallel Resonance

We will now discuss the effect of variation of frequency on the susceptance of the two parallel branches. The variations are shown in Fig. 14.50.

(*i*) **Inductive susceptance** ; $b = -1/X_L = -1/2\pi f L$

It is inversely proportional to the frequency of the applied voltage. Hence, it is represented by a rectangular hyperbola drawn in the fourth quadrant ($\because$ it is assumed negative).

(*ii*) **Capacitive susceptance** ; $b = 1/X_C = \omega C = 2\pi f C$

It increases with increases in the frequency of the applied voltage. Hence, it is represented by a straight line drawn in the first quadrant (it is assumed positive).

(*iii*) **Net Susceptance B**

It is the difference of the two susceptances and is represented by the dotted hyperbola. At point A, net susceptance is zero, hence admittance is minimum (and equal to G). So at point A, line current is minimum.

Obviously, below resonant frequency (corresponding to point A), inductive susceptance predominates, hence line current lags behind the applied voltage. But for frequencies above the resonant frequency, capacitive susceptance predominates, hence line current leads.3

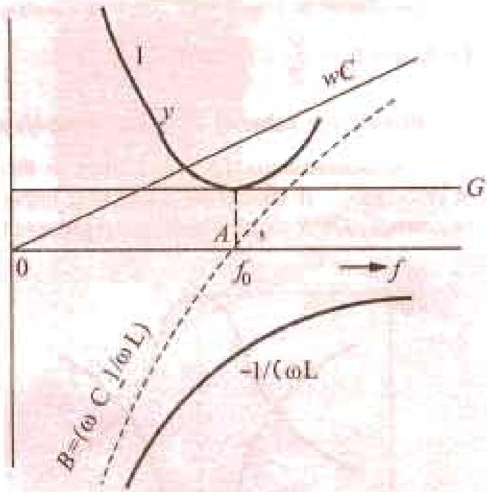


Fig. 14.50

14.11. Points to Remember

Following points about parallel resonance should be noted and compared with those about series resonance. At resonance.

1. net susceptance is zero *i.e.* $1/X_C = X_L/Z^2$ or $X_L \times X_C = Z^2$ or $L/C = Z^2$
2. the admittance equals conductance

3. reactive or wattless component of line current is zero.
4. dynamic impedance = L/CR ohm.
5. line current at resonance is minimum and = $\dfrac{V}{L/CR}$ but is in phase with the applied voltage.
6. power factor of the circuit is unity.

14.12. Bandwidth of a Parallel Resonant Circuit

The bandwidth of a parallel circuit is defined in the same way as that for a series circuit. This circuit also has upper and lower half-power frequencies where power dissipated is half of that at resonant frequency.

At bandwidth frequencies, the net susceptance B equals the conductance. Hence, at f_2, $B = B_{C2} - B_{L2} = G$. At f_1, $B = B_{L1} - B_{C1} = G$. Hence, $Y = \sqrt{G^2 + B^2} = \sqrt{2}.G$ and $\phi = \tan^{-1}(B/G) = \tan^{-1}(1) = 45°$.

However, at off-resonance frequencies, $Y > G$ and $B_C \neq B_L$ and the phase angle is greater than zero.

Comparison of Series and Parallel Resonant Circuits

item	*series circuit (R–L–C)*	*parallel circuit (R–L and C)*
Impedance at resonance	Minimum	Maximum
Current at resonance	Maximum = V/R	Minimum = $V/(L/CR)$
Effective impedance	R	L/CR
Power factor at resonance	Unity	Unity
Resonant frequency	$1/2\pi\sqrt{(LC)}$	$\dfrac{1}{2\pi}\sqrt{\left(\dfrac{1}{LC} - \dfrac{R^2}{L^2}\right)}$
It magnifies	Voltage	Current
Magnification is	$\omega L/R$	$\omega L/R$

14.13. Q-factor of a Parallel Circuit

It is defined as *the ratio of the current circulating between its two branches to the line current drawn from the supply or simply, as the current magnification.* As seen from Fig. 14.51, the circulating current between capacitor and coil branches is I_C

Hence Q-factor = I_C/I

Now $\quad I_C = V/X_C = V/(1/\omega C) = \omega CV$

and $\quad I_C = V/(L/CR)$

$$\therefore Q - \text{factor} = \omega CV \div \frac{V}{L/CR} = \frac{\omega L}{R} = \frac{2\pi f\,L}{R}$$

$= \tan\phi$ (same as for series circuit)

where ϕ is the power factor angle of the *coil.*

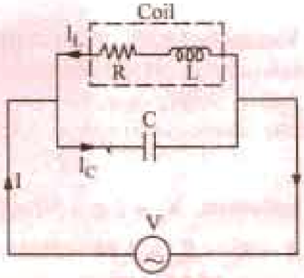


Fig. 14.51

Now, resonant frequency when R is negligible is,

$$f_0 = \frac{1}{2\pi\sqrt{(LC)}}$$

Putting this value above, we get, Q-factor $= \dfrac{2\pi L}{R} \cdot \dfrac{1}{2\pi\sqrt{(LC)}} = \dfrac{1}{R}\sqrt{\dfrac{L}{C}}$

It should be noted that in series circuits, Q-factor gives the voltage magnification, whereas in parallel circuits, it gives the current magnification.

Again, $\quad Q = 2\pi\,\dfrac{\text{maximum stored energy}}{\text{energy dissipated/cycle}}$

Example 14.45. *A capacitor is connected in parallel with a coil having L = 5.52 mH and R = 10 Ω, to a 100-V, 50-Hz supply. Calculate the value of the capacitance for which the current taken from the supply is in phase with voltage.* **(Elect. Machines, A.M.I.E. Sec B, 1992)**

Solution. At resonance, $L/C = Z^2$ or $C = L/Z^2$

$X_L = 2\pi \times 50 \times 5.52 \times 10^{-3} = 1.734\ \Omega$, $Z^2 = 10^2 + 1.734^2 = 10.1\ \Omega$

$C = 5.52 \times 10^{-3}/10.1 =$ **54.6 μF**

Example 14.46. *Calculate the impedance of the parallel-turned circuit as shown in Fig. 14.52 at a frequency of 500 kHz and for bandwidth of operation equal to 20 kHz. The resistance of the coil is 5 Ω.* **(Circuit and Field Theory, A.M.I.E. Sec. B, 1993)**

Solution. At resonance, circuit impedance is L/CR. We have been given the value of R but that of L and C has to be found from the given data.

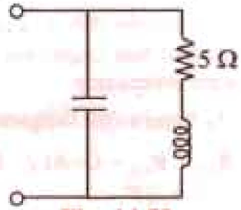


Fig. 14.52

$$BW = \frac{R}{2\pi L}\ 20 \times 10^3 = \frac{5}{2\pi \times L} \quad \text{or} \quad L = 39\ \mu\text{H}$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} = \frac{1}{2\pi}\sqrt{\frac{1}{39 \times 10^{-6} C} - \frac{5^2}{(39 \times 10^{-6})^2}}$$

$\therefore \quad C = 2.6 \times 10^{-9}$ F, $Z = L/CR = 39 \times 10^{-6}/2.6 \times 10^{-9} \times 5 =$ **$3 \times 10^3\ \Omega$**

Example 14.47. *An inductive circuit of resistance 2 ohm and inductance 0.01 H is connected to a 250-V, 50-Hz supply. What capacitance placed in parallel will produce resonance ?*

Find the total current taken from the supply and the current in the branch circuits.

(Elect. Engineering, Kerala Univ. 1987)

Solution. As seen from Art. 14.9, at resonance $C = L/Z^2$

Now, $\quad R = 2\ \Omega$, $X_L = 314 \times 0.01 = 3.14\ \Omega$; $Z = \sqrt{2^2 + 3.14^2} = 3.74\ \Omega$

$C = 0.01/3.74^2 = 714 \times 10^{-6}$ F = **714 μF** ; $I_{RL} = 250/3.74 =$ **66.83 A**

$\tan \phi_L = 3.14/2 = 1.57$; $\phi_L = \tan^{-1}(1.57) = 57.5°$

Hence, current in R-L branch lags the applied voltage by 57.5°

$$\therefore \quad I_C = \frac{V}{X_C} = \frac{V}{1/\omega C} = \omega VC = 250 \times 314 \times 714 \times 10^{-6} = \textbf{56.1 A}$$

This current leads the applied voltage by 90°.

Total current taken from the supply under resonant condition is

$I = I_{RL} \cos \phi_L = 66.83 \cos 57.5° = 66.83 \times 0.5373 =$ **35.9 A** $\left(\text{or} \quad I = \dfrac{V}{L/CR}\right)$

Example 14.48. *Find active and reactive components of the current taken by a series circuit consisting of a coil of inductance 0.1 henry and resistance 8 Ω and a capacitor of 120 μF connected to a 240-V, 50-Hz supply mains. Find the value of the capacitor that has to be connected in parallel with the above series circuit so that the p.f. of the entire circuit is unity.*

(Elect. Technology, Mysore Univ. 1986)

Solution. $X_L = 2\pi \times 50 \times 0.1 = 31.4\ \Omega$, $X_C = I/\omega C = 1/2\pi \times 50 \times 120 \times 10^{-6} = 26.5\ \Omega$

$X = X_L - X_C = 3.14 - 26.5 = 5\ \Omega$, $Z = \sqrt{(8^2 + 5^2)} = 9.43\ \Omega$; $I = V/Z = 240/9.43 = 25.45$ A

$\cos \phi = R/Z = 8/9.43 = 0.848$, $\sin \phi = X/Z = 5/9.43 = 0.53$

active component of current $= I \cos \phi = 25.45 \times 0.848 = 21.58$ A

reactive component of current $= I \sin \phi = 25.45 \times 0.53 = 13.49$ A

Let a capacitor of capacitance C be joined in parallel across the circuit.

$$\mathbf{Z}_1 = R + jX = 8 + j5 \ ; \ \mathbf{Z}_2 = -jX_C \ ;$$

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} = \frac{1}{8 + j5} + \frac{1}{-jX_C}$$

$$= \frac{8 - j5}{89} + \frac{j}{X_C} = 0.0899 - j0.056 + \frac{j}{X_C} \ ; = 0.0899 + j\,(1/X_C - 0.056)$$

For p.f. to be unit , the j-component of $\mathbf{Y}$ must be zero.

$\therefore \quad \dfrac{1}{X_C} - 0.056 = 0$ or $1/X_C = 0.056$ or $\omega C = 0.056$ or $2\pi \times 50C = 0.056$

$\therefore \quad C = 0.056/100\pi = 180 \times 10^{-6}$ F = **180 μF**

Example 14.49. *A coil of resistance 20 Ω and inductance 200 μH is in parallel with a variable capacitor. This combination is in series with a resistor of 8000 Ω. The voltage of the supply is 200 V at a frequency of 10^6 Hz. Calculate*

(i) the value of C to give resonance *(ii) the Q of the coil*

(iii) the current in each branch of the circuit at resonance. Similar

(Similar Question : Bombay Univ. 2000)

Solution. The circuit is shown in Fig. 14.53.

$X_L = 2\pi fL = 2\pi \times 10^6 \times 200 \times 10^{-6} = 1256\ \Omega$

Since coil resistance is negligible as compared to its reactance, the resonant frequency is given by

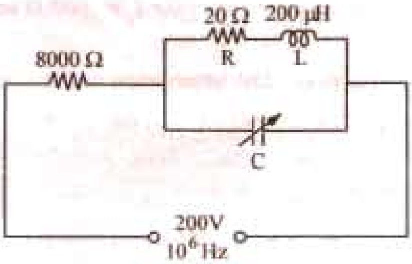


Fig. 14.53

$$f = \frac{1}{2\pi\sqrt{LC}}$$

$$\therefore \quad 10^6 = \frac{1}{2\pi\sqrt{200\times10^{-6}\times C}}$$

(*i*) $\therefore \quad C =$ **125 μF**

(*ii*) $Q = \dfrac{2\pi f L}{R} = \dfrac{2\pi\times10^6\times200\times10^{-4}}{20} =$ **62.8**

(*iii*) Dynamic resistance of the circuit is $= \dfrac{L}{CR} = \dfrac{200\times10^{-6}}{125\times10^{-12}\times20} = 80{,}000\ \Omega$

Total equivalent resistance of the tuned circuit is 80,000 + 8,000 = 88,000 Ω

$\therefore$ Current $\quad I = 200/88{,}000 = 2.27$ mA

p.d. across tuned circuit = current × dynamic resistance = $2.27 \times 10^{-3} \times 80{,}000 = 181.6$ V

Current through inductive branch $= \dfrac{181.6}{\sqrt{10^2 + 1256^2}} = 0.1445$ A = **144.5 mA**

Current through capacitor branch

$$= \frac{V}{1/\omega C} = \omega VC = 181.6 \times 2\pi \times 10^6 \times 125 \times 10^{-12} = \textbf{142.7 mA}$$

Note. It may be noted in passing that current in each branch is nearly 62.8 (*i.e.* Q-factor) times the resultant current taken from the supply.

Example 14.50. *Impedances Z_2 and Z_3 in parallel are in series with and impedance Z_1 across a 100-V, 50-Hz a.c. supply. $Z_1 = (6.25 + j\,1.25)$ ohm ; $Z_2 = (5 + j0)$ ohm and $Z_3 = (5 - j\,X_C)$ ohm. Determine the value of capacitance of X_C such that the total current of the circuit will be in phase with the total voltage. When is then the circuit current and power ?*

(Elect. Engg-I, Nagpur Univ, 1992)

Solution. $\mathbf{Z}_{23} = \dfrac{5\,(5 - jX_C)}{(10 - jX_C)}$, for the circuit in Fig. 14.59.

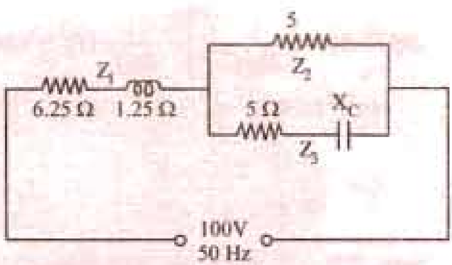


Fig. 14.54

$$= \frac{25 - j5X_C}{(10 - jX_C)} \times \frac{10 + jX_C}{10 + jX_C} = \frac{250 + 5X_C^2}{100 + X_C^2} - j\,\frac{25X_C}{100 + X_C^2}$$

$$\mathbf{Z} = 6.25 + j\,1.25 + \frac{250 + 5X_C^2}{100 + X_C^2} - j\,\frac{25X_C}{100 + X_C^2}$$

$$= \left(6.25\,\frac{250 + 5X_C^2}{100 + X_C^2}\right) - j\left(\frac{25X_C}{100 + X_C^2} - \frac{5}{4}\right)$$

Power factor will be unity or circuit current will be in phase with circuit voltage if the j term in the above equation is zero.

$$\therefore \quad \left(\frac{25X_C}{100+X_C^2}-\frac{5}{4}\right) = 0 \text{ or } X_C = 10 \quad \therefore \quad 1/\omega C = 10 \text{ or } C = 1/314 \times 10 = \mathbf{318\ \mu F}$$

Substituting the value of $X_C = 10\ \Omega$ above, we get

$$Z = 10 - j0 = 10\angle 0^\circ \text{ and } I = 100/10 = \mathbf{10\ A}\text{ ; Power } I^2R = 10^2 \times 10 = \mathbf{1000\ W}$$

Example 14.51. *In the circuit given below, if the value of $R = \sqrt{L/C}$, then prove that the impedance of the entire circuit is equal to R only and is independent of the frequency of supply. Find the value of impedance for L = 0.02 H and C = 100 μF.*

(Communication System, Hyderabad Univ. 1991)

Solution. The impedance of the circuit of Fig. 14.55 is

$$\mathbf{Z} = \frac{(R+j\omega L)(R-j/\omega C)}{2R+j(\omega L-1/\omega C)} = \frac{R^2+(L/C)+jR(\omega L-1/\omega C)}{2R+j(\omega L-1/\omega C)}$$

If $R^2 = L/C$ or $R = \sqrt{L/C}$, then

$$\mathbf{Z} = \frac{R^2+R^2+jR(\omega L-1/\omega C)}{2R+j(\omega L-1/\omega C)}$$

$$= R\left[\frac{2R+j(\omega L-1/\omega C)}{2R+j(\omega L-1/\omega C)}\right] \quad \text{or} \quad \mathbf{Z} = R$$

$$\text{Now, } R = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.02}{100\times10^{-6}}} = \mathbf{14.14\ \Omega}$$

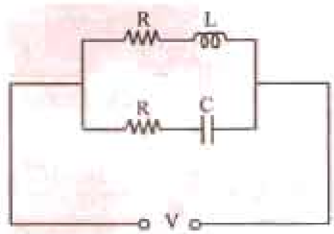


Fig. 14.55

Example 14.52. *Derive an expression for the resonant frequency of the parallel circuit shown in Fig. 14.46.* **(Electrical Circuit, Nagpur Univ, 1993)**

Solution. As stated in Art. 14.9 for resonance of a parallel circuit, total circuit susceptance should be zero. Susceptance of the R-L branch is

$$B_1 = -\frac{X_L}{R_1^2+X_L^2}$$

Similarly, susceptance of the R-C branch is

$$B_2 = \frac{X_C}{R_2^2+X_C^2}$$

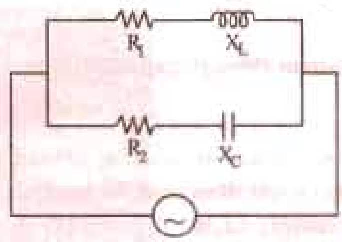


Fig. 14.56

Net susceptance is $B = -B_1 + B_2$

For resonance $B = 0$ or $0 = -B_1 + B_2 \ \therefore\ B_1 = B_2$

$$\text{or} \quad \frac{X_L}{R_1^2+X_L^2} = \frac{X_C}{R_2^2+X_C^2} \quad \text{or} \quad X_L(R_2^2+X_C^2) = X_C(R_1^2+X_L^2)$$

$$2\pi fL\left[R_2^2+\left(\frac{1}{2\pi fC}\right)^2\right] = \frac{1}{2\pi fC}[R_1^2+(2\pi fL)^2]\text{ ; } 4\pi^2f^2LC\left[R_2^2\left(\frac{1}{2\pi fC}\right)^2\right] = [R_1^2+(2\pi fL)^2]$$

$$\therefore \quad 4\pi^2f^2LCR_2^2+\frac{L}{C} = R_1^2+4\pi^2f^2L^2\text{ ; } 4\pi^2f^2[L(L-CR_2^2)] = \frac{L}{C}-R_1^2$$

$$\therefore \quad f_0 = \frac{1}{2\pi}\sqrt{\left(\frac{L/C-R_1^2}{L(L-CR_2^2)}\right)} \quad \therefore \quad f_0 = \frac{1}{2\pi}\sqrt{\left(\frac{L-CR_1^2}{LC(L-CR_2^2)}\right)}\text{ ; } \omega_0 = \frac{1}{\sqrt{LC}}\sqrt{\left(\frac{L-CR_1^2}{L-CR_2^2}\right)}$$

Note. If both R_1 and R_2 are negligible, then $f_0 = \dfrac{1}{2\pi\sqrt{LC}}$ —as in Art. 14.9

Example 14.53. *Calculate the resonant frequency of the network shown in Fig. 14.57.*

Solution. Total impedance of the network between terminals A and B is

$$Z_{AB} = (R_1 \| jX_L) + [R_2 \| (-jX_C)] = \frac{jR_1X_L}{R_1 + jX_L} + \frac{R_2(-jX_C)}{R_2 - jX_C} = \frac{jR_1\omega L}{R_1 + j\omega L} - \frac{jR_2/\omega C}{R_2 - j/\omega C}$$

$$= \frac{R_1\omega^2L^2}{R_1^2 + \omega^2L^2} + \frac{R_2}{\omega C(R_2^2 + 1/\omega^2C^2)} + j\left[\frac{R_1^2\omega L}{R_1^2 + \omega^2L^2} - \frac{R_2^2}{\omega C(R_2^2 + 1/\omega^2C^2)}\right]$$

At resonance, $\omega = \omega_0$ and the j term of Z_{AB} is zero.

$$\therefore \frac{R_1^2\omega_0 L}{R_1^2 + \omega_0^2L^2} - \frac{R_2^2}{\omega_0 C(R_2^2 + 1/\omega_0^2C^2)} = 0$$

$$\text{or } \frac{R_1^2\omega_0 L}{R_1^2 + \omega_0^2L^2} = \frac{R_2^2\omega_0 C}{R_2^2\omega_0^2C^2 + 1}$$

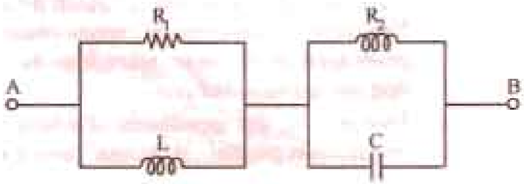


Fig. 14.57

Simplifying the above, we get

$$\omega_0^2 = \frac{G_2^2 - C/L}{LC(G_1^2 - C/L)} \text{ where } G_1 = \frac{1}{R_1} \text{ and } G_2 = \frac{1}{R_2}$$

The resonant frequency of the given network in Hz is

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\left[\frac{G^2 - C/L}{LC\,(G_1^2 - C/L)}\right]}$$

Example 14.54. *Compute the value of C which results in resonance for the circuit shown in Fig. 14.58 when $f = 2500/\pi$ Hz.*

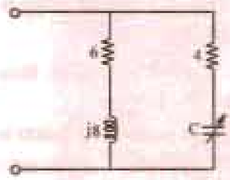


Fig. 14.58

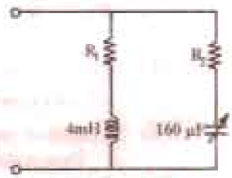


Fig. 14.59

Solution. $\mathbf{Y}_1 = 1/(6 + j8)$

$\mathbf{Y}_2 = 1\ (4 - jX_C)$

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 = \frac{1}{6 + j8} = \frac{1}{4 - jX_C}$$

$$= \left(0.06 + \frac{4}{16 + X_C^2}\right) + j\left(\frac{X_C}{16 + X_C^2} - 0.08\right)$$

For resonance, j part of admittance is zero, *i.e.* the complex admittance is real number.

$\therefore\ X_C/(16 + X_C^2) - 0.08 = 0$ or $0.08\,X_C^2 - X_C + 1.28 = 0$

$\therefore\ X_C = 11.05$ or 1.45 $\therefore\ 1/\omega C = 11.05$ or 1.45

(*i*) $1/5000C = 11.05$ or $C =$ **18 μF** (*ii*) $1/5000\,C = 1.45$ or $C =$ **138 μF**

Example 14.55. *Find the values of R_1 and R_2 which will make the circuit of Fig. 14.59 resonate at all frequencies.*

Solution. As seen from Example 14.42. the resonant frequency of the given circuit is

$$\omega_0 = \frac{1}{\sqrt{LC}}\sqrt{\left(\frac{L - CR_1^2}{L - CR_2^2}\right)}$$

Now, ω_0 can assume any value provided $R_1^2 = R_2^2 = L/C$.

In the present case, $L/C = 4 \times 10^{-3}/60 \times 10^{-6} = 25$. Hence, $R_1 = R_2 = \sqrt{25} =$ **5 ohm.**

Tutorial Problem No. 14.2

1. A resistance of 20 W and a coil of inductance 31.8 mH and negligible resistance are connected in parallel across 230 V, 50 Hz supply. Find (*i*) The line current (*ii*) power factor and (*iii*) The power consumed by the circuit. **[(*i*) 25.73 A (*ii*) 0.44 T lag (*iii*) Z 246 W]** (*F. E. Pune Univ. May 1989*)
2. Two impedances $Z_1 = (150 + j157)$ ohm and $Z_2 = (100 + j\,110)$ ohm are connected in parallel across a 220-V, 50-Hz supply. Find the total current and its power factor.

[24 ∠ − 47°A ; 0.68 (lag)] (*Elect. Engg. & Electronics Bangalore Univ. 1988*)

3. Two impedances $(14 + j5)\Omega$ and $(18 + j10)\ \Omega$ are connected in parallel across a 200-V, 50-Hz supply. Determine (*a*) the admittance of each branch and of the entire circuit ; (*b*) the total current, power, and power factor and (*c*) the capacitance which when connected in parallel with the original circuit will make the resultant power factor unity. **[(*a*) (0.0634 – j0.0226), (0.0424 – j0.023) (0.1058 – j0.0462 S) (*b*) 23.1 A, 4.232 kW, 0.915 (*c*) 147 μF]**
4. A parallel circuit consists of two branches *A* and *B*. Branch A has a resistance of 10 Ω and an inductance of 0.1 H in series. Branch *B* has a resistance of 20 Ω and a capacitance of 100 μF in series. The circuit is connected to a single-phase supply of 250 V, 50 Hz. Calculate the magnitude and the phase angle of the current taken from the supply. Verify your answer by measurement from a phasor diagram drawn to scale. **[6.05 $\angle -15.2°$]** (*F. E. Pune Univ. Nov. 1989*)
5. Two circuits, the impedances of which are given by $Z_1 = (10 + j15)\ \Omega$ and $Z_2 = (6 - j8)\ \Omega$ are connected in parallel. If the total current supplied is 15 A, what is the power taken by each branch ? **[737 W ; 1430 W]** (*Elect. Engg. A.M.A.E. S.I. June 1989*)
6. A voltage of 240 V is applied to a pure resistor, a pure capacitor, and an inductor in parallel. The resultant current is 2.3 A, while the component currents are 1.5, 2.0 and 1.1 A respectively. Find the resultant power factor and the power factor of the inductor. **[0.88 ; 0.5]**
7. Two parallel circuits comprise respectively (*i*) a coil of resistance 20 Ω and inductance 0.07 H and (*ii*) a capacitance of 60 μF in series with a resistance of 50 Ω. Calculate the current in the mains and the power factor of the arrangement when connected across a 200-V, 50-Hz supply. **[7.05 A ; 0.907 lag]** (*Elect. Engg. & Electronics, Bangalore Univ. 1987*)
8. Two circuits having the same numerical ohmic impedance are joined in parallel. The power factor of one circuit is 0.8 lag and that of other 0.6 lag. Find the power factor of the whole circuit. **[0.707]** (*Elect. Engg. Pune Univ. 1988*)
9. How is a current of 10 A shared by three circuits in parallel, the impedances of which are $(2 - j5)\ \Omega$, $(6 + j3)\Omega$ and $(3 + j4)\ \Omega$. **[5.68 A ; 4.57 A, 6.12 A]**
10. A piece of equipment consumes 2,000 W when suppled with 110 V and takes a lagging current of 25 A. Determine the equivalent series resistance and reactance of the equipment.

    If a capacitor is connected in parallel with the equipment to make the power factor unity, find its capacitance. The supply frequency is 100 Hz. **[3.2 Ω, 3.02 Ω, 248 μF]** (*Sheffield Univ. U.K.*)
11. A capacitor is placed in parallel with two inductive loads, one of 20 A at 30° lag and one of 40° A at 60° lag. What must be current in the capacitor so that the current from the external circuit shall be at unity power factor ? **[44.5 A]** (*City & Guilds, London*)
12. An air-cored choking coil is subjected to an alternating voltage of 100 V. The current taken is 0.1 A and the power factor 0.2 when the frequency is 50 Hz. Find the capacitance which, if placed in parallel with the coil, will cause the main current to be a minimum. What will be the impedance of this parallel combination (*a*) for currents of frequency 50 (*b*) for currents of frequency 40 ? **[3.14 μF (*a*) 5000 Ω (*b*) 1940 Ω]** (*London Univ.*)
13. A circuit, consisting of a capacitor in series with a resistance of 10 Ω, is connected in parallel with a coil having $L = 55.2$ mH and $R = 10\ \Omega$, to a 100-V, 50-Hz supply. Calculate the value of the capacitance for which the current take from the supply is in phase with the voltage. Show that for the particular values given, the supply current is independent of the frequency. **[153 μF]** (*London Univ.*)
14. In a series-parallel circuit, the two parallel branches *A* and *B* are in series with *C*. The impedances are $Z_A = (10 - j8)\ \Omega$, $Z_B = (9 - j6)\ \Omega$ and $Z_C = (100 + j0)$. Find the currents I_A and I_B and the phase difference between them. Draw the phasor diagram. **[$I_A = 12.71 \angle -30°58'$ $I_B = 15\angle -35°56'$; 4°58']** (*Elect. Engg. & Electronics Bangalore Univ. 1985*)
15. For the series-parallel circuit shown in Fig. 14.60 calculate (*a*) impedance between points *B* and *C* (*b*) total impedance of the circuit between points *A* and *C* and (*c*) the circuit current. **[(*a*) (0.57 – j0.25) Ω; (*b*) (0.97 + j 0.55) Ω, (*c*) (77.8 – j44.6)A, 89.7 $\angle -29.8$ A]**
16. Calculate the value of the current flowing in the series-parallel circuit of Fig. 14.62. **[(2.02 – j3.07)A ; 3.68 $\angle -56.54°$ A]**

17. Calculate the amount of power developed in each arm of the series parallel circuit shown in Fig. 14.61. **[zero, 400 W, 400 W]**

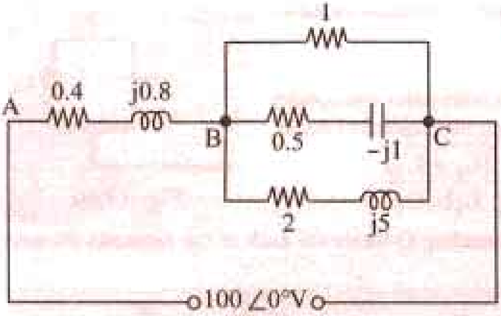


Fig. 14.60

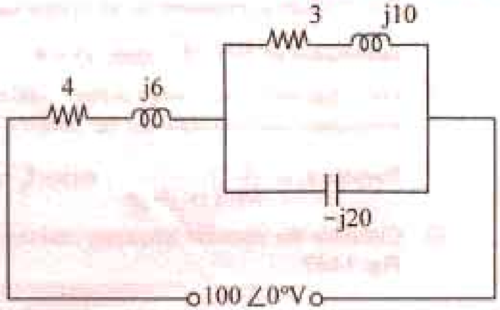


Fig. 14.61

18. Calculate the power developed in each branch series of the parallel circuit shown in Fig. 14.63. **[238.4 W ; 205.7 W ; zero]**

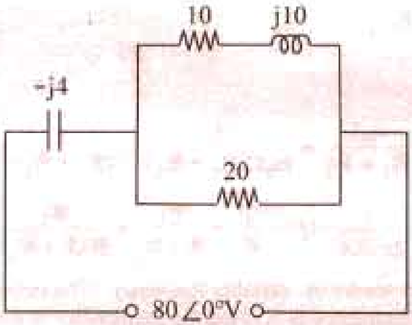


Fig. 14.62

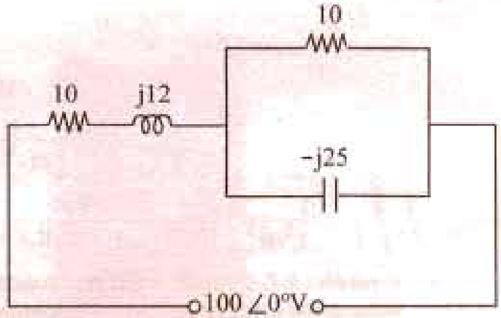


Fig. 14.63

19. Find the equivalent series circuits of the 4-branch parallel circuit shown in Fig. 14.64. **[(4.41 + j2.87) Ω]** [A resistor of 4.415 Ω is series with a 4.57 mH inductor]

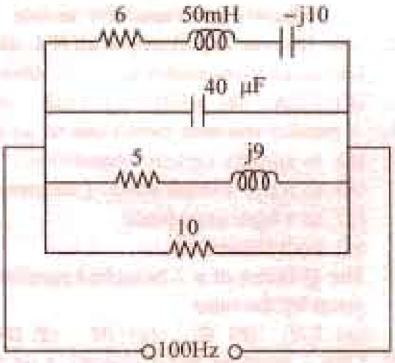


Fig. 14.64

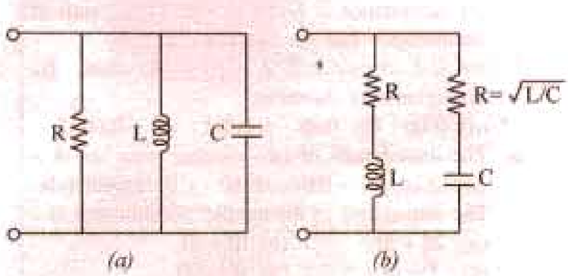


Fig. 14.65

20. A coil of 20 Ω resistance has an inductance of 0.2 H and is connected in parallel with a 100-μF capacitor. Calculate the frequency at which the circuit will act as a non-inductive resistance of *R* ohms. Find also the value of *R*. **[31.8 Hz; 100 Ω]**

21. Calculate the resonant frequency, the impedance at resonance and the Q-factor at resonance for the two circuits shown in Fig. 14.65.

(a) $f_0 = \dfrac{1}{2\pi\sqrt{LC}}$; $Z_0 = R$; $Q_0 = \dfrac{R}{\sqrt{L/C}}$

(b) **Circuit is resonant at all frequencies with a constant resistive impedance of $(\sqrt{L/C})$ ohm, Q = 0.]**

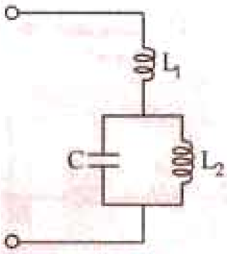


Fig. 14.66

22. Prove that the circuit shown in Fig. 14.66 exhibits both series and parallel resonances and calculate the frequencies at which two resonaces occur.

Parallel $f_0 = \dfrac{1}{2\pi}\sqrt{\dfrac{1}{(L_2C_2)}}$; **series** $f_0 = \dfrac{1}{2\pi}\sqrt{\dfrac{(L_1+L_2)}{L_1L_2C_2}}$

23. Calculate the resonant frequency and the corresponding Q-factor for each of the networks shown in Fig. 14.67.

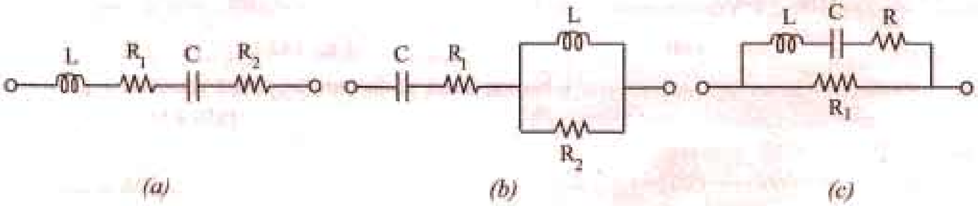


Fig. 14.67

$$\left[(a)\ f_0 = \frac{1}{2\pi\sqrt{LC}};\ Q = \frac{\omega_0 L}{R_1+R_2} = \frac{1}{\omega_0 C(R_1+R_2)} = \frac{1}{(R_1+R_2)}\cdot\sqrt{\left(\frac{L}{C}\right)}\right.$$

$$(b)\ f_0 = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC - L^2/R_2^2}\right)};\ Q = \frac{R_2}{\omega_0(L+R_1R_2C)}\quad (c)\ f_0 = \frac{1}{2\pi\sqrt{LC}};\ Q = \frac{\omega_0 L}{R}\times\frac{R_1}{R+R_1} = \frac{R_1}{R(R+R_1)}\cdot\left.\sqrt{\frac{L}{C}}\right]$$

24. A parallel *R-L-C* circuit is fed by a constant current source of variable frequency. The circuit resonates at 100 kHz and the *Q*-factor measured at this frequency is 5. Find the frequencies at which the amplitude of the voltage across the circuit falls to (*a*) 70.7% (*b*) 50% of the resonant frequency amplitude. **[(*a*) 90.5 kHz ; 110.5 kHz (*b*) 84.18 kHz ; 118.8 kHz]**

OBJECTIVE TYPES – 14

1. Fill in the blanks
   (*a*) unit of admittance is ....
   (*b*) unit of capacitive susceptance is ....
   (*c*) admittance equals the reciprocal of ....
   (*d*) admittance is given by the .... sum of conductance and susceptance.
2. An *R-L* circuit has **Z** = (6 + *j*8) ohm. Its susceptance is -Siemens.
   (*a*) 0.06 (*b*) 0.08 (*c*) 0.1 (*d*) –0.08
3. The impedances of two parallel branches of a circuit are (10 + *j*10) and (10 – *j*10) respectively. The impedance of the parallel combination is
   (*a*) 20 + *j*0 (*b*) 10 + *j*0
   (*c*) 5 – *j*5 (*d*) 0 – *j*20
4. A parallel ac circuit in resonance will
   (*a*) act like a resistor of low value
   (*b*) have a high impedance
   (*c*) have current in each section equal to the line current
   (*d*) have a high voltage developed across each inductive and capacitive section.
5. The dynamic impedance of an *R-L* and *C* parallel circuit at resonance is .... ohm.
   (*a*) *C/LR* (*b*) *L/CR* (*c*) *LC/R* (*d*) *R/LC*
6. A parallel resonant circuit can be used
   (*a*) to amplify certain frequencies
   (*b*) to reject a small band of frequencies
   (*c*) as a high impedance
   (*d*) both (*b*) and (*c*).
7. The *Q*-factor of a 2-branched parallel circuit is given by the ratio
   (*a*) I_C/I_L (*b*) I/I_C (*c*) I/I_L (*d*) *L/C*
8. Like a resonant *R-L-C* circuit, a parallel resonant circuit also
   (*a*) has a power factor of unity
   (*b*) offers minimum impedance
   (*c*) draws maximum current
   (*d*) magnifies current.

ANSWERS

1. (*a*) Siemens (*b*) Siemens (*c*) impedance (*d*) vector **2.** *d* **3.** *b* **4.** *b* **5.** *b* **6.** *d* **7.** *b* **8.** *a*.

15 A.C. NETWORK ANALYSIS

15.1. Introduction

We have already discussed various d.c. network theorems in Chapter 2 of this book. The same laws are applicable to a.c. networks except that instead of resistances, we have impedances and instead of taking algebraic sum of voltages and currents we have to take the phasor sum.

15.2. Kirchhoff's Laws

The statements of Kirchhoff's laws are similar to those given in Art. 2.2 for d.c. networks except that instead of algebraic sum of currents and voltages, we take phasor or vector sums for a.c. networks.

1. ***Kirchhoff's Current Law.*** According to this law, in any elcetrical network, the phasor sum of the currents meeting at a junction is zero.

In other words, $\sum I = 0$ ...at a junction

Put in another way, it simply means that in any electrical circuit the phasor sum of the currents flowing towards a junction is equal to the phasor sum of the currents going away from that junction.

2. ***Kirchhoff's Voltage Law.*** According to this law, the phasor sum of the voltage drops across each of the conductors in any closed path (or mesh) in a network plus the phasor sum of the e.m.fs. connected in that path is zero.

In other words, $\sum IR + \sum \text{e.m.f.} = 0$ ...round a mesh

Example 15.1. *Use Kirchhoff's laws to find the current flowing in each branch of the network shown in Fig. 15.1.*

Solution. Let the current distribution be as shown in Fig. 15.1 (*b*). Starting from point *A* and applying KVL to closed loop *ABEFA*, we get

$$-10(x + y) - 20\,x + 100 = 0 \quad \text{or} \quad 3x + y = 10 \qquad ...(i)$$

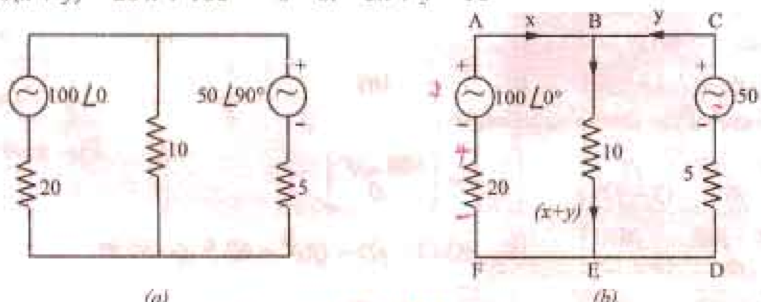


Fig. 15.1

Similarly, considering the closed loop *BCDEB* and starting from point *B*, we have

$$-50\,\angle 90^\circ + 5y + 10\,(x + y) = 0 \quad \text{or} \quad 2x + 3y = j10 \qquad ...(ii)$$

Multiplying Eq. (*i*) by 3 and subtracting it from Eq. (*ii*), we get

$$7x = 30 - j10 \ \text{ or } \ x = 4.3 - j1.4 = 4.52\ \angle{-18^\circ}$$

Substituting this value of *x* in Eq. (*i*), we have

$$y = 10 - 3x = 5.95 \angle 119.15^\circ = -2.9 + j5.2$$

$$\therefore \quad x + y = 4.3 - j1.4 - 2.9 + j5.2 = 1.4 + j3.8$$

Tutorial Problem No. 15.1

1. Using Kirchhoff's Laws, calculate the current flowing through each branch of the circuit shown in Fig. 15.2 **[I = 0.84 ∠47.15° A: I$_1$ = 0.7 ∠ − 88.87° A: I$_2$ = 1.44 ∠ 67.12°A]**

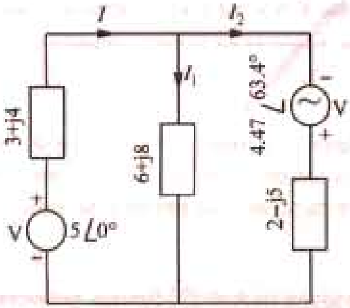


Fig. 15.2

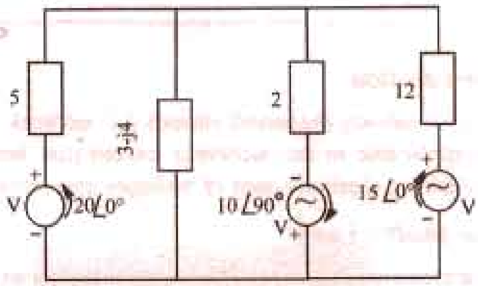


Fig. 15.3

2. Use Kirchhoff's laws to find the current flowing in the capacitive branch of Fig. 15.3 **[5.87 A]**

15.3. Mesh Analysis

It has already been discussed in Art. 2.3. Sign convention regarding the voltage drops across various impedances and the e.m.f.s is the same as explained in Art. 2.3. The circuits may be solved with the help of KVL or by use of determinants and Cramer's rule or with the help of impedance matrix [Z$_m$].

Example 15.2. *Find the power output of the voltage source in the circuit of Fig. 15.4. Prove that this power equals the power in the circuit resistors.*

Solution. Starting from point A in the clockwise direction and applying KVL to the mesh *ABEFA*, we get.

$$-8\,I_1 - (-j6)\,(I_1 - I_2) + 100 \angle 0^\circ = 0$$

or $$I_1\,(8 - j6) + I_2 .\,(j6) = 100 \angle 0^\circ \quad ...(i)$$

Similarly, starting from point *B* and applying *KVL* to mesh *BCDEB*, we get

$$-I_2(3 + j4) - (-j6)\,(I_2 - I_1) = 0$$

or $$I_1\,(j6) + I_2\,(3 - j2) = 0 \quad ...(ii)$$

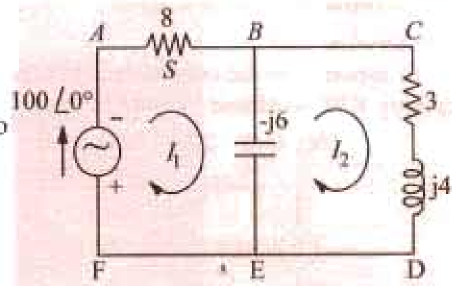


Fig. 15.4

The matrix form of the above equation is

$$\begin{bmatrix} (8-j6) & j6 \\ j6 & (3-j2) \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 100 \angle 0^\circ \\ -0 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} (8-j6) & j6 \\ j6 & (3-j2) \end{bmatrix} = (8 - j6)\,(3 - j2) - (j6)^2 = 62.5 \angle -39.8^\circ$$

$$\Delta_1 = \begin{vmatrix} 100\angle 0^\circ & j6 \\ 0 & (3-j2) \end{vmatrix} = (300 - j\,200) = 360 \angle -26.6^\circ$$

$$\Delta_2 = \begin{vmatrix} (8-j6) & 100\angle 0^\circ \\ j6 & 0 \end{vmatrix} = 600 \angle 90^\circ$$

$$I_1 = \frac{\Delta_1}{\Delta} = \frac{360\angle -26.6^\circ}{62.5\angle -39.8^\circ} = 5.76\angle 13.2^\circ \; ; \; I_2 = \frac{\Delta_2}{\Delta} = \frac{600\angle 90^\circ}{62.5\angle -39.8^\circ} = 9.6\angle 129.8^\circ$$

Example 15.3. *Using Maxwell's loop current method, find the value of current in each branch of the network shown in Fig. 15.5 (a).*

Solution. Let the currents in the two loops be I_1 and I_2 flowing in the clockwise direction as shown in Fig. 15.5 (*b*) Applying KVL to the two loops, we get

Loop No. 1

$$25 - I_1 (40 + j50) - (-j\,100)\,(I_1 - I_2) = 0$$

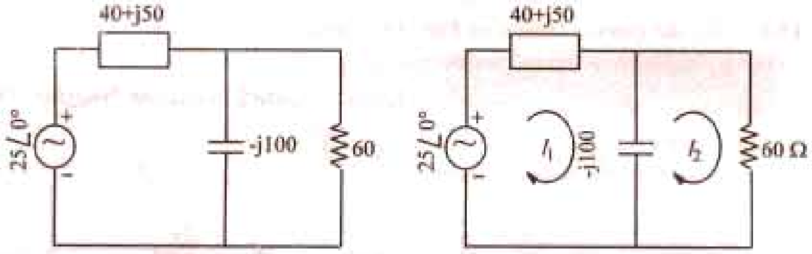
$$\therefore \quad 25 - I_1 (40 - j50) - j\,100\, I_2 = 0 \qquad \text{...}(i)$$

Loop No. 2

$$-\,60\, I_2 - (-j100)\,(I_2 - I_1) = 0$$

$$\therefore \quad -j100\, I_1 - I_2\,(60 - j100) = 0$$

$$\therefore \quad I_2 = \frac{-j100 I_1}{(60 - j100)} = \frac{100\angle{-90^\circ}\, I_1}{116.62\angle{-59^\circ}} = 0.8575\angle 31^\circ I_1 \qquad \text{...}(ii)$$



(*a*) (*b*)

Fig. 15.5

Substituting this value of I_2 in (*i*) above, we get $25 - I_1\,(40 - j50) - j100 \times 0.8575\,\angle 31^\circ\, I_1 = 0$

or $\quad 25 - 40\, I_1 + j50\, I_1 - 85.75\,\angle 59^\circ \quad I_1 = 0\ (j100 = 100\angle 90^\circ)$

or $\quad 25 - I_1\,(84.16 + j\,23.5) = 0.$

$$\therefore \quad I_1 = \frac{25}{(84.16 + j23.5)} = \frac{25}{87.38\angle 15.6^\circ}\ 0.286\angle{-15.6^\circ}\text{ A}$$

Also, $\quad I_2 = 0.8575\angle{-31^\circ}\, I_1 \times 0.286\angle{-15.6^\circ} = 0.2452\angle{-46.6^\circ}\text{A}$

Current through the capacitor $= (I_1 - I_2) = 0.286\angle{-15.6^\circ} - 0.2452\angle 46.6^\circ = 0.107 + j0.1013 = 0.1473\angle 43.43^\circ$ A.

Example 15.4. *Write the three mesh current equations for network shown in Fig. 15.6.*

Solution. While moving along I_1, if we apply KVL, we get

$$-(-j10)\, I_1 - 10(I_1 - I_2) - 5\,(I_1 - I_3) = 0$$

or $\quad I_1\,(15 - j10) - 10\, I_2 - 5 I_3 = 0 \qquad \text{...}(i)$

In the second loop, current through the a.c. source is flowing upwards indicating that its upper end is positive and lower is negative. As we move along I_2, we go from the positive terminal of the voltage source to its negative terminal. Hence, we experience a decrease in voltage which as per Art. would be taken as negative.

$$-j5\, I_2 - 10\angle 30^\circ - 8(I_2 - I_3) - 10\,(I_2 - I_1) = 0$$

or $\quad 10\, I_1 - I_2\,(18 + j5) + 8\, I_3 = 10\angle 30^\circ \quad \text{...}(ii)$

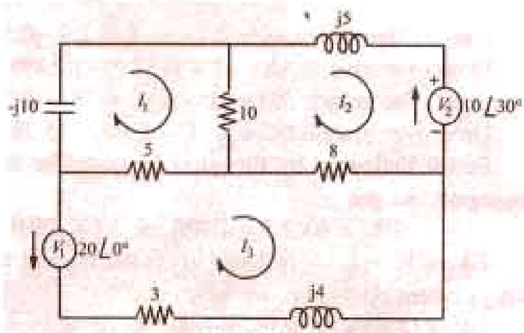


Fig. 15.6

Similarly, from third loop, we get

$$-20\angle 0^\circ - 5(I_3 - I_1) - 8\,(I_3 - I_2) - I_3\,(3 + j4) = 0$$

or $$5\,I_1 + 8\,I_2 - I_3\,(16 + j4) = 20\,\angle 0^\circ \qquad ...(iii)$$

The values of the three currents may be calculated with the help of Cramer's rule. However, the same values may be found with the help of mesh impedance $[Z_m]$ whose different items are as under :

$$Z_{11} = -j\,10 + 10 + 5 = (15 - j10);\ Z_{22} = (18 + j5)$$
$$Z_{33} = (16 + j5);\ Z_{12} = Z_{21} = -10;\ Z_{23} = Z_{32} = -8$$
$$Z_{13} = Z_{31} = -5;\ E_1 = 0;\ E_2 = -10\,\angle 30^\circ;\quad E_3 = -20\,\angle 0^\circ$$

Hence, the mesh equations for the three currents in the matrix form are as given below :

$$\begin{bmatrix} (15-j10) & -10 & -5 \\ -10 & (18+j5) & -8 \\ -5 & -8 & (16+j5) \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -10\angle 30^\circ \\ -20\angle 0^\circ \end{bmatrix}$$

Example 15.5. *For the circuit shown in Fig. 15.7 determine the branch voltage and currents and power delivered by the source using mesh analysis.*

(Elect. Network Analysis Nagpur Univ. 1993)

Solution. Let the mesh currents be as shown in Fig. 15.7. The different items of the mesh resistance matrix $[E_m]$ are :

$$Z_{11} = (2 + j1 + j2 - j1) = (2 + j2)$$
$$Z_{22} = (-j2 + 1 - j1 + j2) = (1 - j1)$$
$$Z_{12} = Z_{21} = -(j2 - j1) = -j1$$

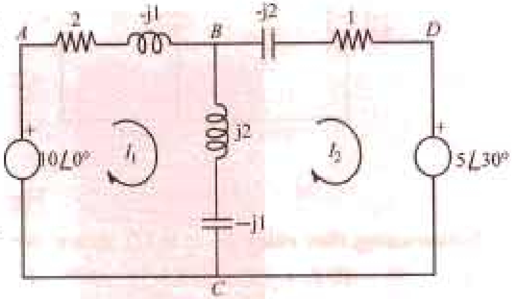


Fig. 15.7

Hence, the mesh equations in the matrix form are

$$\begin{bmatrix} (2+j2) & -j1 \\ -j1 & (1-j1) \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 10\angle 0^\circ \\ -5\angle 30^\circ \end{bmatrix}$$

$$\therefore\quad \Delta = (2 + j2)\,(1 - j1) + 1 = 5$$

$$\Delta_1 = \begin{bmatrix} 10 & -j1 \\ -(4.43 + j2.5 & (1-j1) \end{bmatrix} = 10(1 - j1) - j1(4.43 + j2.5) = 12.5 - j\,14.43 = 19.1\,\angle -49.1^\circ$$

$$\Delta_2 = \begin{bmatrix} (2+j2) & 10 \\ -j1 & -(4.43 + j2.5) \end{bmatrix} = (2 + j2)\,(4.43 + j2.5) + j\,10 = -3.86 - j\,3.86$$
$$= 5.46\angle -135^\circ \text{ or } \angle 225^\circ$$

$$I_1 = \Delta_1/\Delta = 19.1\,\angle -49.1^\circ/5 = 3.82\,\angle -49.1^\circ = 2.5 - j2.89$$
$$I_2 = \Delta_2/\Delta = 5.46\,\angle -135^\circ/5 = 1.1\,\angle -135^\circ = -0.78 - j0.78$$

Current through branch $BC = I_1 - I_2 = 2.5 - j2.89 + 0.78 + j0.78 = 3.28 - j2.11 = 3.49\,\angle -32.75^\circ$

Drop over branch $AB = (2 + j1)(2.5 - j\,2.89) = 7.89 - j\,3.28$

Drop over branch $BD = (1 - j2)\,(-0.78 - j0.78) = 2.34 + j0.78$

Drop over branch $BC = j_1\,(I_1 - I_2) = j1\,(3.28 - j2.11) = 2.11 + j3.28$

Power delivered by the sources would be found by using conjugate method. Using current conjugate, we get

$$VA_1 = 10(2.5 + j2.89) = 25 + j28.9\,;\quad \therefore\ W_1 = 25\text{ W}$$

$VA_2 = V_2 \times I_2$ — because $-I_2$ is the current coming out of the second voltage source. Again, using current conjugate, we have

$VA_2 = (4.43 + j2.5)\,(0.78 - j0.78)$ or $W_2 = 4.43 \times 0.78 + 2.5 \times 0.78 = 5.4$ W

$\therefore$ total power supplied by the two sources = 25 + 5.4 = 30.4 W

Incidentally, the above fact can be verified by adding up the powers dissipated in the three branches of the circuit. It may be noted that there is no power dissipation in the branch BC.

Power dissipated in branch $AB = 3.82^2 \times 2 = 29.2$ W

Power dissipated in branch $BD = 1.1^2 \times 1 = 1.21$ W

Total power dissipated $= 29.2 + 1.21 = 30.41$ W.

Tutorial Problems No. 15.2

1. Using mesh analysis, find current in the capacitor of Fig. 15.8. **[13.1 ∠70.12° A]**

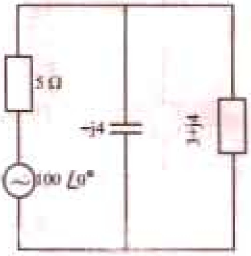


Fig. 15.8

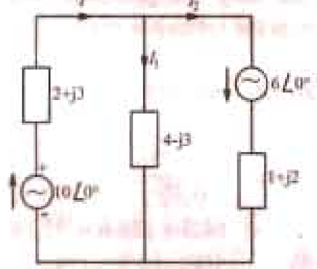


Fig. 15.9

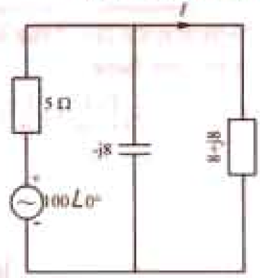


Fig. 15.10

2. Using mesh analysis or Kirchhoff's laws, determine the values of I, I_1 and I_2 (in Fig. 15.9) **[I = 2.7 ∠ − 58.8° A ; I_1 = 0.1 ∠97° A; I_2 = 2.8 ∠−59.6° A]**

3. Using mesh current analysis, find the value of current I and active power output of the voltage source in Fig. 15.10. **[7 ∠ − 50° A ; 645 W]**

4. Find the mesh currents I_1, I_2 and I_3 for the circuit shown in Fig. 15.11. All resistances and reactances are in ohms **[I_1 = (1.168 + j1.281) ; I_2 = (0.527 − j0.135); I_3 = (0.718 + j0.412)]**

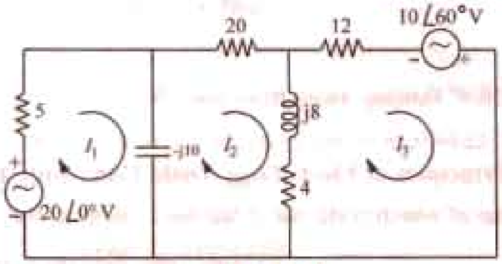


Fig.15.11

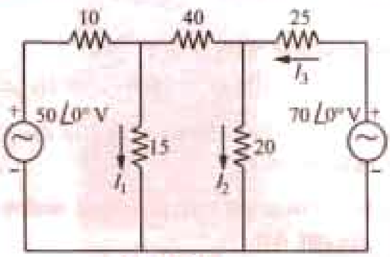


Fig. 15.12

5. Find the values of branch currents I_1, I_2 and I_3 in the circuit shown in Fig. 15.12 by using mesh analysis. All resistances are in ohms. **[I_1 = 2.008 ∠0° ; I_2 = 1.545 ∠0° ; I_3 = 1.564 ∠0°]**

6. Using mesh-current analysis, determine the current I_1, I_2 and I_3 flowing in the branches of the networks shown in Fig. 15.13. **[I_1 = 8.7 ∠−1.37° A; I_2 = 3 ∠−48.7° A ; I_3 = 7 ∠17.25° A]**

7. Apply mesh-current analysis to determine the values of current I_1 to I_5 in different branches of the circuit shown in Fig. 15.14.

[I_1 = 2.4 ∠52.5° A ; I_2 = 1.0 ∠ 46.18° A ; I_3 = 1.4 ∠ 57.17° A ; I_4 = 0.86 ∠166.3° A ; I_5 = 1.0 ∠83.7° A]

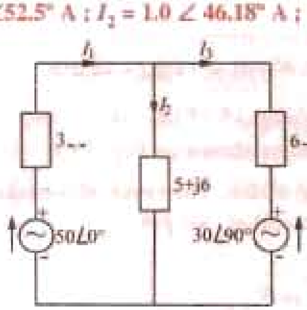


Fig. 15.13

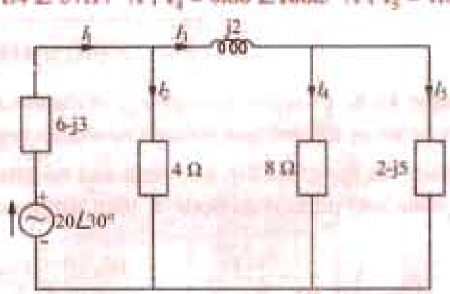


Fig. 15.14

15.4. Nodal Analysis

This method has already been discussed in details in Art. 15. --. This technique is the same although we have to deal with circuit impedances rather than resistances and take phasor sum of voltages and currents rather than algebraic sum.

Example 15.6. *Use Nodal analysis to calculate the current flowing in each branch of the network shown in Fig. 15.15.*

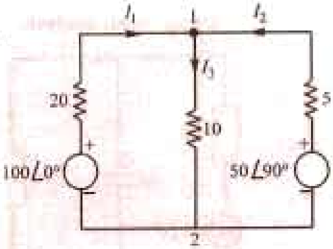


Fig. 15.15

Solution. As seen, there are only two principal nodes out of which node No. 2 has been taken as the reference node. As seen from Art... we have

$$V_1\left(\frac{1}{20}+\frac{1}{10}+\frac{1}{5}\right)-\frac{100\angle 0^\circ}{20}-\frac{50\angle 90^\circ}{5}=0$$

$$\therefore \qquad 0.35\,V_1 = 5+j10;\; V_1 = \frac{5+j10}{0.35}$$

$$= 14.3 + j28.6 = 32\angle 63.4^\circ$$

$$\therefore \qquad I_1 = \frac{100\angle 0^\circ - V_1}{20} = \frac{100-14.3-j28.6}{20}$$

$$= 4.3 - j1.4 = 4.5\angle -18^\circ \text{ flowing towards node No. 1}$$

(or $4.5\angle -18^\circ + 180^\circ = 45\angle 162^\circ$ flowing away from node No. 1)

$$I_3 = \frac{V_1}{10} = \frac{32\angle 63.4^\circ}{10} \leq = 3.2\angle 63.4^\circ = 1.4 + j2.9 \text{ flowing from node No. 1 to node No. 2}$$

$$I_2 = \frac{50\angle 90^\circ - V_1}{5} = \frac{j50-14.3-j28.6}{5} = \frac{-14.3+j21.4}{5} = -2.86 + j4.3 = 5.16\angle 123.6^\circ$$

flowing towards node No. 1

$$\angle 123.6^\circ - 180^\circ = 5.16\angle -56.4^\circ \text{ flowing away from node No. 1).}$$

Example15.7. *Find the current I in the j10 Ω branch of the given circuit shown in Fig. 15.16 using the Nodal Method.* **(Principles of Elect. Engg. Delhi Univ. June 1985)**

Solution. There are two principal nodes out of which node No. 2 has been taken as the reference node. As per Art.

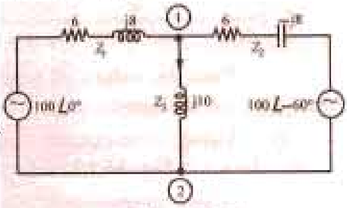


Fig. 15.16

$$V_1\left(\frac{1}{6+j8}+\frac{1}{6-j8}+\frac{1}{j10}\right)-\frac{100\angle 0^\circ}{6+j8}\;\frac{100\angle -60^\circ}{6-j8}=0$$

$$V_1(0.06 - j0.08 + 0.06 + j0.08 - j0.1) = 6 - j8 + 9.93 - j1.2$$
$$= 18.4\angle -30^\circ$$

$$\therefore \qquad V_1(0.12 - j0.1) = 18.4\angle 30^\circ \text{ or}$$

$$V_1 \times 0.156\angle -85.6^\circ = 18.4\angle -30^\circ$$

$$\therefore \qquad V_1 = 18.4\angle -30^\circ/0.156\angle -85.6^\circ = 118\angle 55.6^\circ\text{V}$$

$$\therefore \qquad V = V_1/j10 = 118\angle 55.6^\circ/j10 = 11.8\angle -34.4^\circ\text{A}$$

Example 15.8. *Find the voltage V_{AB} in the circuit of Fig. 15.17 (a). What would be the value of V_1 if the polarity of the second voltage source is reversed as shown in Fig. 15.17 (b).*

Solution. In the given circuit, there and no principle nodes. However, if we take point *B* as the reference node and point *A* as node 1, then using nodal mathod, we get

$$V_1\left(\frac{1}{10}+\frac{1}{8+j4}\right) - \frac{10\angle 0^\circ}{10}-\frac{10\angle 30^\circ}{8+j4}=0$$

$$V_1 \times 0.2\angle -14.1^\circ = 1 + 1.116 + j0.066 = 4.48\angle 1.78^\circ$$

$$\therefore \qquad V_1 = 4.48\angle 1.78^\circ/0.2\angle -14.1^\circ = 22.4\angle 15.88^\circ$$

When source polarity is Reversed

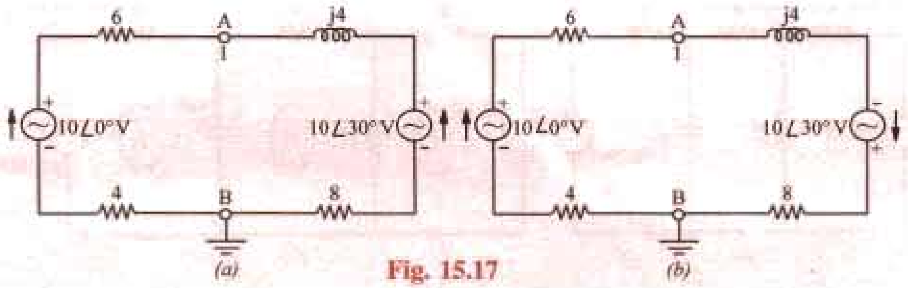


Fig. 15.17

$$V_1\left(\frac{1}{10}+\frac{1}{8+j4}\right)-\frac{10\angle 0^\circ}{10}+\frac{10\angle 30^\circ}{8+j4}=0 \text{ or } V_1=0.09\angle 223.7^\circ$$

Example 15.9. *Write the nodal equations for the network shown in Fig. 15.18.*

Solution. Keeping in mind the guidance given in Art. 2.10, it would be obvious that since current of the second voltage source is flowing away from node 1, it would be taken as negative. Hence, the term containing this source will become positive because it has been reversed twice. As seen, node 3 has been taken as the reference node. Considering node 1, we have

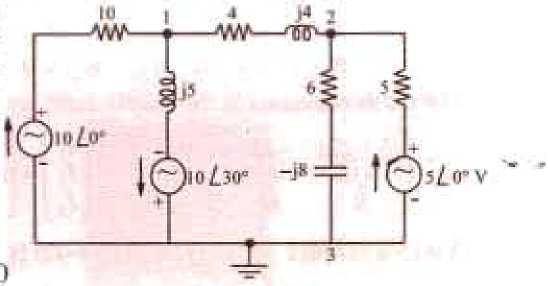


Fig. 15.18

$$V_1\left(\frac{1}{10}+\frac{1}{4+j4}+\frac{1}{j5}\right)-\frac{V_2}{4+j4}-\frac{10\angle 0^\circ}{10}+\frac{10\angle 30^\circ}{j5}=0$$

Similarly, considering node 2, we have

$$V_2\left(\frac{1}{4+j4}+\frac{1}{5}+\frac{1}{6-j8}\right)-\frac{V_1}{4+j4}-\frac{5\angle 0^\circ}{5}=0$$

Example 15.10. *In the network of Fig. 15.19 determine the current flowing through the branch of 4 Ω resistance using nodal analysis.* **(Network Analysis Nagpur Univ. 1993)**

Solution. We will find voltages V_A and V_B by using Nodal analysis and then find the current through 4 Ω resistor by dividing their difference by 4.

$$V_2\left(\frac{1}{5}+\frac{1}{4}+\frac{1}{j2}\right)-\frac{V_B}{4}-\frac{50\angle 30^\circ}{5}=0 \qquad \text{...for node A}$$

$\therefore \quad V_A(9-j10)-5V_B=200\angle 30^\circ$ ...(*i*)

Similarly, from node *B*, we have

$$V_B\left(\frac{1}{4}+\frac{1}{2}+\frac{1}{-j2}\right)-\frac{V_A}{4}-\frac{50\angle 90^\circ}{2}=0 \;\therefore\; V_B(3+j2)-V_A=100\angle 90^\circ=j\,100 \qquad \text{...(ii)}$$

V_A can be eliminated by multiplying. Eq. (*ii*) by (9–j10) and adding the result.

$$\therefore \quad V_B(42-j12)=1173+j1000 \quad \text{or } V_B=\frac{1541.4\angle 40.40^\circ}{43.68\angle -15.9^\circ}=35.29\angle 56.3^\circ=19.58+j29.36$$

Substituting this value of V_B in Eq. (*ii*), we get

$$V_A=V_B(3+j2)-j100 = (19.58+j29.36)(3+j2)-j100=j27.26$$

$$\therefore \quad V_A-V_B = j\,27.26-19.58-j29.36=-19.58-j2.1=19.69\angle 186.12^\circ$$

$$\therefore \quad I_2=(V_A-V_B)/4 = 19.69\angle 186.12^\circ/4 = 4.92\angle 186.12^\circ$$

For academic interest only, we will solve the above question with the help of following two methods :

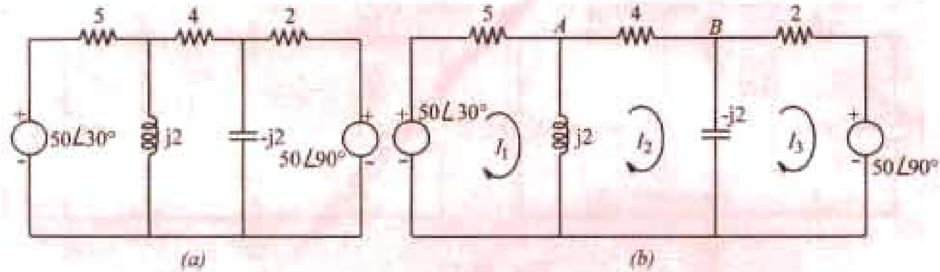


Fig. 15.19

Solution by using Mesh Resistance Matrix

Let the mesh currents I_1, I_2 and I_3 be as shown in Fig. 15.19 (b). The different items of the mesh resistance matrix $[R_m]$ are as under :

$$R_{11} = (5 + j2)\ ;\ R_{22} = 4\ ;\ R_{33} = (2 - j2)\ ;\ R_{12} = R_{21} = -j2;$$

$$R_{23} = R_{32} = j2\ ;\ R_{31} = R_{13} = 0$$

The mesh equations in the matrix form are :

$$\begin{bmatrix} (5+j2) & -j2 & 0 \\ -j2 & 4 & j2 \\ 0 & j2 & (2-j2) \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 50\angle 30^\circ \\ 0 \\ -j50 \end{bmatrix}$$

$$\Delta = (5 + j2)\,[4(2 - j2) - (j2 \times j2) - (-j2)\,[(-j2)\,(2-j2)] = 84 - j24 = 87.4\ \angle{-15.9^\circ}$$

$$\Delta_2 = \begin{bmatrix} (5+j2) & (43.3+j25) & -0 \\ -j2 & 0 & j2 \\ 0 & -j50 & (2-j2) \end{bmatrix} = (5 + j2)\,[-j2\,(-j50)] +$$

$$j2\,[43.3 + 25)\,(2 - j2)] = -427 + j73 = 433\angle 170.3^\circ$$

$$\therefore \qquad I_2\ \Delta_2/\Delta \ = \ 433\angle 170.3^\circ/87.4\angle{-15.9^\circ} \ = \ 4.95\angle 186.2^\circ$$

Solution by using Thevenin's Theorem

When the 4 Ω resistor is disconnected, the given figure becomes as shown in Fig. 15.20 (*a*). The voltage V_A is given by the drop across $j2$ reactance. Using the voltage-divider rule, we have

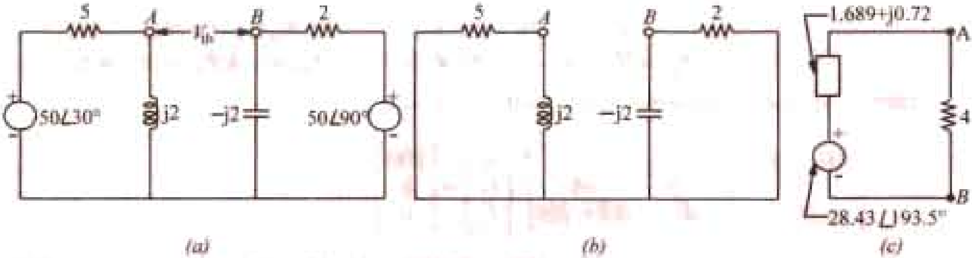
$$V_A = 50\angle 30^\circ \times \frac{j2}{5 + j2} \ = \ 18.57\angle 98.2^\circ = -2.65 + j18.38$$

Similarly,
$$V_B \ = \ 50\angle 90^\circ\ \frac{-j2}{2 - j2} = 35.36\angle 45^\circ = 25 + j25$$

$$\therefore \qquad V_{th} = V_A - V_B \ = \ -2.65 + 18.38 - 25 - j25 = 28.43\angle 193.5^\circ$$

$$R_{th} = 5 \| j2 + 2 \| (-j2) \ = \ \frac{j10}{5 + j2} + \frac{-j4}{2 - j2} + 1.689 + j0.72$$

The Thevenin's equivalent circuit consists of a voltage source of 28.43 ∠ 193.5° V and an impedance of (1.689 + j0.72) Ω as shown in Fig. 15.20 (*c*). Total resistance is 4 + (1.689 + j0.72) = 5.689 + j 0.72 = 5.73∠7.2°. Hence, current through the 4 Ω resistor is 28.43 ∠193.5°/5.73∠7.20° = 4.96 ∠ 186.3°.



(a) (b) (c)

Fig. 15.20

Note. The slight variations in the answers are due to the approximations made during calculations.

Example 15.11. *Using any suitable method, calculate the current through 4 ohm resistance of the network shown in Fig. 15.21.* **(Network Analysis AMIE Sec. B Summer 1990)**

Solution. We will solve this queestion with the help of (*i*) Kirchhoff's laws (*ii*) Mesh analysis and (*iii*) Nodal analysis.

(*i*) **Solution by using Kirchhoff's Laws**

Let the current distribution be as shown in Fig. 15.21 (*b*). Using the same sign convention as given in Art. we have

First Loop $-10(I_1 + I_2 + I_3) - (-j5)\,I_1 + 100 = 0$

or $I_1\,(10 - j5) + 10\,I_3 + 10I_2 = 100$ ...(*i*)

Second Loop $-5(I_2 + I_3) - 4I_2 + (-j\,5)\,I_1 = 0$

or $j5I_1 + 9\,I_2 + 5I_3 = 0$ ...(*ii*)

Third Loop $-I_3\,(8 + j6) + 4I_2 = 0$

or $0I_1 + 4I_2 - I_3\,(8 + j6) = 0$ ...(*iii*)

The matrix form of the above three equations is

$$\begin{bmatrix} (10-j5) & -10 & 10 \\ j5 & 9 & 5 \\ 0 & 4 & -(8+j6) \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

$$\Delta = (10 - j\,5)\,[-9\,(8 + j6) - 20] - j5\,[-10(8 + j6) - 40]$$

$$= -\,1490 + j\,520 = 1578\angle 160.8°$$

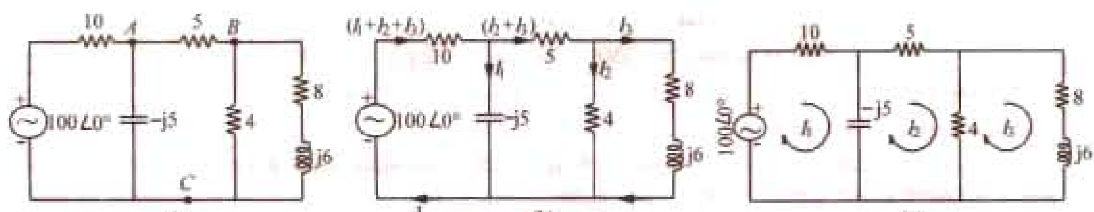
Since we are interested in finding I_2 only, we will calculate the value of Δ_2.

$$\Delta_2 = \begin{bmatrix} (10-j5) & 100 & 10 \\ j5 & 0 & 5 \\ 0 & 0 & -(8+j6) \end{bmatrix}$$

$$= -j\,5\,(-800 - j\,600) = -\,3000 + j\,4000 = 5000\angle 126.9°$$

$$I_2 = \frac{\Delta_2}{\Delta} = \frac{500.0\angle 126.9°}{1578\angle 160.8°} = 3.17\angle -33.9°\text{ A}$$

(*ii*) **Solution by using Mesh Impedance Matrix**



(a) (b) (c)

Fig. 15.21

Let the mesh currents I_1, I_2 and I_3 be as shown in Fig. 15.21 (*c*). From the inspection of Fig. 15.21 (*c*), the different items of the mesh impedance matrix $[Z_m]$ are as under :

$$Z_{11} = (10 - j5) ; Z_{22} = (9 - j5); Z_{33} = (12 + j6)$$
$$Z_{21} = Z_{12} = -(-j5) = j5; Z_{23} = Z_{32} = -4; Z_{31} = Z_{13} = 0$$

Hence, the mesh equations in the matrix form are :

$$\begin{bmatrix} (10-j5) & j5 & 0 \\ j5 & (9-j5) & -4 \\ 0 & -4 & (12+j6) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$
$$\Delta = (10 - j5)\,[(9 - j5)(12 + j6) - 16] - j5\,(j60 - 30)$$
$$= 1490 - j520 = 1578\angle 19.2^\circ$$

It should be noted that the current passing through 4 Ω resistance is the vector diference $(I_2 - I_3)$. Hence, we will find I_2 and I_3 only.

$$\Delta_2 = \begin{bmatrix} (10-j5) & 100 & 0 \\ j5 & 0 & -4 \\ 0 & 0 & (12+j6) \end{bmatrix} = j5\,(1200 + j600) = 3000 - j6000 = 6708\angle -63.4^\circ$$

$$\Delta_2 = \begin{bmatrix} (10-j5) & j5 & 100 \\ j5 & (9-j5) & 0 \\ 0 & -4 & 0 \end{bmatrix} = -j5\,(400) = -j2000 = 2000\angle -90^\circ$$

$$\therefore \quad I_2 = \frac{\Delta_2}{\Delta} = \frac{6708\angle -63.4^\circ}{1578\angle -19.2^\circ} = 4.25\angle -44.2^\circ = 3.05 - j2.96$$

$$I_3 = \frac{\Delta_3}{\Delta} = \frac{2000\angle -90^\circ}{1578\angle -19.2^\circ} = 1.27\angle -70.8^\circ = 0.42 - j1.2$$

Current $\qquad (I_2 - I_3) = 2.63 - j1.76 = 3.17\angle -33.9^\circ$

(iii) Solution by Nodal Analysis

The current passing through 4 Ω resistance can be found by finding the voltage V_B of node B with the help of Nodal analysis. For this purpose point C in Fig. 15.21 (a) has been taken as the reference node. Using the Nodal technique as explained in Art. we have

$$V_A\left(\frac{1}{10} + \frac{1}{5} + \frac{1}{-j5}\right) - \frac{V_B}{5} - \frac{100\angle 0^\circ}{10} = 0 \qquad \text{...for node A}$$

$$V_A(3 + j2) - 2V_B = 100 \qquad ...(i)$$

Similarly, for node B, we have

$$V_B\left(\frac{1}{5} + \frac{1}{4} + \frac{1}{(8+j6)}\right) - \frac{V_A}{5} = 0 \quad \text{or} \quad V_B(53 - j6) - 20\,V_A = 0 \qquad ...(ii)$$

Estimating V_A from Eq. (i) and (ii), we have

$$V_B(131 + j88) = 2000 \quad \text{or} \quad V_B = 12.67\angle -33.9^\circ$$

Current through 4 Ω resistor $12.67\angle -33.9^\circ/4 = 3.17\angle -33.9^\circ$

Tutorial Problems No. 15.3.

1. Apply nodal analysis to the network of Fig. 15.22 to determine the voltage at node A and the active power delivered by the voltage source. **[8∠3.7° V; 9.85 W]**

2. Using nodal analysis, determine the value of voltages at models 1 and 2 in Fig. 15.23. **[$V_1 = 88.1\angle 33.88^\circ$ A; $V_2 = 58.7\angle 72.34^\circ$A]**

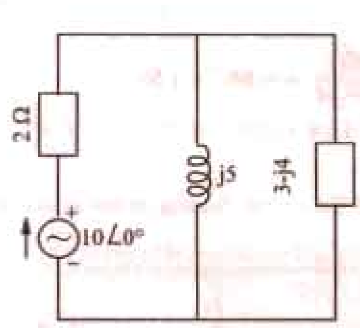


Fig. 15.22

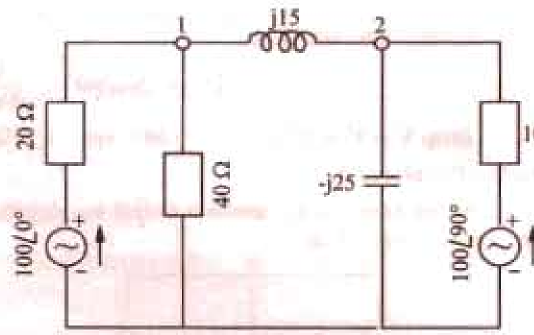


Fig. 15.23

3. Using Nodal analysis, find the nodal voltages V1 and V2 in the circuit shown in Fig. 15.24. All resistances are given in terms of siemens [V_1 = 1.64 V : V_2 = 0.38 V]

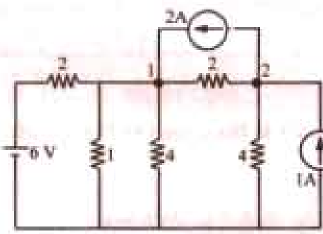


Fig. 15.24

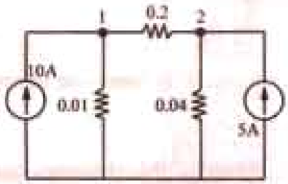


Fig. 15.25

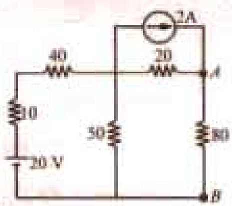


Fig. 15.26

4. Find the values of nodal voltages V_1 and V_2 in the circuit of Fig. 15.25. Hence, find the current going from nodel 1 to node 2. All resistances are given in siemens. [V_1 = 327 V; V_2 = 293.35 V ; 6.73 A]

5. Using Nodal analysis, find the voltage across points *A* and and *B* in the circuit of Fig. 15.26: Check your answer by using mesh analysis. [32 V]

15.5. Superposition Theorem

As applicable to a.c. networks, it states as follows :

In any network made up of linear impedances and containing more than one source of e.m.f., the current flowing in any branch is the phasor sum of the currents that would flow in that branch if each source were considered separately, all other e.m.f. sources being replaced for the time being, by their respective internal impedances (if any).

Note. It may be noted that independent sources can be 'killed' *i.e.* removed leaving behind their internal impedances (if any) but dependent sources should not be killed.

Example 15.12. *Use Superposition theorem to find the voltage V in the network shown in Fig. 15.27.*

Solution. When the voltage source is killed, the circuit becomes as shown in the Fig. 15.27 (*b*) Using current-divider rule,

$$I = 10\angle 0^{\circ} \times \frac{-j4}{(3+j4)-j4}\text{, Now, } V' = I\,(3+j4)$$

$$\therefore \qquad V' = 10\ \frac{-j4\,(3+j4)}{3} = 53.3 - j\,40$$

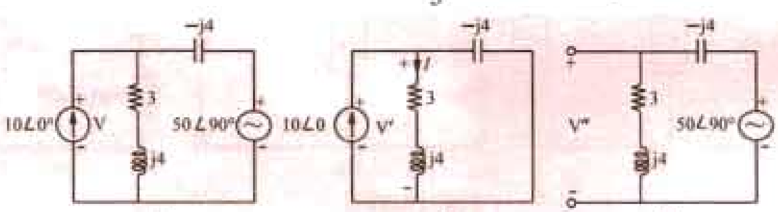


Fig. 15.27

Now, when current source is killed, the circuit becomes as shown in Fig. 15.27 (c). Using the voltage-divider rule, we have

$$V'' = 50\angle 90^\circ \times \frac{(3+j4)}{(3+j4)-j4} = -66.7 + j50$$

$\therefore$ drop $V = V' + V'' = 53.3 - j40\ (-66.7 + j50) = -13.4 + j10 = 16.7 \angle 143.3^\circ\ V$

Tutorial Problems No. 15.4

1. Using Superposition theorem to find the magnitude of the current flowing in the branch AB of the circuit shown in Fig. 15.28.

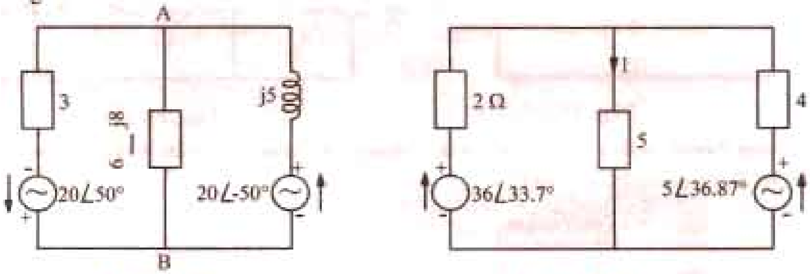


Fig. 15.28 Fig. 15.29

2. Apply Superposition theorem to determine the circuit I in the circuit of Fig. 15.29. **[0.53 ∠ 5.7° A]**

15.6. Thevenin's Theorem

As applicable to a.c. networks, this theorem may be stated as follows :

The current through a load impedance Z_L connected across any two terminals A and B of a linear network is given by $V_{th}/(Z_{th} + Z_L)$ wher V_{th} is the open-circuit voltage across A and B and Z_{th} is the internal impedance of the network as viewed from the open-circuited terminals A and B with all voltage sources replaced by their internal impedances (if any) and current sources by infinite impedance.

Example 15.13. *In the network shown in Fig. 15.30.*

$Z_1 = (8 + j8)\ \Omega$; $Z_2 = (8 - j8)\ \Omega$; $Z_3 = (2 + j20)$; $V = 10 \angle 0^\circ$ and $Z_L = j\,10\ \Omega$

Find the current through the load Z_L using Thevenin's theorem.

Solution. When the load impedance Z_L is removed, the circuit becomes as shown in Fig. 15.30 (b). The open-circuit voltage which appears across terminals A and B represents the Thevenin voltage V_{th}. This voltage equals the drop across Z_2 because there is no current flow through Z_3.

Current flowing through Z_1 and Z_2 is

$$I = V(Z_1 + Z_2) = 10 \angle 0^\circ\ [(8 + j8) + (8 - j8)] = 10 \angle 0^\circ/16 = 0.625 \angle 0^\circ$$

$$V_{th} = IZ_2 = 0.625\ (8 - j8) = (5 - j5) = 7.07 \angle -45^\circ$$

The Thevenin impedance Z_{th} is equal to the impedance as viewed from open terminals A and B with voltage source shorted.

$$\therefore \quad Z_{th}\ Z_3 + Z_1 \,||\, Z_2 = (2 + j20) + (8 + j8) \,||\, (8 - j8) = (10 + j20)$$

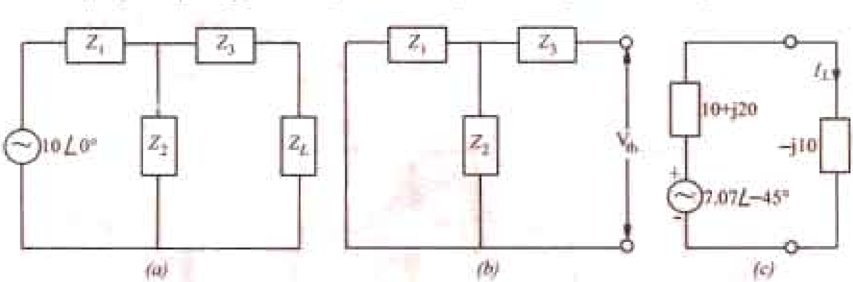
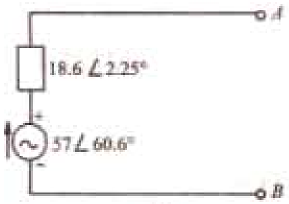


Fig. 15.30

(*iii*) From the above information, we can find V_{th} and Z_{th}



(*c*)
Fig. 15.34

$V_{th} = V_{CD} = 75.5\angle 19.1° - 50\angle -30° = 57\angle 60.6°$

$Z_{th} = Z_{CD} = 10\angle -30° + 7.55\angle 10.9° + 5\angle 60° = 18.6\angle 2.25°$

The Thevenin equivalent will respect to the terminals *A* and *B* is shown in Fig. 15.34 (*c*).

For finding V_{AB} *i.e.* voltage at point *A* will respect to point *B*, we start from point B in Fig. 15.34 (*b*) and go to point *A* and calculate the phasor sum of the voltages met on the way.

$$\therefore\ V_{AB} = 75.1\angle 19.1° - 50\angle -30° = 57\angle 60.6°$$

$$Z_{AB} = 10\angle 30° + 7.55\angle 10.9° + 5\angle 60° = 18.6\angle 2.25°$$

Example 15.17. *For the network shown, determine using Thevenin's theorem, voltage across capacitor in. Fig. 15.35.* **(Elect. Network Analysis Nagpur Univ. 1993)**

$Z_{CD} = j5 \| (10 + j5) = 1.25 + j3.75$. This impedance is in series with the 10Ω resistance. Using voltage divider rule, the drop over Z_{CD} is

Solution. When load of –j5 Ω is removed the circuit becomes as shown in Fig. 15.35 (*b*). Thevenin is voltage is given by the voltage drop produced by 100-V source over (5 + *j*5)impedance. It can be calculated as under.

$$V_{CD} = 100\frac{(1.25 + j3.75)}{10 + (1.25 + j3.75)} = \frac{125 + j375}{11.25 + j3.75}$$

This V_{CD} is applied across j5 reactance as well as across the series combination of 5Ω and (5 + *j*5) Ω. Again, using voltage-divider rule for V_{CD}, we get

$$V_{AB} = V_{th} = V_{CD} \times \frac{5 + j5}{10 \times j5} = \frac{(125 + j375)}{11.25 + j3.75} \times \frac{5 + j5}{10 + j5} = 21.1\angle 71.57° = 6.67 + j20$$

As looked into terminals *A* and *B*, the equivalent impedance is given by

$$R_{AB} = R_{th} = (5 + j5) \,\|\, (5 + 10 \,\|\, j5) = (5 + j5) \,\|\, (7 + j4) = 3 + j2.33$$

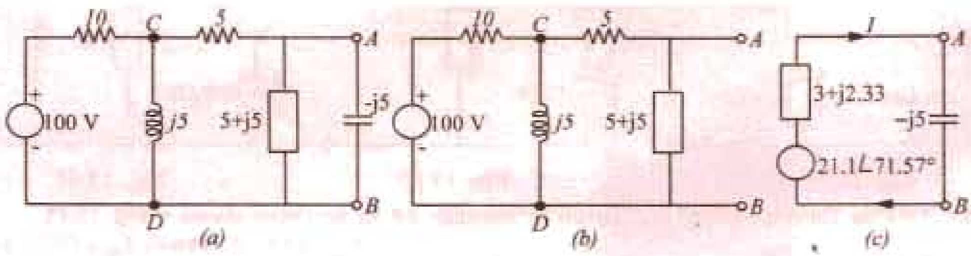


Fig. 15.35

The equivalent Thevenin's source along with the load is shown in Fig. 15.35 (*c*).

$$\text{Total impedance} = 3 + j2.33 - j5 = 3 - j2.67 = 4.02\angle -41.67°$$

$$\therefore\ I = 21.1\angle 71.57° / 4.02\angle -41.67° = 5.25\angle 113.24°$$

Solution by Mesh Resistance Matrix

The different items of the mesh resistance matrix $[R_m]$ are as under :

$R_{11} = 10 + j5$; $R_{22} = 10 + j10$; $R_{33} = 5$;

$R_{12} = R_{21} = -j5$;

$R_{23} = R_{32} = -(5 + j5)$; $R_{31} = R_{13} = 0$. Hence, the mesh equations in the matrix form are as given below

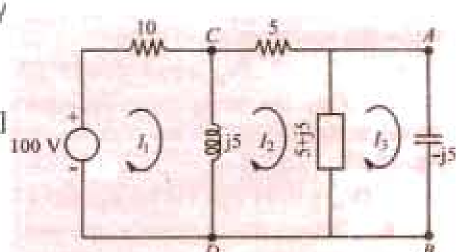


Fig. 15.36

The equivalent Thevenin circuit is shown in Fig. 15.30 (*c*) across which the load impedance has been reconnected. The load current is given by

$$\therefore \qquad I_L = \frac{V_{th}}{Z_{th} + Z_L} = \frac{(5 - j5)}{(10 + j20) + (-j10)} = \frac{-j}{2}$$

Example 15.13 A. *Find the Thevenin equivalent circuit at terminals AB of the circuit given in Fig. 15.31 (a).*

Solution. For finding $V_{th} = V_{AB}$, we have to find the phasor sum of the voltages available on the way as we go from point B to point A because V_{AB} means voltage of point A with respect to that of point B (Art.). The value of current $I = 100\angle 0^\circ/(6 - j8) = (6 + j8)A$.

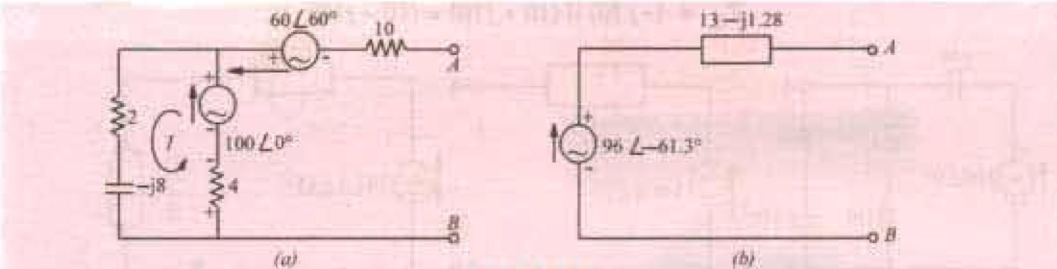


Fig. 15.31

Drop across 4 Ω resistor = 4 (6 + j 8) = (24 + j 32)

$$\therefore \qquad V_{th} = V_{AB} = -(24 + j32) + (100 + j0) - 60(0.5 + j0.866)$$
$$= 46 - j84 = 96\angle -61.3^\circ$$
$$Z_{AB} = Z_{th} = [10 + [4 \,||\, (2 - j8)] = (13 - j1.28)$$

The Thevenin equivalent circuit is shown in Fig. 15.31 (*b*).

Example 15.14. *Find the Thevenin's equivalent of the circuit shown in Fig. 15.32 and hence calculate the value of the current which will flow in an impedance of (6 + j 30) Ω connected across terminals A and B. Also calculate the power dissipated in this impedance.*

Solution. Let us first find the value of V_{th} *i.e.* the Thevenin voltage across open terminals A and B. With terminals A and B open, there is no potential drop across the capacitor. Hence, V_{th} is the drop across the pure inductor $j3$ ohm.

$$\text{Drop across the inductor} = \frac{10 + j0}{(4 + j3)} \times j3 = \frac{j30}{4 + j3} = \frac{j30(4 - j3)}{4^2 + 3^2} = (3.6 + j4.8)V$$

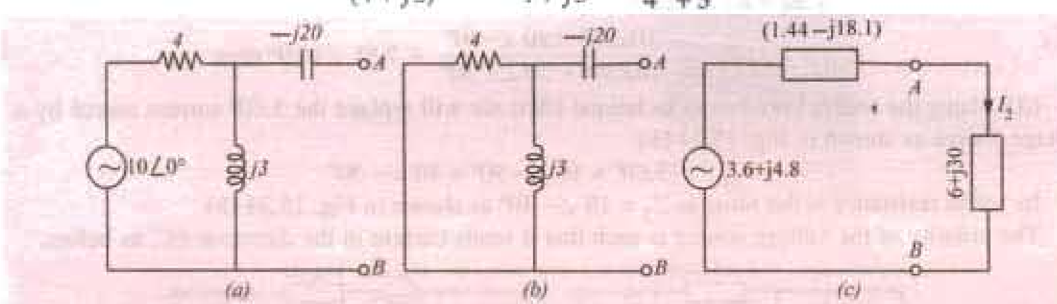


Fig. 15.32

Let us now find the impedance of the circuit as viewed from terminals A and B after replacing the voltage source by a short circuit as shown in Fig. 15.32 (*a*).

$$Z_{th} = -j20 + 4 \,||\, j3 = -j20 + 1.44 + j1.92 = 1.44 - j18.1$$

The equivalent Thevenin circuit along with the load impedance of (6 + j30) is shown in Fig. 15.32 (*c*).

$$\text{Load current} = \frac{(3.6 + j4.8)}{(1.44 - j18.1) + (6 + j30)} = \frac{(3.6 + j4.8)}{(7.44 + j11.9)} = \frac{6\angle 53.1^\circ}{14\angle 58^\circ} = 0.43\angle -4.9^\circ$$

The current in the load is 0.43 A and lags the supply voltage by 4.9°

Power in the load impedance is $0.43^2 \times 6 =$ **1.1 W**

Example 15.15. *Using Thevenin's theorem, calculate the current flowing through the load connected across terminals A and B of the circuit shown in Fig. 15.33 (a). Also calculate the power delivered to the load.*

Solution. The first step is to remove the load from the terminals A and B. $V_{th} = V_{AB} =$ drop across $(10 + j10)$ ohm with A and B open.

$$\text{Circuit current } I = \frac{100}{-j10+10+j10} = 10\angle 0°$$

$$\therefore \quad V_{th} = 10(10+j10) = 141.4\angle 45°$$

$$Z_{th} = (-j10) \,||\, (10+j10) = (10-j10)$$

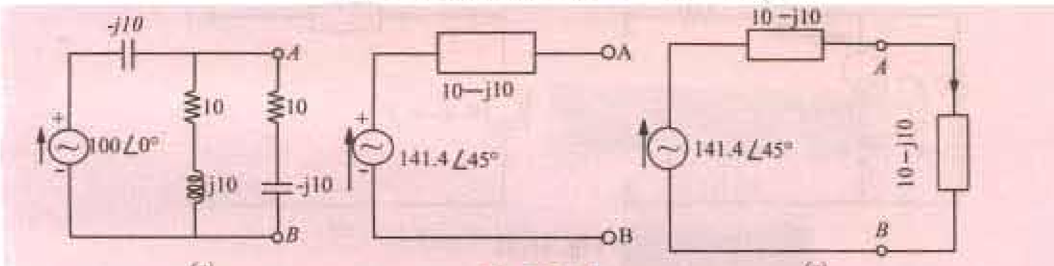


Fig. 15.33

The equivalent Thevenin's source is shown in Fig. 15.33 (*b*). Let the load be re-connected across A and B shown in Fig. 15.33 (*c*).

$$\therefore \quad I_L = \frac{141.4\angle 45°}{(10-j10)+(10-j10)} = \frac{141.4\angle 45°}{20-j20} = \frac{141.4\angle 45°}{28.3°\angle -45°} = 5\angle 90°$$

Power delivered to the load $= \mathbf{I_L}^2\mathbf{R_L} = 5^2 \times 10 = 250$ W

Example 15.16. *Find the Thevenin's equivalent across terminals A and B of the networks shown in Fig. 15.34. (a).*

Solution. The solution of this circuit involves the following steps :

(*i*) Let us find the equivalent Thevenin voltage V_{CD} and Thevenin impedance Z_{CD} as viewed from terminals C and D.

$$V_{CD} = V\frac{Z_2}{Z_1+Z_2} = \frac{100\angle 0° \times 20\angle -30°}{10\angle 30° + 20\angle -30°} = 75.5\angle 19.1°V$$

$$Z_{CD} = Z_1 \,||\, Z_2 = \frac{10\angle 30° \times 20\angle -30°}{10\angle 30° + 20\angle -30°} = 7.55\angle 10.9° \text{ ohm}$$

(*ii*) Using the source conversion technique (Art) we will replace the $5\angle 0°$ current source by a voltage source as shown in Fig. 15.34 (*b*).

$$V_{EC} = 5\angle 0° \times 10\angle -30° = 50\angle -30°$$

Its series resistance is the same as $Z_3 = 10\angle -30°$ as shown in Fig. 15.34 (*b*).

The polarity of the voltage source is such that it sends current in the direction EC, as before.

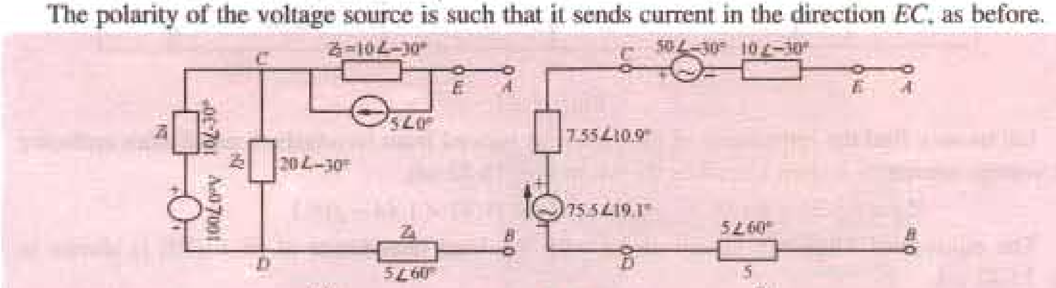


Fig. 15.34

(*iii*) From the above information, we can find V_{th} and Z_{th}

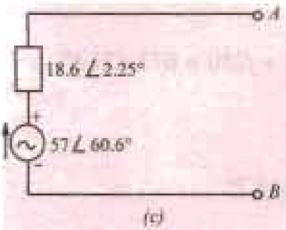


Fig. 15.34

$V_{th} = V_{CD} = 75.5\angle 19.1° - 50\ \angle -30° = 57\ \angle 60.6°$

$Z_{th} = Z_{CD} = 10\angle -30° + 7.55\angle 10.9° + 5\angle 60° = 18.6\ \angle 2.25°$

The Thevenin equivalent will respect to the terminals *A* and *B* is shown in Fig. 15.34 (*c*).

For finding V_{AB} *i.e.* voltage at point *A* will respect to point *B*, we start from point B in Fig. 15.34 (*b*) and go to point *A* and calculate the phasor sum of the voltages met on the way.

$$\therefore\ V_{AB} = 75.1\angle 19.1° - 50\angle -30° = 57\angle 60.6°$$

$$Z_{AB} = 10\angle 30° + 7.55\angle 10.9° + 5\angle 60° = 18.6\angle 2.25°$$

Example 15.17. *For the network shown, determine using Thevenin's theorem, voltage across capacitor in. Fig. 15.35.* **(Elect. Network Analysis Nagpur Univ. 1993)**

$Z_{CD} = j5 \| (10 + j5) = 1.25 + j3.75$. This impedance is in series with the 10Ω resistance. Using voltage divider rule, the drop over Z_{CD} is

Solution. When load of –j5 Ω is removed the circuit becomes as shown in Fig. 15.35 (*b*). Thevenin is voltage is given by the voltage drop produced by 100-V source over (5 + *j*5)impedance. It can be calculated as under.

$$V_{CD} = 100\frac{(1.25 + j3.75)}{10 + (1.25 + j3.75)} = \frac{125 + j375}{11.25 + j3.75}$$

This V_{CD} is applied across j5 reactance as well as across the series combination of 5Ω and (5 + *j*5) Ω. Again, using voltage-divider rule for V_{CD}, we get

$$V_{AB} = V_{th} = V_{CD} \times \frac{5 + j5}{10 \times j5} = \frac{(125 + j375)}{11.25 + j3.75} \times \frac{5 + j5}{10 + j5} = 21.1\angle 71.57° = 6.67 + j20$$

As looked into terminals *A* and *B*, the equivalent impedance is given by

$$R_{AB} = R_{th} = (5 + j5) \| (5 + 10 \| j5) = (5 + j5) \| (7 + j4) = 3 + j2.33$$

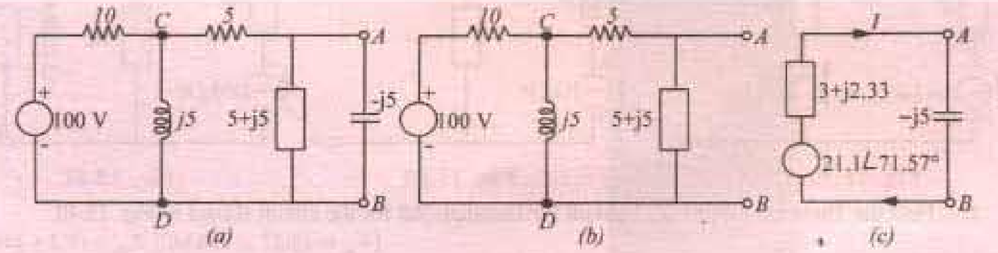


Fig. 15.35

The equivalent Thevenin's source along with the load is shown in Fig. 15.35 (*c*).

$$\text{Total impedance} = 3 + j2.33 - j5 = 3 - j2.67 = 4.02\ \angle -41.67°$$

$\therefore$ $I = 21.1\angle 71.57°/4.02\angle -41.67° = 5.25\angle 113.24°$

Solution by Mesh Resistance Matrix

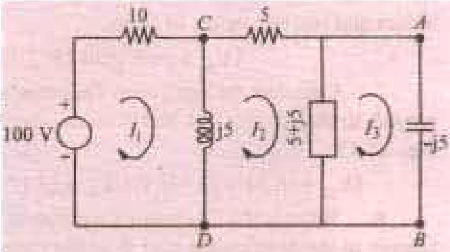


Fig. 15.36

The different items of the mesh resistance matrix $[R_m]$ are as under :

$R_{11} = 10 + j5$; $R_{22} = 10 + j10$; $R_{33} = 5$;

$R_{12} = R_{21} = -j5$;

$R_{23} = R_{32} = -(5 + j5)$; $R_{31} = R_{13} = 0$. Hence, the mesh equations in the matrix form are as given below

$$\begin{bmatrix} (10+j5) & -j5 & 0 \\ -j5 & (10+j110) & -(5+j5) \\ 0 & -(5+j5) & 5 \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad \Delta = (10 + j5)\,[5(10 + j10) - (5 + j5)(5 + j5)] + j5\,(-j25) = 625 + j250 = 673\angle 21.8°$

$$\begin{bmatrix} (10+j5) & -j5 & 100 \\ -j5 & (10+j10) & 0 \\ 0 & -(5+j5) & 0 \end{bmatrix} = j5\,(500 + j500) = 3535\angle 135°$$

$\therefore \quad I_3 = \Delta_3/\Delta = 3535\angle 135°/673\angle 21.8° = 5.25\angle 113.2°$

Tutorial Problems No. 15.5

1. Determine the Thevenin's equivalent circuit with respect to terminals AB of the circuit shown in Fig. 15.37. **[$V_{th} = 14.3\angle 6.38°$, $Z_{th} = (4 + j0.55)\ \Omega$]**

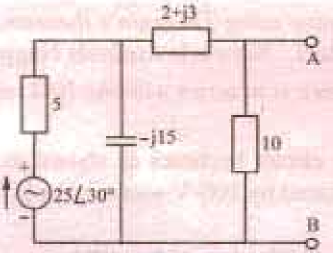


Fig. 15.37

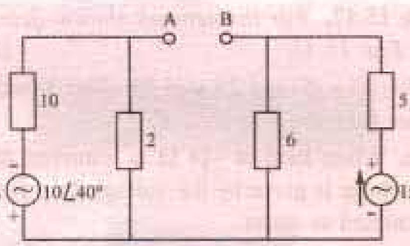


Fig. 15.38

2. Determine Thevenin's equivalent circuit with respect to terminals AB in Fig. 15.38. **[$V_{th} = 9.5\angle 6.46°$; $Z_{th} = 4.4\angle 0°$]**

3. The e.m.fs. of two voltage source shown in Fig. 15.39 are in phase with each other. Using Thevenin's theorem, find the current which will flow in a 16 Ω resistor connected across terminals A and B. **[$V_{th} = 100$ V ; $Z_{th} = (48 + j32)$; $I = 1.44\angle -26.56°$]**

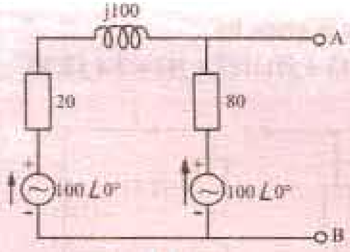


Fig. 15.39

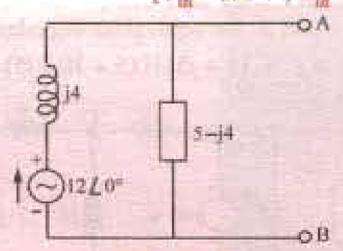


Fig. 15.40

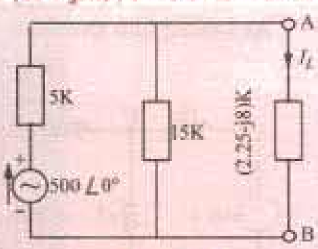


Fig. 15.41

4. Find the Thevenin's equivalent circuit for terminals AB for the circuit shown in Fig. 15.40. **[$V_{th} = 15.37\angle -38.66°$; $Z_{th} = (3.2 + j4)\ \Omega$]**

5. Using Thevenin's theorem, find the magnitude of the load current I_L passing through the load connected across terminals AB of the circuit shown in Fig. 15.41. **[37.5 mA]**

6. By using Thevenin's theorem, calculate the current flowing through the load connected across terminals A and B of circuit shown in Fig. 15.42. All resistances and reactances are in ohms. **[$V_{th} = 56.9\angle 50.15°$; $3.11\angle 85.67°$]**

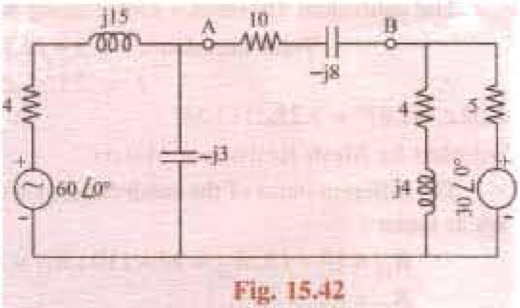


Fig. 15.42

7. Calculate the equivalent Thevenin's source with respect to the terminals A and B of the circuit shown in Fig. 15.43. **[$V_{th} = (6.34 + j2.93)$ V ; $Z_{th} = (3.17 - j5.07)\ \Omega$]**

8. What is the Thevenin's equivalent source with respect to the terminals A and B of the circuit shown in Fig. 15.44 ? **[$V_{th} = (9.33 + j8)$ V ; $Z_{th} = (8 - j11)\ \Omega$]**

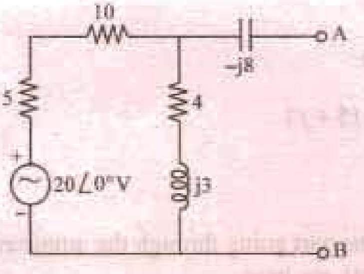


Fig. 15.43

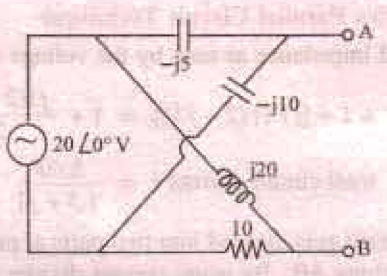


Fig. 15.44

9. What is the Thevenin's equivalent soure with respect to terminals A and B the circuit shown in Fig. 15.45 ? Also, calculate the value of impedance which should be connected across AB for MPT. All resistances and reactances are in ohms. **[V_{th} = (16.87 + j15.16) V; Z_{th} = (17.93 − j1.75) Ω ; (17.93 + j1.75 Ω)]**

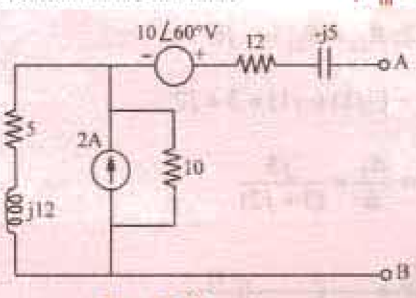


Fig. 15.45

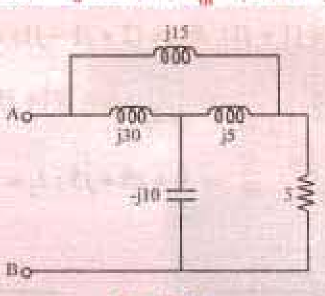


Fig.15.46

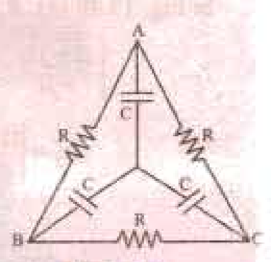


Fig. 15.47

10. Find the impedance of the network shown in Fig. 15.46, when viewed form the terminals A and B. All resistances and reactances are in ohms. **[(4.435 + j6.878)]**

11. Find the value of the impedance that would be measured across terminals BC of the circuit shown in Fig. 15.47.

$$\left[\frac{2R}{9+\omega^2C^2R^2}(3 - j\omega CR)\right]$$

15.7. Reciprocity Theorem

This theorem applies to networks containing linear bilateral elements and a single voltage source or a single current source. This theorem may be stated as follows :

If a voltage source in branch A of a network causes a current of 1 branch B, then shifting the voltage source (but not its impedance) of branch B will cause the same current I in branch A.

It may be noted that currents in other branches will generally not remain the same. A simple way of stating the above theorem is that if an ideal voltage source and an ideal ammeter are interchanged, the ammeter reading would remain the same. The ratio of the input voltage in branch A to the output current in branch B is called the transfer impedance.

Similarly, if a current source between nodes I and 2 causes a potential difference of V between nodes 3 and 4, shifting the current source (but not its admittance) to nodes 3 and 4 causes the same voltage V between nodes 1 and 2.

In other words, the interchange of an ideal current source and an ideal voltmeter in any linear bilateral network does not change the voltmeter reading.

However, the voltages between other nodes would generally not remain the same. The ratio of the input current between one set of nodes to output voltage between another set of nodes is called the transfer admittance.

Example 15.18. *Verify Reciprocity theorem for V & I in the circuit shown in Fig. 15.48.*

[Elect. Network Analysis, Nagpur Univ. 1993]

Solution. We will find the value of the current I as read by the ammeter first by applying series parallel circuits technique and then by using mesh resistance matrix (Art.)

1. Series Parallel Circuit Technique

The total impedance as seen by the voltage source is

$$= 1 + [j1 \parallel (2 - j1)] = 1 + \frac{j1(2-j1)}{2} = 15 + j1$$

$\therefore$ total circuit current $i = \dfrac{5\angle 0^\circ}{1.5 + j1}$

This current gets divided into two parts at point A, one part going through the ammeter and the other going along AB. By using current-divider rule. (Art), we have

$$I = \frac{5}{1.5 + j1} \times \frac{j1}{(2 + j1 - j1)} = \frac{j5}{3 + j2}$$

2. Mesh Resistance Matrix

In Fig. 15.48 (b), $R_{11} = (1 + j1)$, $R_{22} = (2 + j1 - j1) = 2$; $R_{12} = R_{21} = -j1$

$$\therefore \quad \begin{vmatrix} (1+j1) & -j1 \\ -j1 & 2 \end{vmatrix} \begin{vmatrix} I_1 \\ I_2 \end{vmatrix} = \begin{vmatrix} 5 \\ j0 \end{vmatrix} ; \Delta = 2(1 + j1) - (-j1)(-j1) = 3 + j2$$

$$\Delta_2 = \begin{vmatrix} (1+j1) & 5 \\ -j1 & 0 \end{vmatrix} = 0 + j5 = j5 ; I_2 = I = \frac{\Delta_2}{\Delta} = \frac{j5}{(3 + j2)}$$

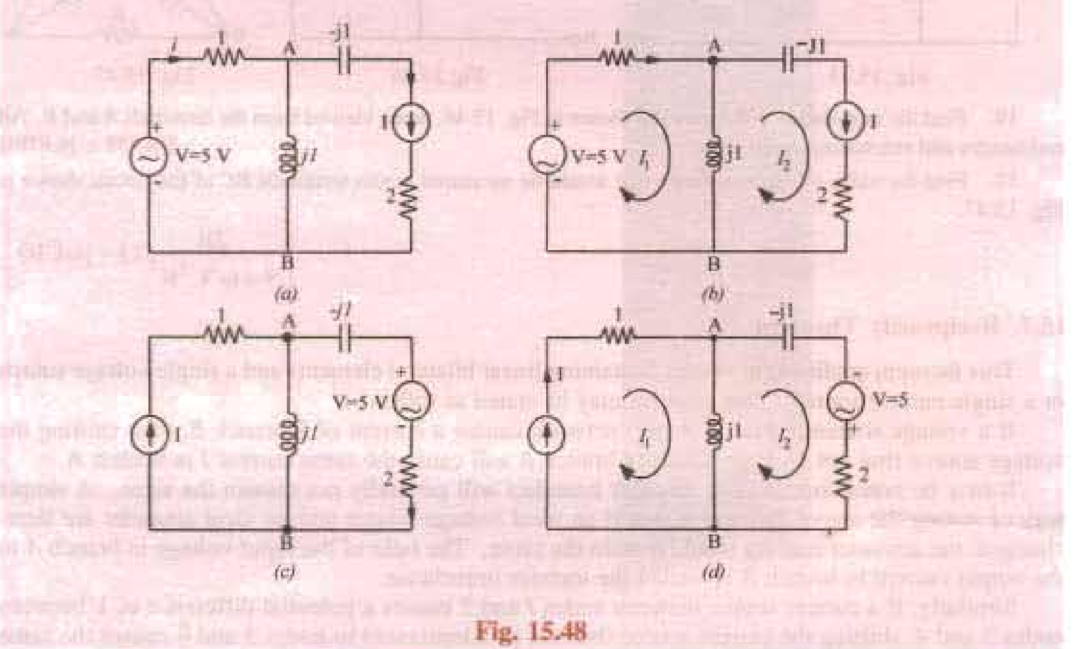


Fig. 15.48

As shown in Fig. 15.48 (c), the voltage source has been interchanged with the ammeter. The polarity of the voltage source should be noted in particular. It looks as if the voltage source has been pushed along the wire in the counterclockwise direction to its new position, thus giving the voltage polarity as shown in the figure. We will find the value of I in the new position of the ammeter by using the same two techniques as above.

1. Series Parallel Circuit Technique

As seen by the voltage source from its new position, the total circuit impedance is

$$= 2(2 - j1) + j1 \parallel 1 = \frac{3 + j2}{1 + j1}$$

The total circuit current $i = 5 \times \dfrac{1+j1}{3+j2}$

This current i gets divided into two parts at point B as per the current-divider rule.

$$I = \frac{5(1+j1)}{3+j2} \times j\frac{1}{1+j1} = \frac{j5}{3+j2}$$

2. Mesh Resistance Matrix

As seen from Fig. 15.48 (*d*).

$$\begin{vmatrix} (1+j1) & -j1 \\ -j1 & 2 \end{vmatrix} \begin{vmatrix} I_1 \\ I_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 5 \end{vmatrix};\ \Delta = 2(1+j1)+1 = 3+j2$$

$$\Delta = \begin{vmatrix} 0 & -j1 \\ 5 & 2 \end{vmatrix} = j5;\ I = I_1 = \frac{\Delta_1}{\Delta} = \frac{j5}{3+j2}$$

The reciprocity theorem stands verified from the above results.

Tutorial problem No. 15.6

1. State reciprocity theorem. Verfity if for the circuit Fig. 15.49, with the help of any suitable current through any element. **(Elect. Network Analysis Nagpur Univ. 1993**

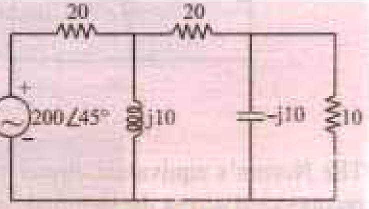


Fig. 15.49

15.8. Norton's Theorem

As applied to a.c. networks, this theorem can be stated as under :

Any two terminal active linear network containing voltage sources and impedances when viewed from its output terminals is equivalent to a constant current source and a parallel impedance. The constant current is equal to the current which would flow in a short-circuit placed across the terminals and the parallel impedance is the impedance of the network when viewed from open-circuited terminals after voltage sources have been replaced by their internal impedances (if any) and current sources by infinite impedance.

Example 15.19. *Find the Norton's equivalent of the circuit shown in Fig. 15.50. Also find the current which will flow through an impedance of (10 − j 20) Ω across the terminals A and B.*

Solution. As shown in Fig. 15-50 (*b*), the terminals A and B have been short-cicuited.

$\therefore\ I_{SC} = I_N = 25/(10 + j\,20) = 25/22.36\ \angle 63.4° = 1.118\ \angle{-63.4°}$

When voltage source is replaced by a short, then the internal resistance of the circuit, as viewed from open terminals A and B, is $R_N = (10 + j20)\Omega$. Hence, Norton's equivalent circuit becomes as shown in Fig. 15.50(*c*).

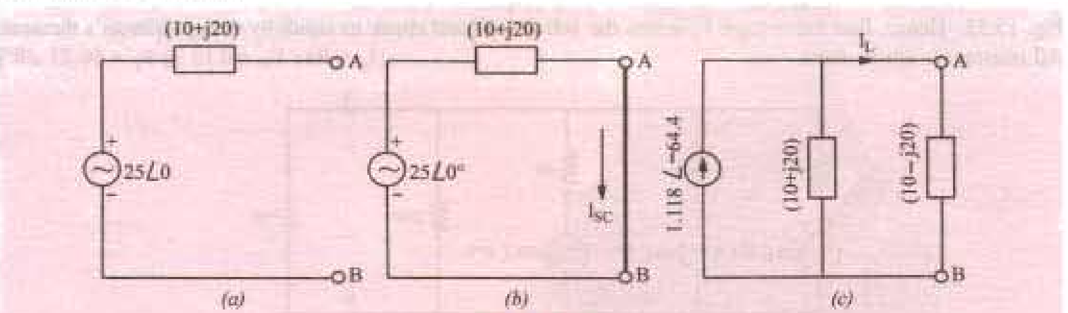


Fig. 15.50

When the load impedance of (10 − *j*20) is applied across the terminals A and B, current through it can be found with the help of current-divider rule.

$$\therefore \qquad I_L = 1.118 \angle{-63.4^\circ} \times \frac{(10 + j20)}{(10 + j20) + (10 - j20)} = 1.25 \text{ A}$$

Example 15.20. *Use Norton's theorem to find current in the load connected across terminals A and B of the circuit shown in Fig. 15.51 (a).*

Solution. The first step is to short-circuit terminals A and B as shown in Fig. 15.51 (a)*. The short across A and B not only short-circuits the load but the $(10 + j10)$ impedance as well.

$$I_N = 100 \angle 0^\circ / (-j\,10) = j\,10 = 10 \angle 90^\circ$$

Since the impedance of the Norton and Thevenin equivalent circuits is the same, $Z_N = 10 - j10$.

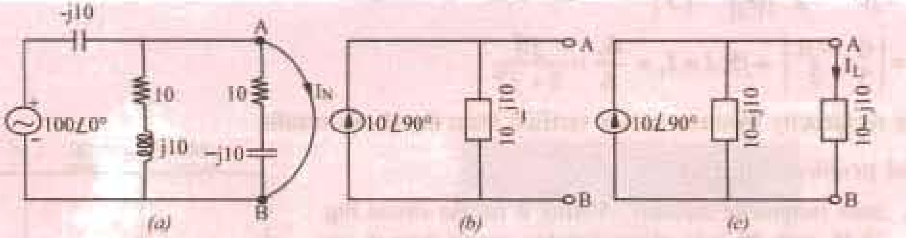
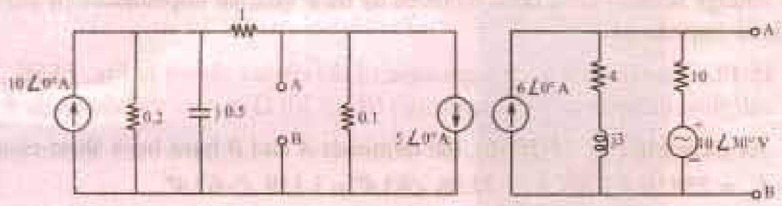


Fig. 15.51

The Norton's equivalent circuit is shown in Fig. 15.51 (b). In Fig. 15.51 (c), the load has been reconnected across the terminals A and B. Since the two impedances are equal, current through each is half of the total current *i.e.* $10\angle 90^\circ/2 = 5 \angle 90^\circ$.

Tutorial Problems No. 15.7

1. Find the Norton's equivalent sorce with respect to terminals A and B of the networks shown in Fig. 15.51 (a) (b). All resistances and reactances are expressed in siemens in Fig. 15.51 (a) and in ohms in Fig. 15.52. **[(a) $I_N = -(2.1 - j3)$ A; $1/Z_N = (0.39 + j0.3)$S (b) $I_N = (6.87 + j0.5)$ A; $1/Z_N = (3.17 + j1.46$S]**



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2. Find the Nortons equivalent source with respect to terminals A and B'for the circuit shown in Fig. 15.53. Hence, find the voltage V_L across the 100 Ω load and check its result by using Millman's theorem. All resistances are in ohms. **[$I_N = 9$A; $Y_N = 0.15$ S; $V_L = 56.25 \angle 0^\circ$]**

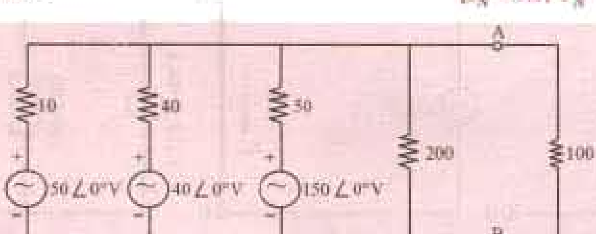


Fig. 15.54

* For finding I_N, we may or may not remove the load from the terminals (because, in either case, it would be short-circuited) but for finding R_N, it has to be removed as in the case of Thevenin's theorem.

3. Find the Norton's equivalent network at terminals *AB* of the circuit shown in Fig. 15.55.

[I_{SC} = 2.17 ∠–44° A; Z_N = (2.4 + j1.47) Ω]

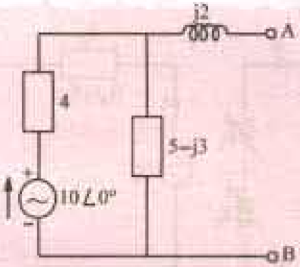


Fig. 15.55

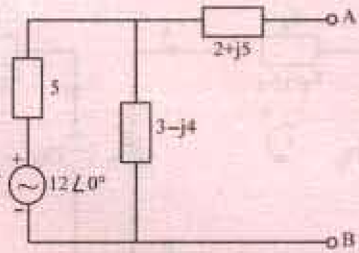


Fig. 15.56

4. What is the Norton equivalent circuit at terminals AB of the network shown in Fig. 15.56

[I_{SC} = 1.15 ∠–66.37° ; Z_N = (4.5 + j3.75) Ω]

15.9. Maximum Power Transfer Theorem

As explained earlier in Art. this theorem is particularly useful for analysing communication networks where the goals is transfer of maximum power between two circuits and not highest efficiency.

15.10. Maximum Power Transfer Theorems – General Case

We will consider the following maximum power transfer theorems when the source has a fixed complex impedance and delivers power to a load consisting of a variable resistance or a variable complex impedance.

Case 1. When load consists only of a variable resistance R_L [Fig. 15.57 (*a*)]. The circuit current is

$$I = \frac{V_g}{\sqrt{(R_g + R_L)^2 + X_g^2}}$$

Power delivered to R_L is P_L

$$= \frac{V_g^2 R_L}{(R_g + R_L)^2 + X_g^2}$$

To determine the value of R_L for maximum transfer of power, we should set the first derivative dP_L/dR_L to zero.

$$\frac{dP_L}{dR_L} = \frac{d}{dR_L}\left[\frac{V_g^2 R_L}{(R_g + R_L)^2 + X_g^2}\right] = V_g^2 \left\{\frac{[(R_g + R_L)^2 + X_g^2] - R_L(2)(Rg + R_L)}{[(R_g + R_L)^2 + X_g^2]^2}\right\} = 0$$

or $\quad R_g^2 + 2R_g R_L + R_L^2 + X_g^2 - 2R_L R_g - 2R_L^2 = 0$ and $R_g^2 + X_g^2 = R_L^2$

$$\therefore \qquad R_L = \sqrt{R_g^2 + X_g^2} = |Z_g|$$

It means that with a variable pure resistive load, maximum power is delivered across the terminals of an active network only when the load resistance is equal to the absolute value of the impedance of the active network. Such a match is called magnitude match.

Moreover, if X_g is zero, then for maximum power transfer $R_L = R_g$

Case 2. Load impedance having both variable resistance and variable reactance [Fig. 15.57 (*b*)].

The circuit current is $I = \dfrac{V_g}{\sqrt{(R_g + R_L)^2 + (X_g + X_L)^2}}$

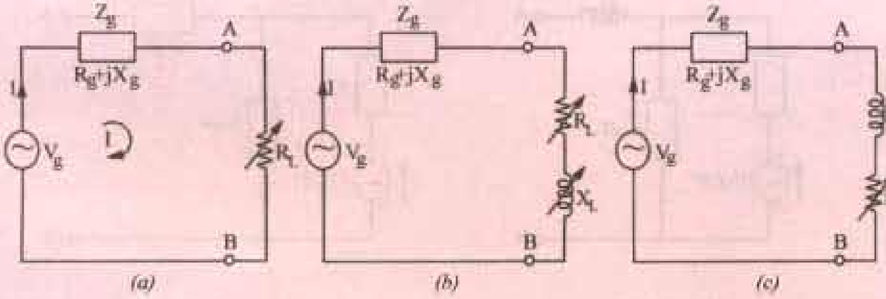


Fig. 15.57

The power delivered to the load is $= P_L = I^2 R_L = \dfrac{V_g^2 R_L}{(R_g + R_L)^2 + (X_g + X_L)^2}$

Now, if R_L is held fixed, P_L is maximum when $X_g = -X_L$. In that case $P_{Lmax} = \dfrac{V_g^2 R_L}{(R_g + R_L)^2}$

If on the other hand, R_L is variable then, as in **Case** 1 above, maximum power is delivered to the load when $R_L = R_g$. In that case if $R_L = R_g$ and $X_L = -X_g$, then $Z_L = Z_g$. Such a match is called conjugate match.

From the above, we come to the conclusion that in the case of a load impedance having both variable resistance and variable reactance, maximum power transfer across the terminals of the active network occurs when Z_L equals the complex conjugate of the network impedance Z_g *i.e.* the two impedances are conjugately matched.

Case 3. Z_L with variable resistance and fixed reactance [Fig. 15.57 (*c*)]. The equations for current I and power P_L are the same as in **Case** 2 above except that we will consider X_L to remain constant. When the first derivative of P_L with respect to R_L is set equal to zero, it is found that

$$R_L^2 = R_g^2 + (X_g + X_L)^2 \text{ and } R_L = |Z_g + jZ_L|$$

Since Z_g and X_L are both fixed quantities, these can be combined into a single impedance. Then with R_L variable, **Case** 3 is reduced to **Case** 1 and the maximum power transfer takes place when R_L equals the absolute value of the network impedance.

Summary

The above facts can be summarized as under :

1. When load is purely resistive and adjustable, MPT is achieved when $R_L = |Z_g| = \sqrt{R_g^2 + X_g^2}$.

2. When both load and source impedances are purely resistive (*i.e.* $X_L = X_g = 0$), MPT is achieved when $R_L = R_g$.

3. When R_L and X_L are both independently adjustable, MPT is achieved when $X_L = -X_g$ and $R_L = R_g$.

4. When X_L is fixed and R_L is adjustable, MPT is achieved when $R_L = \sqrt{[R_g^2 + (X_g + X_L)^2]}$

Example 15.21. *In the circuit of Fig. 15.58, which load impedance of p.f. = 0.8 lagging when connected across terminals A and B will draw the maximum power from the source. Also find the power developed in the load and the power loss in the source.*

Solution. For maximum power transfer $|Z_L| = |Z_1| = \sqrt{(3^2+5^2)} = 5.83\ \Omega$.

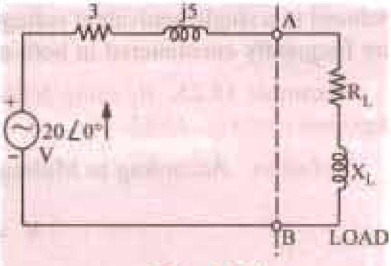


Fig. 15.58

For p.f. = 0.8, $\cos\phi = 0.8$ and $\sin\phi = 0.6$.

$\therefore\ R_L = Z_L \cos\phi = 5.83 \times 0.8 = 4.66\ \Omega$.

$X_L = Z_L \sin\phi = 5.83 \times 0.6 = 3.5\ \Omega$.

Total circuit impedance $Z = \sqrt{[(R_1+R_L)^2+(X_1+X_L)^2]}$

$= \sqrt{[(3+4.66)^2+(5+3.5)^2]} = 11.44\ \Omega$

$\therefore\ I = V/Z = 20/11.44 = 1.75$ A.

Power in the load $= I^2R_L = 1.75^2 \times 4.66 = 14.3$ W

Power loss in the source $= 1.75^2 \times 3 = 9.2$ W.

Example 15.22. *In the network shown in Fig. 15.59 find the value of load to be connected across terminals AB consisting of variable resistance R_L and capacitive reactance X_C which would result in maximum power transfer.* **(Network Analysis, Nagpur Univ. 1993)**

Solution. We will first find the Thevenin's equivalent circuit between terminals A and B. When the load is removed , the circuit become as shown in Fig. 15.59 (b).

$$V_{th} = \text{drop across } (2 + j10) = 50\angle 45° \times \frac{2+j10}{7+j10}$$
$$= 41.8 \times 68.7° = 15.2 + j\,38.9$$
$$R_{th} = 5\,||\,(2+j10) = 4.1\angle 23.7° = 3.7 + j\,1.6$$

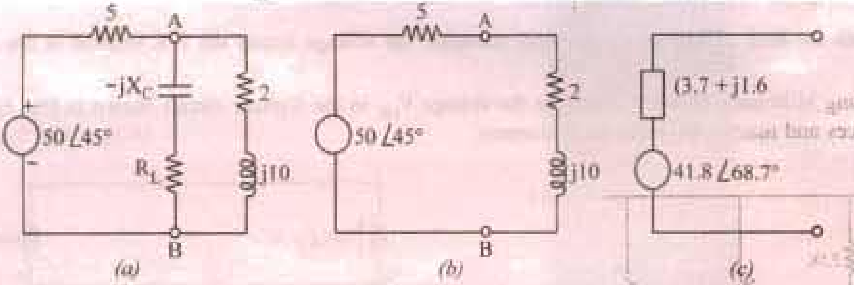


Fig. 15.59

The Thevenin's equivalent source is shown in Fig. 15.59 (*c*)

Since for MPT, conjugate match is required hence, $X_C = 1.6\ \Omega$ and $R_L = 3.7\ \Omega$.

Tutorial Problem No. 15.8

1. In the circuit of Fig. 15.60 the load consists of a fixed inductance having a reactance of $j10\ \Omega$ and a variable load resistor R_L. Find the value of R_L for MPT and the value of this power.

[58.3 Ω ; 46.2 W]

2. In the circuit of Fig. 15.61, the source resistance R_g is variable between 5 Ω and 50 Ω but R_L has a fixed value of 25 Ω. Find the value of R_g for which maximum power is dissipated in the load and the value of this power.**[5 Ω ; 250 W]**

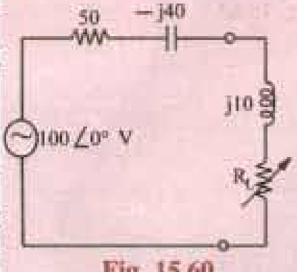


Fig. 15.60

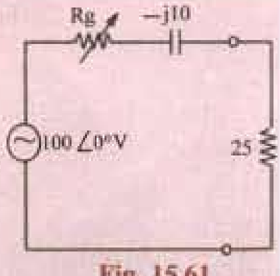


Fig. 15.61

15.11. Millman's Theroem

It permits any number of parallel branches consisting of voltage sources and impedances to be reduced to a single equivalent voltage source and equivalent impedance. Such multi-branch circuits are frequently encountered in both electronics and power applications.

Example 15.23. *By using Millman's theorem, calculate node voltatge V and current in the j6 impedance of Fig. 15.62.*

Solution. According to Millman's theorem as applicable to voltage sources.

$$V = \frac{\pm V_1Y_1 \pm V_2Y_2 \pm V_3Y_3 \pm .... \pm V_nY_n}{Y_1 + Y_2 + Y_3 + ... + Y_n}$$

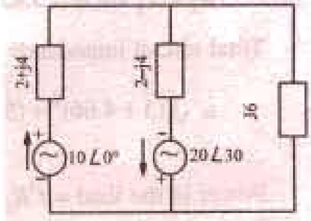


Fig. 15.62

$$Y_1 = \frac{1}{2+j4} = \frac{2-j4}{20} = 0.01 - j0.02 = 0.022\angle -63.4^\circ$$

$$Y_2 = \frac{1}{2-j4} = \frac{2+j4}{20} = 0.01 + j0.02 = 0.022\angle 63.4^\circ$$

$$Y_3 = \frac{1}{j6} = 0.167\angle -90^\circ = 0 - j0.167$$

$$Y_1 + Y_2 + Y_3 = 0.02 = j\,0.167$$

In the present case, $V_3 = 0$ and also V_2Y_2 would be taken as negative because current due to V_2 flows away from the node.

$$\therefore \quad V = \frac{V_1Y_1 - V_2Y_2}{Y_1+Y_2+Y_3} = \frac{10\angle 0^\circ \times 0.22\angle -63.4^\circ - 20\angle 30^\circ \times 0.022\angle 63.4^\circ}{0.02 - j0.167}$$

$$= 3.35\angle 177^\circ$$

Current through $j\,6$ impedance $= 3.35\angle 177^\circ / 6\angle 90^\circ = 0.56\angle 87^\circ$

Tutorial Problems No. 15.9

1. With the help of Millman's theorem, calculate the voltage across the 1 K resistor in the circuit of Fig. 15.63. **[2.79 V]**

2. Using Millman's theorem, calculate the voltage V_{ON} in the 3-phase circuit shown in Fig. 15.64. All load resistances and reactances are in milli-siemens. **[V_{ON} = 69.73 ∠113.53°]**

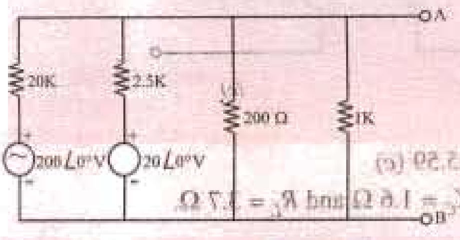


Fig. 15.63

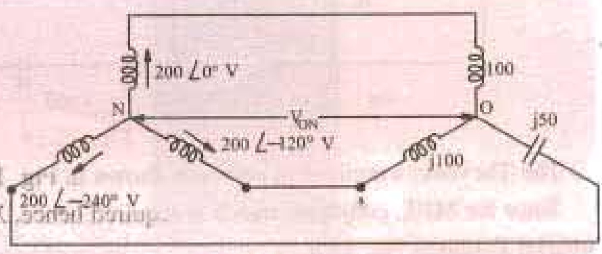


Fig. 15.64

16 A.C. BRIDGES

16.1. A.C. Bridges

Resistances can be measured by direct-current Wheatstone bridge, shown in Fig. 16.1 (*a*) for which the condition of balance is that

$$\frac{R_1}{R_2}=\frac{R_4}{R_3} \quad \text{or} \quad R_1R_3=R_2R_4^{*}$$

Inductances and capacitances can also be measured by a similar four-arm bridge, as shown in Fig. 16.1 (*b*); instead of using a source of direct current, alternating current is employed and galvanometer is replaced by a vibration galvanometer (for commercial frequencies or by telephone detector if frequencies are higher (500 to 2000 Hz)).

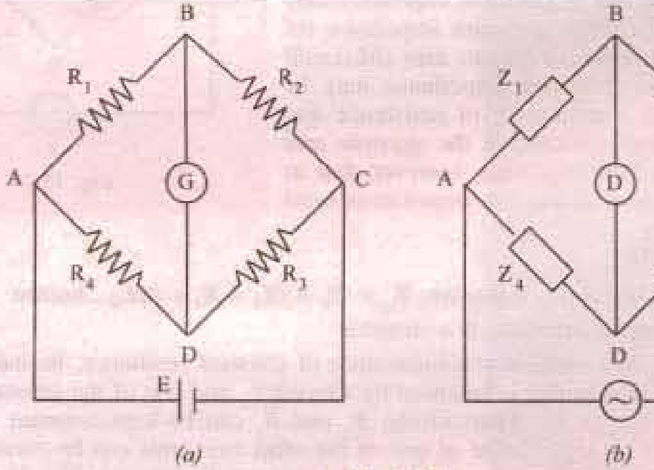


Fig. 16.1

The condition for balance is the same as before but instead of resistances, impedances are used *i.e.*

$$\mathbf{Z}_1/\mathbf{Z}_2=\mathbf{Z}_4/\mathbf{Z}_3 \quad \text{or} \quad \mathbf{Z}_1\mathbf{Z}_3=\mathbf{Z}_2\mathbf{Z}_4$$

But there is one important difference *i.e.* not only should there be balance for the magnitudes of the impedances but also a phase balance. Writing the impedances in their polar form, the above condition becomes

$$Z_1\angle\phi_1 . Z_3\angle\phi_3 = Z_2\angle\phi_2 . Z_4\angle\phi_4 \text{ or } Z_1Z_3\angle\phi_1+\phi_3 = Z_2Z_4\angle\phi_2+\phi_4$$

Hence, we see that, in fact, there are two balance conditions which must be satisfied simultaneously in a four-arm a.c. impedance bridge.

(*i*) $Z_1Z_3 = Z_2Z_4$ ... for magnitude balance

(*ii*) $\phi_1+\phi_3=\phi_2+\phi_4$ ... for phase angle balance

* Products of opposite arm resistances are equal.

In this chapter, we will consider a few of the numerous bridge circuits used for the measurement of self-inductance, capacitance and mutual inductance, choosing as examples some bridges which are more common.

16.2. Maxwell's Inductance Bridge

The bridge circuit is used for medium inductances and can be arranged to yield results of considerable precision. As shown in Fig. 16.2, in the two arms, there are two pure resistances so that for balance relations, the phase balance depends on the remaining two arms. If a coil of an unknown impedance Z_1 is placed in one arm, then its positive phase angle ϕ_1 can be compensated for in either of the following two ways:

(*i*) A known impedance with an equal positive phase angle may be used in either of the *adjacent* arms (so that $\phi_1 = \phi_2$ or $\phi_1 = \phi_4$). remaining two arms have zero phase angles (being pure resistances). Such a network is known as Maxwell's a.c. bridge or L_1/L_4 bridge.

(*ii*) Or an impedance with an equal *negative* phase angle (*i.e.* capacitance) may be used in *opposite* arm (so that $\phi_1 + \phi_3 = 0$). Such a network is known as Maxwell-Wien bridge (Fig. 16.5) or Maxwell's L/C bridge.

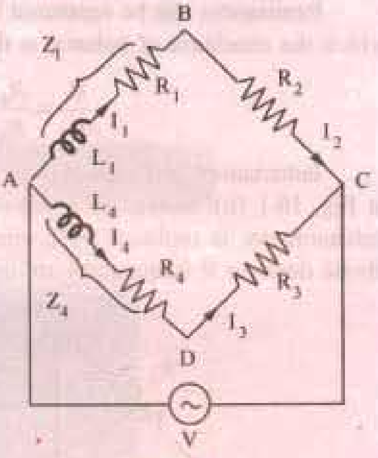


Fig. 16.2

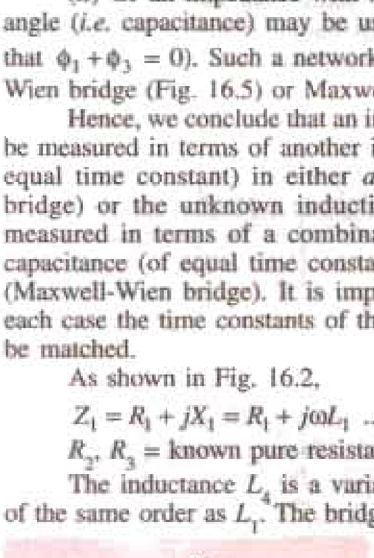
Hence, we conclude that an inductive impedance may be measured in terms of another inductive impedance (of equal time constant) in either *adjacent* arm (Maxwell bridge) or the unknown inductive impedance may be measured in terms of a combination of resistance and capacitance (of equal time constant) in the *opposite* arm (Maxwell-Wien bridge). It is important, however, that in each case the time constants of the two impedances must be matched.

As shown in Fig. 16.2,

$Z_1 = R_1 + jX_1 = R_1 + j\omega L_1$ ... unknown; $\mathbf{Z}_4 = R_4 + jX_4 = R_4 + j\omega L_4$ ...known

R_2, R_3 = known pure resistances; D = detector

The inductance L_4 is a variable self-inductance of constant resistance, its inductance being of the same order as L_1. The bridge is balanced by varying L_4 and one of the resistances R_2 or R_3. Alternatively, R_2 and R_3 can be kept constant and the resistance of one of the other two arms can be varied by connecting an additional resistance in that arm (Ex. 16.1).



$I_2R_2 = I_3R_3$; $I_1R_1 = I_4R_4$

Fig. 16.3

The balance condition is that $\mathbf{Z}_1\mathbf{Z}_3 = \mathbf{Z}_2\mathbf{Z}_4$

$$\therefore \quad (R_1 + j\omega L_1)R_3 = (R_4 + j\omega L_4)R_2$$

Equating the real and imaginary parts on both sides, we have

$$R_1R_3 = R_2R_4 \text{ or } R_1 / R_4 = R_2 / R_3$$

(*i.e.* products of the resistances of opposite arms are equal).

and $$\omega L_1 R_3 = \omega L_4 R_2 \text{ or } L_1 = L_4 \frac{R_2}{R_3}$$

We can also write that $L_1 = L_4 . \dfrac{R_1}{R_4}$*.

Hence, the unknown self-inductance can be measured in terms of the known inductance L_4 and the two resistors. Resistive and reactive terms balance independently and the conditions are independent of frequency. This bridge is often used for measuring the iron losses of the transformers at audio frequency.

---

* Or $\dfrac{L_1}{R_1} = \dfrac{L_4}{R_4}$ *i.e.*, the time constants of the two coils are matched.

The balance condition is shown vectorially in Fig. 16.3. The currents I_4 and I_3 are in phase with I_1 and I_2. This is, obviously, brought about by adjusting the impedances of different branches, so that these currents lag behind the applied voltage V by the same amount. At balance, the voltage drop V_1 across branch 1 is equal to that across branch 4 and $I_3 = I_4$. Similarly, voltage drop V_2 across branch 2 is equal to that across branch 3 and $I_1 = I_2$.

Example 16.1. *The arms of an a.c. Maxwell bridge are arranged as follows: AB and BC are non-reactive resistors of 100 Ω each. DA is a standard variable reactor L_1 of resistance 32.7 Ω and CD comprises a standard variable resistor R in series with a coil of unknown impedance. Balance was obtained with L_1 = 47.8 mH and R = 1.36 Ω. Find the resistance and inductance of the coil.*

(Elect. Inst. & Meas. Nagpur Univ. 1993)

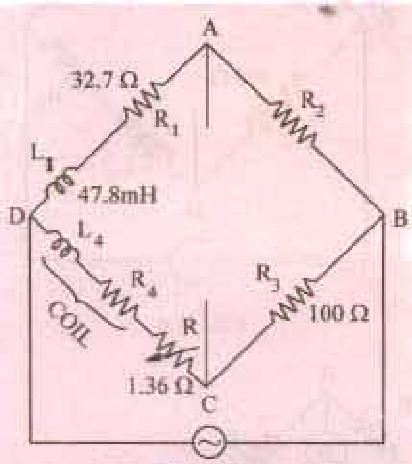


Fig. 16.4

Solution. The a.c. bridge is shown in Fig. 16.4.

Since the products of the resistances of opposite arms are equal

$\therefore\ 32.7 \times 100 = (1.36 + R_4)100$

$\therefore\ 32.7 = 1.36 + R_4$ or $R_4 = 32.7 - 1.36 =$ **31.34Ω**

Since $L_1 \times 100 = L_4 \times 100\ \ \therefore\ L_4 = L_1 =$ **47.8 mH**

or because time constants are the same, hence

$L_1/32.7 = L_4/(31.34 + 1.36)\ \therefore\ L_4 = 47.8$ mH

16.3. Maxwell-Wien Bridge or Maxwell's L/C Bridge

As referred to in Art. 16.2, the *positive* phase angle of an inductive impedance may be compensated by the *negative* phase angle of a capacitive impedance put in the *opposite* arm. The unknown inductance then becomes known in terms of this capacitance.

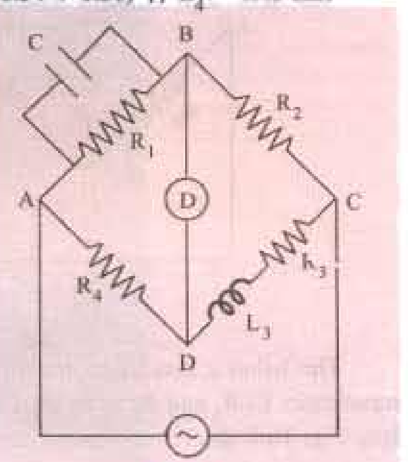


Fig. 16.5

Let us first find the combined impedance of arm 1.

$$\frac{1}{Z_1} = \frac{1}{R_1} + \frac{1}{-jX_C} = \frac{1}{R_1} + \frac{j}{X_C} = \frac{1}{R_1} + j\omega C = \frac{1 + j\omega CR_1}{R_1}$$

$$\therefore\quad \mathbf{Z}_1 = \frac{R_1}{1 + j\omega CR_1};\ \mathbf{Z}_2 = R_2$$

$$\mathbf{Z}_3 = R_3 + j\omega L_3 \text{ and } Z_4 = R_4$$

Balance condition is $\mathbf{Z}_1\mathbf{Z}_3 = \mathbf{Z}_2\mathbf{Z}_4$

or
$$\frac{R_1(R_3 + j\omega L_3)}{1 + j\omega CR_1} = R_2R_4 \text{ or } R_1R_3 + j\omega L_3 R_1 = R_2R_4 + j\omega CR_1R_2R_4$$

Separating the real and imaginaries, we get

$$R_1R_3 = R_2R_4 \text{ and } L_3 R_1 = C R_1R_2R_4;\ R_3 = \frac{R_2R_4}{R_1} \text{ and } L_3 = C R_2R_4$$

Example 16.2. *The arms of an a.c. Maxwell bridge are arranged as follows: AB is a non-inductive resistance of 1,000 Ω in parallel with a capacitor of capacitance 0.5 μF. BC is a non-inductive resistance of 600 Ω. CD is an inductive impedance (unknown) and DA is a non-inductive resistance of 400 Ω. If balance is obtained under these conditions, find the value of the resistance and the inductance of the branch CD.*

[Elect. & Electronic Meas, Madras Univ. 1986]

Solution. The bridge is shown in Fig. 16.6. The conditions of balance have already been derived in Art. 16.3 above.

Since $R_1R_3 = R_2R_4 \quad \therefore \quad R_3 = R_2R_4 / R_1$

$$\therefore \qquad R_3 = \frac{600 \times 400}{1000} = \mathbf{240\ \Omega}$$

Also $\quad L_3 = CR_2R_4$

$$= 0.5 \times 10^{-6} \times 400 \times 600$$

$$= 12 \times 10^{-2} = 0.12 \text{ H}$$

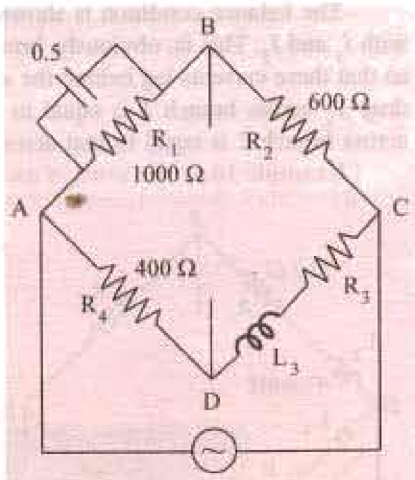


Fig. 16.6

16.4. Anderson Bridge

It is a very important and useful modification of the Maxwell-Wien bridge described in Art. 16.3. In this method, the unknown inductance is measured in terms of a known capacitance and resistance, as shown in Fig. 16.7.

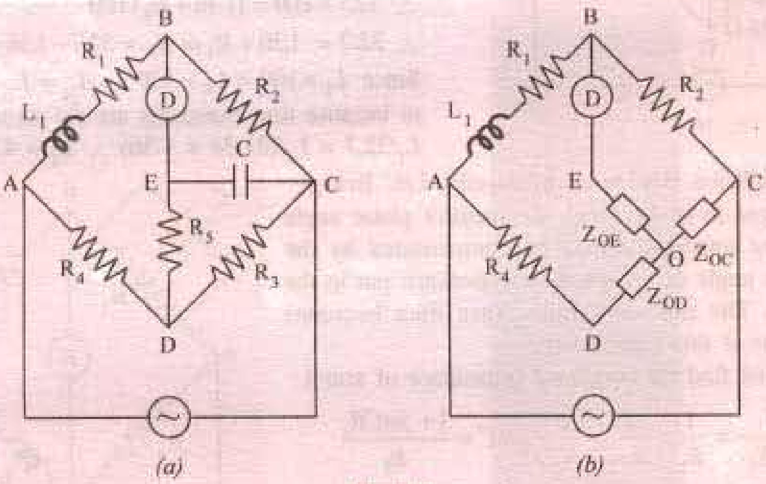


Fig. 16.7

The balance conditions for this bridge may be easily obtained by converting the mesh of impedances C, R_5 and R_3 to an equivalent star with star point O by Δ / Y transformation. As seen from Fig. 16.7 (*b*).

$$\mathbf{Z}_{OD} = \frac{R_3R_5}{(R_3 + R_5 + 1/j\omega C)}; \ \mathbf{Z}_{OC} = \frac{R_3 / j\omega C}{(R_3 + R_5 + 1/j\omega C)} = \mathbf{Z}_3$$

With reference to Fig. 16.7 (*b*) it is seen that

$$\mathbf{Z}_1 = (R_1 + j\omega L_1); \mathbf{Z}_2 = R_2; \mathbf{Z}_3 = \mathbf{Z}_{OC} \text{ and } \mathbf{Z}_4 = R_4 + \mathbf{Z}_{OD}$$

For balance $\mathbf{Z}_1\mathbf{Z}_3 = \mathbf{Z}_2\mathbf{Z}_4 \qquad \therefore \ (R_1 + j\omega L_1) \times \mathbf{Z}_{OC} = R_2(R_4 + \mathbf{Z}_{OD})$

$$\therefore \ (\mathrm{R}_1 + \mathrm{j}\omega \mathrm{L}_1)\frac{\mathrm{R}_3 / \mathrm{j}\omega \mathrm{C}}{(\mathrm{R}_3 + \mathrm{R}_5 + 1/\mathrm{j}\omega \mathrm{C})} = \mathrm{R}_2\left(\mathrm{R}_4 + \frac{\mathrm{R}_3\mathrm{R}_5}{\mathrm{R}_3 + \mathrm{R}_5 + 1/\mathrm{j}\omega \mathrm{C}}\right)$$

Further simplification leads to $R_2R_3R_4 + R_2R_4R_5 - j\dfrac{R_2R_4}{\omega C} + R_2R_3R_5 = -j\dfrac{R_1R_3}{\omega C} + \dfrac{R_3L_1}{C}$

$$\therefore \quad \frac{-jR_2R_4}{\omega C} = -\frac{jR_1R_3}{\omega C} \text{ or } R_1 = R_2R_4 / R_3$$

$$\text{Also } \frac{R_3L_1}{\omega C} = R_2R_3R_4 + R_2R_3R_5 + R_2R_4R_5 \quad \therefore \; L_1 = CR_2\left(R_4 + R_5 + \frac{R_4R_5}{R_3}\right)$$

This method is capable of precise measurements of inductances over a wide range of values from a few micro-henrys to several henrys and is one of the commonest and the best bridge methods.

Example 16.3. *An alternating current bridge is arranged as follows: The arms AB and BC consists of non-inductive resistances of 100-ohm each, the arms BE and CD of non-inductive variable resistances, the arm EC of a capacitor of 1 μF capacitance, the arm DA of an inductive resistance. The alternating current source is connected to A and C and the telephone receiver to E and D. A balance is obtained when resistances of arms CD and BE are 50 and 2,500 ohm respectively. Calculate the resistance and inductance of arm DA.*

Draw the vector diagram showing voltage at every point of the network.

(Elect. Measurements, Pune Univ. 1985)

Solution. The circuit diagram and voltage vector diagram are shown in Fig. 16.8. As seen, I_2 is vector sum of I_C and I_3. Voltage $V_2 = I_2 R_2 = I_C X_C$. Also, vector sum of V_1 and V_2 is V as well as that of V_3 and V_4. I_C is at right angles to V_2.

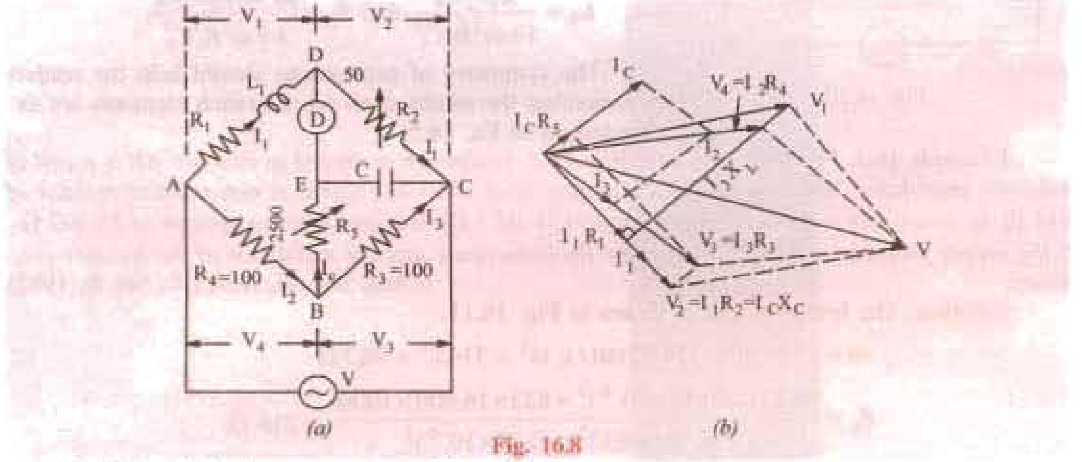


Fig. 16.8

Similarly, V_3 is the vector sum of V_2 and $I_C R_5$.

As shown in Fig. 16.8, $R_1 = R_2 . R_4 / R_3 = 50 \times 100/100 = 50\,\Omega$

The inductance is given by $L = CR_2 (R_4 + R_5 + R_4R_5 / R_3)$

$$\therefore \; L = 1 \times 10^{-6} \times 50(100 + 2500 + 100 \times 2500 / 100)$$
$$= \mathbf{0.2505 \; H}$$

Example 16.4. *Fig. 16.9 gives the connection of Anderson's bridge for measuring the inductance L_1 and resistance R_1 of a coil. Find R_1 and L_1 if balance is obtained when $R_3 = R_4 = 2000$ ohms, $R_2 = 1000$ ohms $R_5 = 200$ ohms and $C = 1 \mu F$. Draw the vector diagram for the voltages and currents in the branches of the bridge at balance.*

(Elect. Measurements, AMIE Sec. B Summer 1990)

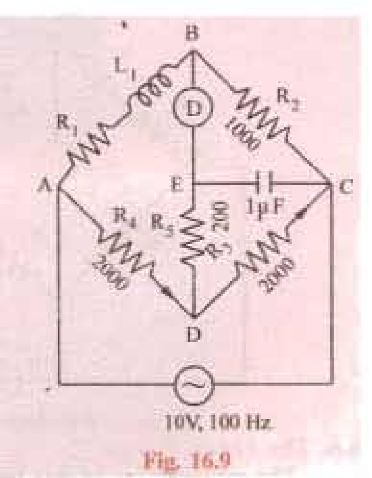


Fig. 16.9

Solution. $R_1 = R_2 R_4 / R_3 = 1000 \times 2000 / 2000 =$ **1000 Ω**

$$L_1 = CR_2\left(R_4 + R_5 + \frac{R_4R_5}{R_3}\right)$$

$$= 1 \times 10^{-6} \times 1000\left(2000 + 200 + \frac{2000 \times 200}{2000}\right) = \mathbf{2.4 \; H}$$

16.5. Hay's Bridge

It is also a modification of the Maxwell-Wien bridge and is particularly useful if the phase angle of the inductive impedance $\phi_m = \tan^{-1}(\omega L / R)$ is large. The network is shown in Fig. 16.10. It is seen that, in this case, a comparatively smaller series resistance R_1 is used instead of a parallel resistance (which has to be of a very large value).

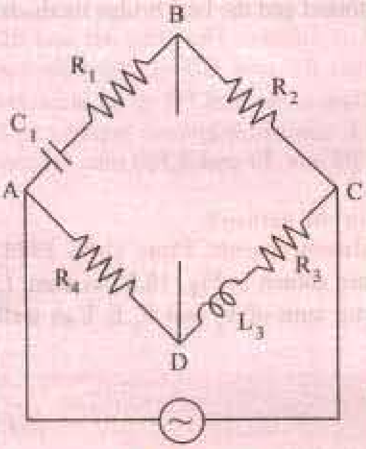


Fig. 16.10

Here $\mathbf{Z}_1 = R_1 - \frac{j}{\omega C_1}; \mathbf{Z}_2 = R_2$

$\mathbf{Z}_3 = R_3 + j\omega L_3; \mathbf{Z}_4 = R_4$

Balance condition is $\mathbf{Z}_1\mathbf{Z}_3 = \mathbf{Z}_2\mathbf{Z}_4$

$$\text{or} \quad \left(R_1 - \frac{j}{\omega C_1}\right)(R_3 + j\omega L_3) = R_2R_4$$

Separating the reals and the imaginaries, we obtain

$$R_1R_3 + \frac{L_3}{C_1} = R_2R_4 \text{ and } \omega L_3R_1 - \frac{R_3}{\omega C_1} = 0$$

Solving these simultaneous equations, we get

$$L_3 = \frac{C_1R_2R_4}{1+\omega^2R_1^2C_1^2} \text{ and } R_3 = \frac{\omega^2C_1^2R_1R_2R_4}{1+\omega^2R_1^2C_1^2}$$

The symmetry of expressions should help the readers to remember the results even when branch elements are exchanged, as in Ex. 16.5.

Example 16.5. *The four arms of a Hay's a.c. bridge are arranged as follows: AB is a coil of unknown impedance; BC is a non-reactive resistor of 1000 Ω; CD is a non-reactive resistor of 833 Ω in series with a standard capacitor of 0.38 μF : DA is a non-reactive resistor of 16,800 Ω. If the supply frequency is 50 Hz, determine the inductance and the resistance at the balance condition.* **(Elect. Measu. A.M.I.E. Sec B, 1992)**

Solution. The bridge circuit is shown in Fig. 16.11.

$$\omega = 277 \times 50 = 314.22 \text{ rad/s}; \ \omega^2 = 314.2^2 = 98{,}721$$

$$R_1 = \frac{98{,}721 \times (0.38 \times 10^{-6})^2 \times 833 \times 16{,}800 \times 1000}{1 + 98{,}721 \times 833^2 \times (0.38 \times 10^{-6})^2} = 210\ \Omega$$

$$L_1 = \frac{16{,}800 \times 1000 \times 0.38 \times 10^{-6}}{1 + 98{,}721 \times 833^2 \times (0.38 \times 10^{-6})^2} = 6.38\ \text{H}$$

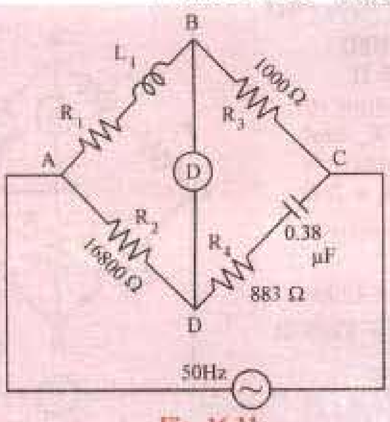


Fig. 16.11

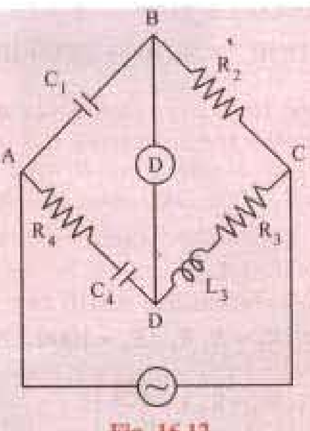


Fig. 16.12

6.6. The Owen Bridge

The arrangement of this bridge is shown in Fig. 16.12. In this method, also, the inductance is determined in terms of resistance and capacitance. This method has, however, the advantage of being useful over a very wide range of inductances with capacitors of reasonable dimensions.

Balance condition is $\mathbf{Z}_1\mathbf{Z}_3 = \mathbf{Z}_2\mathbf{Z}_4$

Here $\mathbf{Z}_1 = -\dfrac{j}{\omega C_1}$; $\quad \mathbf{Z}_2 = R_2$; $\mathbf{Z}_3 = R_3 + j\omega L_3$; $\mathbf{Z}_4 = R_4 - \dfrac{j}{\omega C_4}$

$$\therefore \qquad -\frac{j}{\omega C_1}(R_3 + j\omega L_3) = R_2\left(R_4 - \frac{j}{\omega C_4}\right)$$

Separating the reals and imaginaries, we get $R_3 = R_2\dfrac{C_1}{C_4}$ and $L_3 = C_1R_2R_4$.

Since ω does not appear in the final balance equations, hence the bridge is unaffected by frequency variations and wave-form.

16.7. Heavisible-Campbell Equal Ratio Bridge

It is a mutual inductance bridge and is used for measuring self-inductance over a wide range in terms of mutual inductometer readings. The connections for Heaviside's bridge employing a standard variable mutual inductance are shown in Fig. 16.13. The primary of the mutual inductometer is inserted in the supply circuit and the secondary having self-inductance L_2 and resistance R_2 is put in arm 2 of the bridge. The unknown inductive impedance having self-inductance of L_1 and resistance R_1 is placed in arm 1. The other two arms have pure resistances of R_3 and R_4.

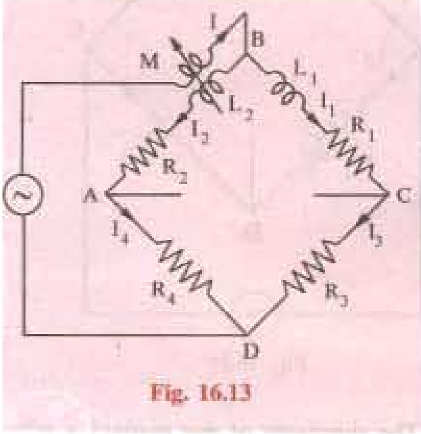


Fig. 16.13

Balance is obtained by varying mutual inductance M and resistances R_3 and R_4.

For balance, $I_1R_3 = I_2R_4$ ... (*i*)

$$I_1(R_1 + j\omega L_1) = I_2(R_2 + j\omega L_2) + j\omega MI \quad \text{... } (ii)$$

Since $I = I_1 + I_2$, hence putting the value of I in equation (*ii*), we get

$$I_1[R_1 + j\omega(L_1 - M)] = I_2[R_2 + j\omega(L_2 + M)] \quad \text{... } (iii)$$

Dividing equation (*iii*) by (*i*), we have $\dfrac{R_1 + j\omega(L_1 - M)}{R_3} = \dfrac{R_2 + j\omega(L_2 + M)}{R_4}$

$$\therefore \qquad R_3[R_2 + j\omega(L_2 + M)] = R_4[R_1 + j\omega(L_1 - M)]$$

Equating the real and imaginaries, we have $R_2R_3 = R_1R_4$ ... (*iv*)

Also, $R_3(L_2 + M) = R_4(L_1 - M)$. If $R_3 = R_4$ then $L_2 + M = (L_1 - M)$ $\therefore$ $L_1 - L_2 = 2M$ ... (*v*)

This bridge, as modified by Campbell, is shown in Fig. 16.14. Here $R_3 = R_4$. A balancing coil or a test coil of self-inductance equal to the self-inductance L_2 of the secondary of the inductometer and of resistance slightly greater than R_2 is connected in series with the unknown inductive impedance (R_1 and L_1) in arm 1. A non-inductive resistance box along with a constant-inductance rheostat are also introduced in arm 2 as shown.

Balance is obtained by varying M and r. Two readings are taken; one when Z_1 is in circuit and second when Z_1 is removed or short-circuited across its terminals.

With unknown impedance Z_1 still in circuit, suppose for balance the values of mutual inductance and r are M_1 and r_1. With Z_1 short-circuited, let these values be M_2 and r_2. Then

$$L_1 = 2(M_1 - M_2) \text{ and } R_1 = r_1 - r_2$$

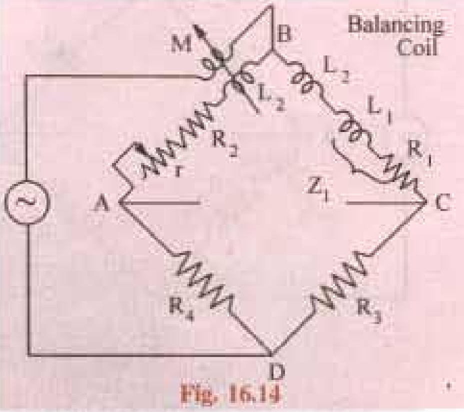


Fig. 16.14

By this method, the self-inductance and resistance of the leads are eliminated.

Example 16.6. *The inductance of a coil is measured by using the Heaviside-Campbell equal ratio bridge. With the test coil short-circulated, balance is obtained when adjustable non-reactive resistance is 12.63 Ω and mutual inductometer is set at 0.1 mH. When the test coil is in circuit, balance is obtained when the adjustable resistance is 25.9 Ω and mutual inductometer is set at 15.9 mH. What is the resistance and inductance of the coil?*

Solution. With reference to Art. 16.7 and Fig. 16.14, $r_1 = 25.9\ \Omega$, $M_1 = 15.9$ mH

With test coil short-circuited $\quad r_2 = 12.63\ \Omega$; $M_2 = 0.1$ mH

$$L_1 = 2\,(M_1 - M_2) = 2\,(15.9 - 0.1) = \mathbf{31.6\ mH}$$

$$R_1 = -r_1 - r_2 = 25.9 - 12.63 = \mathbf{13.27\ \Omega}$$

16.8. Capacitance Bridges

We will consider only De Sauty bridge method of comparing two capacitances and Schering bridge used for the measurement of capacitance and dielectric loss.

16.9. De Sauty Bridge

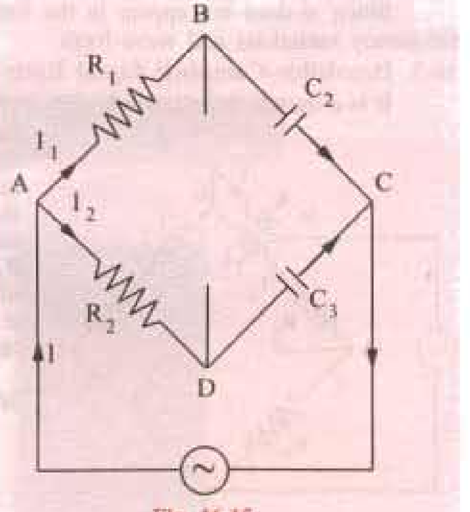


Fig. 16.15

With reference to Fig. 16.15, let

C_2 = capacitor whose capacitance is to be measured

C_3 = a standard capacitor

R_1, R_2 = non-inductive resistors

Balance is obtained by varying either R_1 or R_2. For balance, points B and D are at the same potential.

$$\therefore \quad I_1R_1 = I_2R_2 \text{ and } \frac{-j}{\omega C_2}.I_1 = \frac{-j}{\omega C_3}.I_2$$

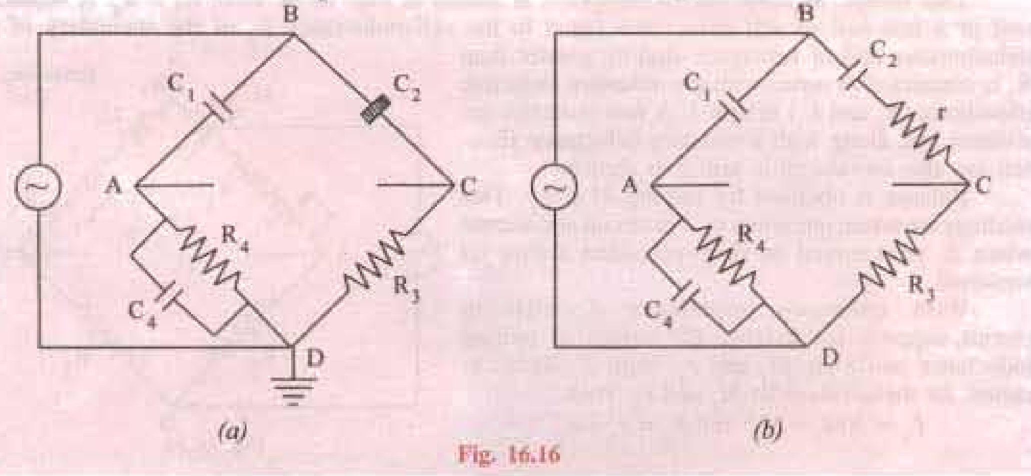
Dividing one equation by the other, we get

$$\frac{R_1}{R_2} = \frac{C_2}{C_3};\ C_2 = C_3\frac{R_1}{R_2}$$

The bridge has maximum sensitivity when $C_2 = C_3$. The simplicity of this method is offset by the impossibility of obtaining a perfect balance if both the capacitors are not free from the dielectric loss. A perfect balance can only be obtained if air capacitors are used.

16.10. Schering Bridge

It is one of the very important and useful methods of measuring the capacitance and dielectric loss of a capacitor. In fact, it is a device for comparing an imperfect capacitor C_2 in terms of a loss-free standard capacitor C_1 [Fig. 16.16 (*a*)]. The imperfect capacitor is represented by its equivalent loss-free capacitor C_2 in series with a resistance r [Fig. 16.16 (*b*)].



(*a*) (*b*)

Fig. 16.16

For high voltage applications, the voltage is applied at the junctions shown in the figure. The junction between arms 3 and 4 is earthed. Since capacitor impedances at lower frequencies are much higher than resistances, most of the voltage will appear across capacitors. Grounding of the junction affords safety to the operator form the high-voltage hazards while making balancing adjustment in arms 3 and 4.

Now $$\mathbf{Z}_1 = \frac{-j}{\omega C_1};\mathbf{Z}_2 = r - \frac{j}{\omega C_2};\mathbf{Z}_3 = R_3;\mathbf{Z}_4 = \frac{1}{(1/R_4)+j\omega C_4} = \frac{R_4}{1+j\omega C_4 R_4}$$

For balance, $\mathbf{Z}_1\mathbf{Z}_3 = \mathbf{Z}_2\mathbf{Z}_4$

or $$\frac{-jR_3}{\omega C_1} = \left(r - \frac{j}{\omega C_2}\right)\left(\frac{R_4}{1+j\omega C_4 R_4}\right) \text{ or } \frac{-jR_3}{\omega C_1}(1+\omega_s C_4 R_4) = R_4\left(r - \frac{1}{\omega C_2}\right)$$

Separating the real and imaginaries, we have $C_2 = C_1(R_4/R_3)$ and $r = R_3.(C_4/C_1)$.

The quality of a capacitor is usually expressed in terms of its phase defect angle or dielectric loss angle which is defined as the angle by *which current departs from exact quadrature from the applied voltage i.e.* the complement of the phase angle. If ϕ is the actual phase angle and δ the defect angle, then $\phi + \delta = 90^\circ$. For small values of δ, tan δ = sin δ = cos ϕ (approximately). Tan δ is usually called the *dissipation factor* of the *R–C* circuit. For low power factors, therefore, dissipation factor is approximately equal to the power factor.

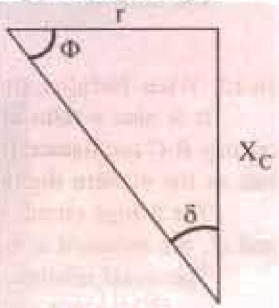


Fig. 16.17

As shown in Fig. 16.17.

Dissipation factor = power factor = tan δ

$$= \frac{r}{X_C} = \frac{r}{1/\omega C_2} = \omega r C_2$$

Putting the value of rC_2 from above.

Dissipation factor = $\omega r C_2 = \omega C_4 R_4$ = power factor.

Example 16.7. *In a test on a bakelite sample at 20 kV, 50 Hz by a Schering bridge, having a standard capacitor of 106 μF, balance was obtained with a capacitance of 0.35 μF in parallel with a non-inductive resistance of 318 ohms, the non-inductive resistance in the remaining arm of the bridge being 130 ohms. Determine the capacitance, the p.f. and equivalent series resistance of the specimen. Derive any formula used. Indicate the precautions to be observed for avoiding errors.* **(Elect. Engg. Paper I, Indian Engg. Services 1991)**

Solution. Here C_1 = 106 pF, $C_4 = 0.35\ \mu F$, $R_4 = 318\ \Omega$, $R_3 = 130\,\Omega$.

$$C_2 = C_1.(R_4/R_3) = 106\times 318/130 = \mathbf{259.3\ pF}$$

$$r = R_3.(C_4/C_1) = 130\times 0.35\times 10^{-6}/106\times 10^{-12} = \mathbf{0.429\ M\Omega}$$

$$\text{p.f.} = \omega_r C_2 = (2\pi\times 50)\times 0.429\times 10^{+6}\times 259.3\times 10^{-12} = \mathbf{0.035}$$

Example 16.8. *A losy capacitor is tested with a Schering bridge circuit. Balance obtained with the capacitor under test in one arm, the succeeding arms being, a non-inductive resistor of 100 Ω, a non-reactive resistor of 309 Ω in parallel with a pure capacitor of 0.5 μF and a standard capacitor of 109 $\mu\mu F$. The supply frequency is 50 Hz. Calculate from the equation at balance the equivalent series capacitance and power factor (at 50 Hz) of the capacitor under test.* **(Measu. & Instru., Nagpur Univ. 1992)**

Solution. Here, we are given

$$C_1 = 109\,\text{pF};\ R_3 = 100\Omega;\ C_4 = 0.5\ \mu\text{F};\ R_4 = 309\ \Omega$$

Equivalent capacitance is $C_2 = 109\times 309/100 = \mathbf{336.8\ pF}$

$$p.f. = \omega C_4 R_4 = 314\times 0.5\times 10^{-6}\times 309 = \mathbf{0.0485}$$

16.11. Wien Series Bridge

It is a simple ratio bridge and is used for audio-frequency measurement of capacitors over a wide range. The bridge circuit is shown in Fig. 16.18.

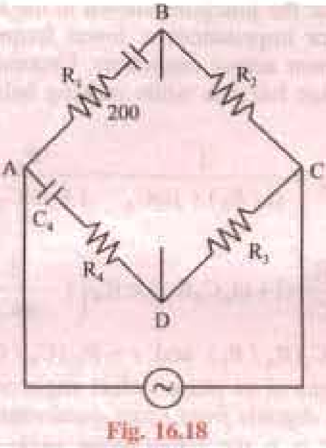


Fig. 16.18

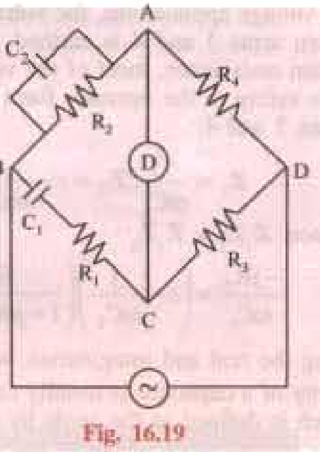


Fig. 16.19

The balance conditions may be obtained in the usual way. For balance

$$R_1 = R_2 R_4 / R_3 \text{ and } C_1 = C_4 (R_3 / R_2)$$

16.12. Wien Parallel Bridge

It is also a ratio bridge used mainly as the feedback network in the wide-range audio-frequency R-C oscillators. It may be used for measuring audio-frequencies although it is not as accurate as the modern digital frequency meters.

The bridge circuit is shown in Fig. 16.19. In the simple theory of this bridge, capacitors C_1 and C_2 are assumed to be loss-free and resistances R_1 and R_2 are separate resistors.

The usual relationship for balance gives

$$R_4\left(R_1 - \frac{j}{\omega C_1}\right) = R_3\left(\frac{R_2}{1 + j\omega C_2 R_2}\right) \text{ or } R_4\left(R_1 - \frac{j}{\omega C_1}\right)(1 + j\omega C_2 R_2) = R_2 R_3$$

Separating the real and imaginary terms, we have

$$R_1 R_4 + R_2 R_4 \frac{C_2}{C_1} = R_2 R_3 \quad \text{or} \quad \frac{C_2}{C_1} = \frac{R_3}{R_4} - \frac{R_1}{R_2} \qquad \text{... (i)}$$

and

$$\omega C_2 R_2 R_4 - \frac{R_4}{\omega C_1} = 0 \quad \text{or} \quad \omega^2 = \frac{1}{R_1 R_2 C_1 C_2} \qquad \text{... (ii)}$$

or

$$f = \frac{1}{2\pi\sqrt{R_1 R_2 C_1 C_2}} \text{ Hz}$$

Note. Eq. (*ii*) may be used to find angular frequency ω of the source if terms are known. For such purposes, it is convenient to make $C_1 = 2C_2, R_3 = R_4$ and $R_2 = 2R_1$. In that case, the bridge has equal ratio arms so that Eq. (*i*) will always be satisfied. The bridge is balanced simultaneously by adjusting R_2 and R_1 (though maintaining $R_2 = 2R_1$). Then, as seen from Eq. (*ii*) above

$$\omega^2 = 1/(R_1 . 2R_1 . 2C_2 . C_2) \text{ or } \omega = 1/(2R_1 C_2)$$

Example 16.9. *The arms of a four-arm bridge ABCD, supplied with a sinusoidal voltage, have the following values:*

AB : 200 ohm resistance in parallel with 1 μF capacitor; BC : 400 ohm resistance; CD : 1000 ohm resistance and DA : resistance R in series with a 2μF capacitor.

Determine (i) the value of R and (ii) the supply frequency at which the bridge will be balanced. **(Elect. Meas. A.M.I.E. Sec. 1991)**

Solution. The bridge circuit is shown in Fig. 16.20.

(*i*) As discussed in Art. 16.12, for balance we have

$$\frac{C_2}{C_1} = \frac{R_3}{R_4} - \frac{R_1}{R_2} \text{ or } \frac{2}{1} = \frac{1000}{4000} - \frac{R_1}{200}$$

$\therefore\ R_1 = 200 \times 0.5 = 100\ \Omega$

(*ii*) The frequency at which bridge is balanced is given by

$$f = \frac{1}{2\pi\sqrt{R_1 R_2 C_1 C_2}} \text{ Hz}$$

$$= \frac{10^6}{2\pi\sqrt{100 \times 200 \times 1 \times 2}} = 796 \text{ Hz}$$

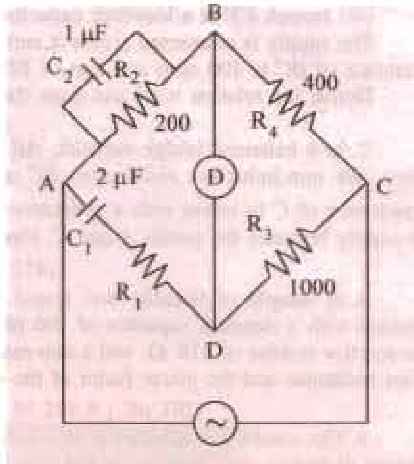


Fig. 16.20

Tutorial Problems No. 16.1

1. In Anderson a.c. bridge, an impedance of inductance L and resistance R is connected between A and B. For balance following data is obtained. An ohmic resistance of 1000Ω each in arms AD and CD, a non-inductive resistance of 500Ω in BC, a pure resistance of 200Ω between points D and E and a capacitor of $2\mu F$ between C and E. The supply is 10 volt (A.C.) at a frequency of 100 Hz and is connected across points A and C. Find L and R. [1.4 H; 500 Ω]

2. A balanced bridge has the following components connected between its five nodes, A, B, C, D and E:

Between A and B :	1,000 ohm resistance;	Between B and C :	1,000 ohm resistance
Between C and D :	an inductor;	Between D and A :	218 ohm resistance
Between A and E :	469 ohm resistance;	Between E and B :	10 μ F capacitance
Between E and C :	a detector;	Between B and D :	a power supply (a.c.)

Derive the equations of balance and hence deduce the resistance and inductance of the inductor.

[R = 218 Ω, L = 7.89 H] (*Elect. Theory and Meas. London Univ.*)

3. An a.c. bridge is arranged as follows: The arms AB and BC consist of non-inductive resistance of 100Ω, the arms, BE and CD of non-inductive variable resistances, the arm EC of a capacitor of 1 μF capacitance, the arm DA of an inductive resistance. The a.c. source is connected to A and C and the telephone receiver to E and D. A balance is obtained when the resistances of the arms CD and BE are 50 Ω and 2500 Ω respectively.

Calculate the resistance and the inductance of the arm DA. What would be the effect of harmonics in the waveform of the alternating current source? [50 Ω; 0.25 H]

4. For the Anderson's bridge of Fig. 16.21, the values are underbalance conditions. Determine the values of unknown resistance R and inductance L. [R = 500 Ω; L = 1.5 H]

(*Elect. Meas & Inst. Madras Univ. Nov. 1978*)

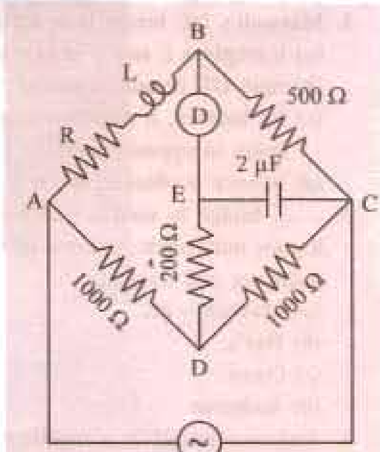


Fig. 16.21

5. An Anderson's bridge is arranged as under and balanced for the following values of the bridge components:

Branch AB – unknown coil of inductance L and resistance R
Branch BC – non-inductive resistance of 500 Ω
Branches AD & CD – non-inductive resistance of 100 Ω each
Branch DE – non-inductive resistance of 200 Ω
Branch EB – vibration galvanometer
Branch EC – 2.0 μF capacitance
Between A and C is 10 V, 100-Hz a.c. supply. Find the values of R and L of the unknown coil.

[R = 500 Ω; L = 0.5 H] (*Elect. Meas & Meas. Inst., Gujarat Univ. Oct. 1977*)

6. An a.c. Anderson bridge is arranged as follows:

(*i*) branches BC and ED are variable non-reactive resistors
(*ii*) branches CD and DA are non-reactive resistors of 200 ohm each

(*iii*) branch *CE* is a loss-free capacitor of 1 μF capacitance.

The supply is connected across *A* and *C* and the detector across *B* and *E*. Balance is obtained when the resistance of *BC* is 400 ohm and that of *ED* is 500 ohm. Calculate the resistance and inductance of *AB*.

Derive the relation used and draw the vector diagram for balanced condition of the bridge.

[400 Ω ; 0.48 H] (*Elect. Measurements, Poona Univ. May 1979*)

7. In a balanced bridge network, *AB* is a resistance of 500 ohm in series with an inductance of 0.18 henry, the non-inductive resistances *BC* and *DA* have values of 1000 ohm and arm *CD* consists of a capacitance of *C* in series with a resistance *R*. A potential difference of 5 volts at a frequency $5000/2\pi$ is the supply between the points *A* and *C*. Find out the values of *R* and *C* and draw the vector diagram.

[472 Ω ; 0.235 μF] (*Elect. Measurements, Poona Univ., April 1979*)

8. A sample of bakelite was tested by the Schering bridge method at 25 kV, 50-Hz. Balance was obtained with a standard capacitor of 106 pF capacitance, a capacitor of capacitance 0.4 μF in parallel with a non-reactive resistor of 318 Ω and a non-reactive resistor of 120 Ω. Determine the capacitance, the equivalent series resistance and the power factor of the specimen. Draw the phase or diagram for the balanced bridge.

[281 pF ; 0.452 M Ω ; 0.04] (*Elect. Measurements-II; Bangalore Univ., Jan. 1981*)

9. The conditions at balance of a Schering bridge set up to measure the capacitance and loss angle of a paper dielectric capacitor are as follows:

f = 500 Hz
Z_1 = a pure capacitance of 0.1 μF
Z_2 = a resistance of 500 Ω shunted by a capacitance of 0.0033 μF
Z_3 = pure resistance of 163 Ω
Z_4 = the capacitor under test

Calculate the approximate values of the loss resistance of the capacitor assuming–
(*a*) series loss resistance (*b*) shunt loss resistance. **[5.37 Ω , 197,000 Ω]** (*London Univ.*)

OBJECTIVE TESTS – 16

1. Maxwell-Wien bridge is used for measuring
   (*a*) capacitance
   (*b*) dielectric loss
   (*c*) inductance
   (*d*) phase angle
2. Maxwell's *L/C* bridge is so called because
   (*a*) it employs *L* and *C* in two arms
   (*b*) ratio *L/C* remains constant
   (*c*) for balance, it uses two opposite impedances in opposite arms
   (*d*) balance is obtained when *L* = *C*
3. .... bridge is used for measuring an unknown inductance in terms of a known capacitance and resistance.
   (*a*) Maxwell's *L/C*
   (*b*) Hay's
   (*c*) Owen
   (*d*) Anderson
4. Anderson bridge is a modification of .... bridge.
   (*a*) Owen
   (*b*) Hay's
   (*c*) De Sauty
   (*d*) Maxwell-Wien
5. Hay's bridge is particularly useful for measuring
   (*a*) inductive impedance with large phase angle
   (*b*) mutual inductance
   (*c*) self inductance
   (*d*) capacitance and dielectric loss
6. The most useful ac bridge for comparing capacitances of two air capacitors is .... bridge.
   (*a*) Schering (*b*) De Sauty
   (*c*) Wien series (*d*) Wien parallel
7. Heaviside-Campbell Equal Ratio bridge is used for measuring
   (*a*) self-inductance in terms of mutual inductance
   (*b*) capacitance in terms of inductance
   (*c*) dielectric loss of an imperfect capacitor
   (*d*) phase angle of a coil
8. The capacitance and dielectric loss of a capacitor is generally measured with the help of .... bridge.
   (*a*) De Sauty
   (*b*) Schering
   (*c*) Wien Series
   (*d*) Anderson

ANSWERS

1. *c* 2. *c* 3. *d* 4. *d* 5. *a* 6. *b* 7. *a* 8. *b* 9. *a* 10. *c*

17 A.C. FILTER NETWORKS

17.1. Introduction

The reactances of inductors and capacitors depend on the frequency of the a.c. signal applied to them. That is why these devices are known as frequency-selective. By using various combinations of resistors, inductors and capacitors, we can make circuits that have the property of passing or rejecting either low or high frequencies or bands of frequencies. These frequency-selective networks, which alter the amplitude and phase characteristics of the input a.c. signal, are called filters. Their performance is usually expressed in terms of how much attenuation a band of frequencies experiences by passing through them. Attenuation is commonly expressed in terms of decibels (dB).

17.2. Applications

A.C. filters find application in audio systems and television etc. Bandpass filters are used to select frequency ranges corresponding to desired radio or television station channels. Similarly, bandstop filters are used to reject undesirable signals that may contaminate the desirable signal. For example, low-pass filters are used to eliminate undesirable hum in d.c. power supplies.

No loudspeaker is equally efficient over the entire audible range of frequencies. That is why high-fidelity loudspeaker systems use a combination of low-pass, high-pass and bandpass filters (called crossover networks) to separate and then direct signals of appropriate frequency range to the different loudspeakers making up the system. Fig. 17.1 shows the output circuit of a

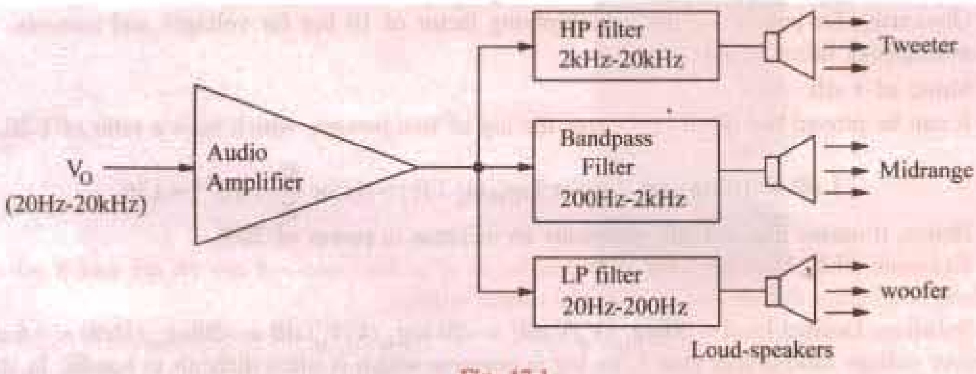


Fig. 17.1

high-fidelity audio amplifier, which uses three filters to separate, the low, mid-range and high frequencies, for feeding them to individual loudspeakers, best able to reproduce them.

17.3. Different Types of Filters

A.C. filter networks are divided into two major categories: (*i*) active networks and (*ii*) passive networks.

Active filter networks usually contain transistors and/or operational amplifiers in combination with *R*, *L* and *C* elements to obtain the desired filtering effect. These will not be discussed in this book. We will consider passive filter networks only which usually consist of series-parallel combinations of *R*, *L* and *C* elements. There are four types of such networks, as described below:

1. Low-Pass Filter. As the name shows, it allows only low frequencies to pass through, but attenuates (to a lesser or greater extent) all higher frequencies. The maximum frequency which it allows to pass through, is called cutoff frequency f_c (also called break frequency). There are R_L and R_C low-pass filters.

2. High-Pass Filter. It allows signals with higher frequencies to pass from input to output while rejecting lower frequencies. The minimum frequency it allows to pass is called cutoff frequency f_c. There are R_L and R_C high-pass filters.

3. Bandpass Filter. It is a resonant circuit which is tuned to pass a certain band or range of frequencies while rejecting all frequencies below and above this range (called passband).

4. Bandstop Filter. It is a resonant circuit that rejects a certain band or range of frequencies while passing all frequencies below and above the rejected band. Such filters are also called wavetraps, notch filters or band-elimination, band-separation or band-rejection filters.

17.4. Octaves and Decades of Frequency

A filter's performance is expressed in terms of the number of decibels the signal is increased or decreased per frequency octave or frequency decade. An octave means a doubling or halving of a frequency whereas a decade means tenfold increase or decrease in frequency.

17.5. The Decibel System

These system of logarithmic measurement is widely used in audio, radio, TV and instrument industry for comparing two voltages, currents or power levels. These levels are measured in a unit called bel (B) or decibel (dB) which is 1/10th of a bel.

Suppose we want to compare the output power P_0 of a filter with its input power P_i. The power level change is $= 10 \log_{10}{}^{th} P_0/P_i$ dB

It should be noted that dB is the unit of power change (*i.e.* increase or decrease) and not of power itself. Moreover, 20 dB is not twice as much power as 10 dB.

However, when voltage and current levels are required, then the expressions are:

$$\text{Current level} = 20 \log_{10} (I_0/I_i) \text{ dB}$$

$$\text{Similarly, voltage level} = 20 \log V_0/V_i \text{ dB}$$

Obviously, for power, we use a multiplying factor of 10 but for voltages and currents, we use a multiplying factor of 20.

17.6. Value of 1 dB

It can be proved that 1 dB represents the log of two powers, which have a ratio of 1.26.

$$1 \text{ dB} = 10\log_{10}(P_2 / P_1) \text{ or } \log_{10}(P_2 / P_1) = 0.1 \text{ or } \frac{P_2}{P_1} = 10^{0.1} = 1.26$$

Hence, it means that + 1 dB represents an increase in power of 26%.

Example 17.1. *The input and output voltages of a filter network are 16 mV and 8 mV respectively. Calculate the decibel level of the output voltage.*

Solution. Decibel level = $20\log_{10}(V_0/V_i)$dB = $-20 \log_{10}(V_i/V_0)$ dB = $-20\log_{10}(16/8) = -6$ dB. Whenever voltage ratio is less than 1, its log is negative which is often difficult to handle. In such cases, it is best to invert the fraction and then make the result negative, as done above.

Example 17.2. *The output power of a filter is 100 mW when the signal frequency is 5 kHz. When the frequency is increased to 25 kHz, the output power falls to 50 mW. Calculate the dB change in power.*

Solution. The decibel change in power is

$= 10 \log_{10}(50/100) = -10 \log_{10}(100/50) = -10 \log_{10} 2 = -10 \times 0.3 = -3$ dB

Example 17.3. *The output voltage of an amplifier is 10 V at 5 kHz and 7.07 V at 25 kHz. What is the decibel change in the output voltage?*

Solution. Decibel change $= 20 \log_{10}(V_0/V_i) = 20 \log_{10}(7.07/10) = -20 \log_{10}(10/7.07)$

$= -20 \log_{10}(1.4/4)$

$= -20 \times 0.15 = -3$ dB

17.7. Low-Pass RC Filter

A simple low-pass *RC* filter is shown in Fig. 17.2 (*a*). As stated earlier, it permits signals of low frequencies upto f_c to pass through while attenuating frequencies above f_c. The range of frequencies upto f_c is called the passband of the filter. Fig. 17.2 (*b*) shows the frequency response curve of such a filter. It shows how the signal output voltage V_0 varies with the signal frequency. As seen at f_c, output signal voltage is reduced to 70.7% of the input voltage. The output is said to be – 3 dB at f_c. Signal outputs beyond f_c roll-off or attenuate at a fixed rate of – 6 dB/octave or – 20 dB/decade. As seen from the frequency-phase response curve of Fig. 17.2 (*c*), the phase angle between V_0 and V_i is 45° at cutoff frequency f_c.

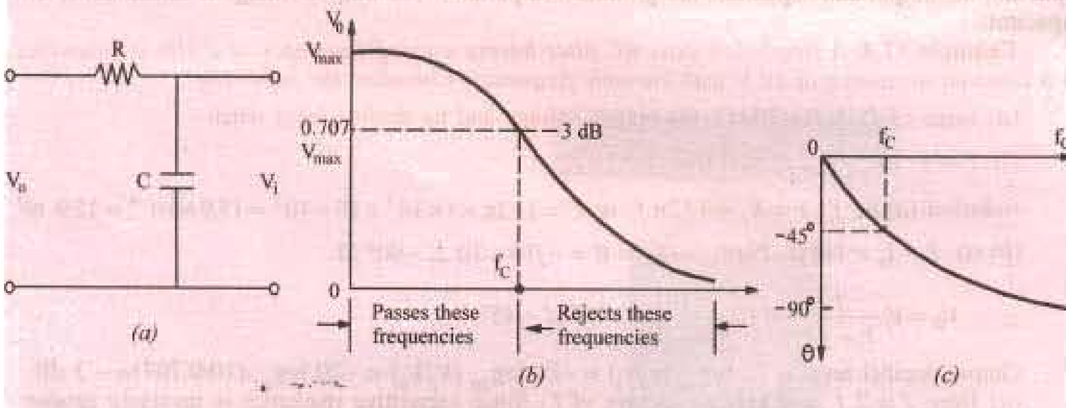


Fig. 17.2

By definition cutoff frequency f_c occurs where (*a*) $V_0 = 70.7\%\ V_i$ *i.e.* V_0 is – 3 dB down from V_i (*b*) $R = X_c$ and $V_R = V_C$ in magnitude. (*c*) The impedance phase angle $\theta = -45°$. The same is the angle between V_0 and V_i.

As seen, the output voltage is taken across the capacitor. Resistance *R* offers fixed opposition to frequencies but the reactance offered by capacitor *C* decreases with increase in frequency. Hence, low-frequency signal develops over *C* whereas high-frequency signals are grounded. Signal frequencies above f_c develop negligible voltage across *C*. Since *R* and *C* are in series, we can find the low-frequency output voltage V_0 developed across *C* by using the voltage-divider rule.

$$\therefore \qquad V_0 = V_i \frac{-jX_C}{R - jX_C} \text{ and } f_c = \frac{1}{2\pi CR}$$

17.8. Other Types of Low-Pass Filters

There are many other types of low-pass filters in which instead of pure resistance, series chokes are commonly used alongwith capacitors.

(*i*) **Inverted–L Type.** It is shown in Fig. 17.3 (*a*). Here, inductive reactance of the choke blocks higher frequencies and *C* shorts them to ground. Hence, only low frequencies below f_c (for which *X* is very low) are passed without significant attenuation.

(*ii*) **T-Type.** It is shown in Fig. 17.3 (*b*). In this case, a second choke is connected on the output side which improves the filtering action.

(*iii*) **π-Type.** It is shown in Fig. 17.3 (*c*). The additional capacitor further improves the filtering action by grounding higher frequencies.

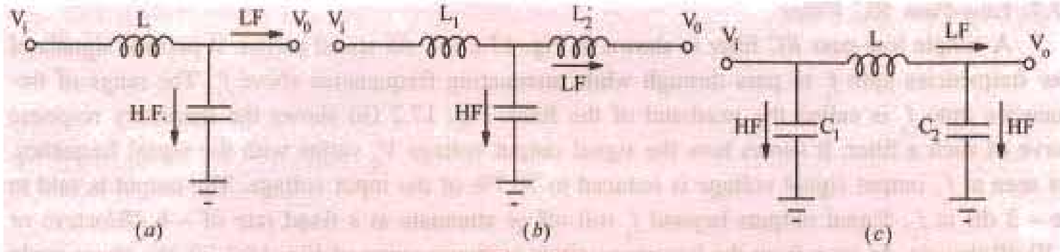


Fig. 17.3

It would be seen from the above figures that choke is always connected in series between the input and the output and capacitors are grounded in parallel. The output voltage is taken across the capacitor.

Example 17.4. *A simple low-pass RC filter having a cutoff frequency of 1 kHz is connected to a constant ac source of 10 V with variable frequency. Calculate the following :*

(a) value of C if $R = 10\,k\Omega$ (b) output voltage and its decibel level when

(i) $f = f_c$ (ii) $f = 2f_c$ and (iii) $f = 10 f_c$.

Solution. (*a*) At f_c, $r = X_c = 1/2\pi f_c$ or $C = 1/2\pi\times1\times10^3\times10\times10^3 = 15.9\times10^{-9} = 15.9$ nF

(*b*) (*i*) $f = f_c = 1\text{kHz}$. Now, $-jX_c = R = -j10 = 10\angle-90°\ \Omega$

$$\therefore\quad V_0 = V_i\frac{-jX_c}{R-jX_c} = 10\frac{-j10}{10-j10} = 7.07\angle-45°$$

Output decibel level = 20 $\log_{10}(v_o/v_i) = -20\log_{10}(V_i/V_0) = -20\log_{10}(10/0.707) = -3$ dB

(*ii*) Here, $f = 2f_c = 2$ kHz *i.e.* octave of f_c. Since capacitive reactance is inversely proportional to frequency, $\therefore X_{c2} = C_{c1}(f_1/f_2) = -j10(1/2) = -j5 = 5\angle-90°\,k\Omega$

$$\therefore\quad V_0 = \frac{5\angle-90°}{10-j5} = \frac{5°\angle-90°}{11.18\angle-26.6°} = 4.472\angle-63.4°$$

Decibel level = $-20\log_{10}(V_i/V_0) = -20\log_{10}(10/4.472) = -6.98$ dB

(*iii*) $X_{c3} = X_{c1}(f_1/f_3) = -j10\,(1/10) = j1 = 1\angle-90°\ \ k\Omega$

$$\therefore\quad V_0 = 10\frac{10\angle-90°}{10-j1} = 1\angle-84.3°$$

Decibel level = $-20\log_{10}(10/1) = -20$ dB

17.9. Low-Pass RL Filter

It is shown in Fig. 17.4 (*a*). Here, coil offers high reactance to high frequencies and low reactance to low frequencies. Hence, low frequencies upto f_c can pass through the coil without much opposition. The output voltage is developed across *R*. Fig 17.4 (*b*) shows the frequency-output response curve of the filter. As seen at f_c, $V_0 = 0.707\,V_i$ and its attenuation level is -3 *dB* with respect to V_0 *i.e.* the voltage at $f = 0$.

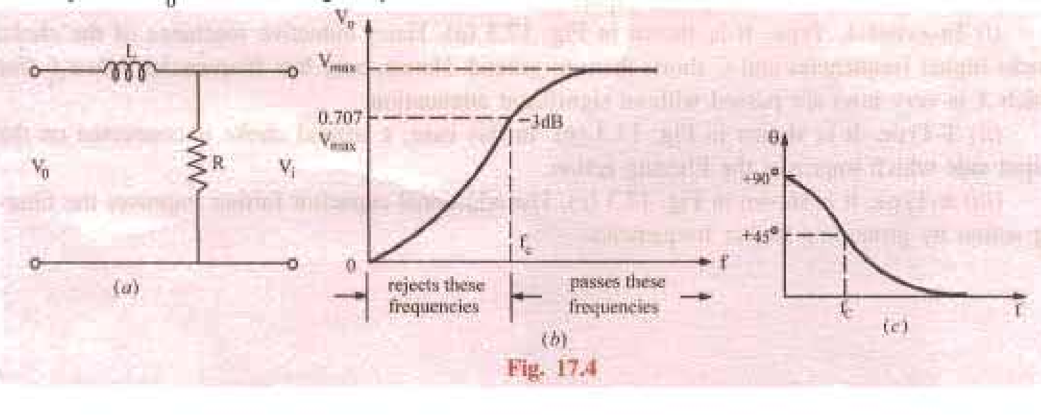


Fig. 17.4

However, it may be noted that being an *RL* circuit, the impedance phase angle is +45° (and not –45° as in low-pass *RC* filter). Again at f_c, $R = X_L$.

Using the voltage-divider rule, the output voltage developed across *R* is given by

$$V_0 = V_i \frac{R}{R + jX_L} \text{ and } f_c = \frac{R}{2\pi L}$$

Example 17.5. *An ac signal having constant amplitude of 10 V but variable frequency is applied across a simple low-pass RL circuit with a cutoff frequency of 1 kHz. Calculate (a) value of L if R = 1 kΩ (b) output voltage and its decibel level when (i) $f = f_c$ and (iii) $f = 10 f_c$.*

Solution. (*a*) $L = R/2\pi f_c = 1\times10^3/2\times10^3 = 159.2$ mH

(*b*) (*i*) $f = f_c = 1k\,Hz$; $jX_L = R = j1; V_0 = 10\frac{1}{(1+j1)} = 7.07\angle -45°\,V$

Decibel decrease = $-20 \log_{10} (V_i / V_0) = -20 \log_{10} 10/7.07 = -3$ dB

(*ii*) $f = 2f_c = 2$ kHz. Since X_L varies directly with f, $X_{L2} = X_{L1} (f_2/f_1) = 1 \times 2/1 = 2$ kΩ

$$\therefore \quad V_0 = 10\frac{1}{(1+j2)} = \frac{10}{2.236\angle 63.4°} = 4.472\angle -63.4°$$

Decibel decrease = $-20 \log_{10} (10/4.472) = -6.98$ dB

(*iii*) $f = 10 f_c = 10$ kHz ; $X_{L3} = 1 \times 10/1 = 10\,\Omega$, $V_0 = \frac{1}{(1+j10)} = 1\angle -84.3°$

Decibel decrease = $-20 \log_{10}(10/1) = -20$ dB

17.10. High-Pass RC Filter

It is shown in Fig. 17.5 (*a*). Lower frequencies experience considerable reactance by the capacitor and are not easily passed. Higher frequencies encounter little reactance and are easily passed. The high frequencies passing through the filter develop output voltage V_o across *R*. As seen from the frequency response of Fig. 13.5 (*b*), all frequencies above f_c are passed whereas those below it are attenuated. As before, f_c corresponds to –3 dB output voltage or half-power point. At f_c, $R = X_c$ and the phase angle between V_o and V_i is +45° as shown in Fig. 17.5 (*c*). It may be noted that high-pass *RC* filter can be obtained merely by interchanging the positions of *R* and *C* in the low-pass *RC* filter of Fig. 17.5 (*a*).

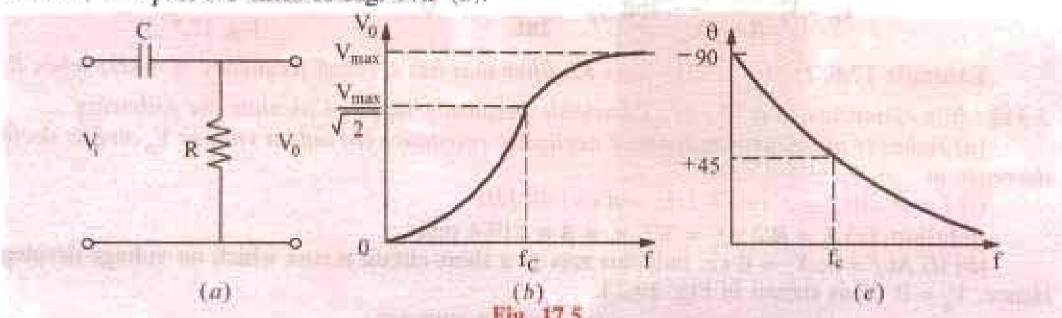


Fig. 17.5

Since *R* and *C* are in series across the input voltage, the voltage drop across *R*, as found by the voltage-divider rule, is

$$V_0 = V_i \frac{R}{R - jX_c} \text{ and } f_c = \frac{1}{2\pi CR}$$

A very common application of the series capacitor high-pass filter is a coupling capacitor between two audio amplifier stages. It is used for passing the amplified audio-signal from one stage to the next and simultaneously block the constant d.c. voltage.

Other high-pass RC filter circuits exist besides the one shown in Fig. 17.5 (*a*). These are shown in Fig 17.6.

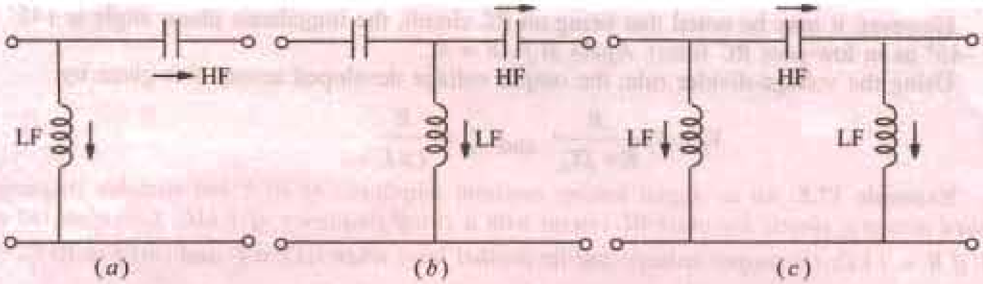


Fig. 17.6

(*i*) **Inverted-L Type.** It is so called because the capacitor and inductor from an upside down *L*. It is shown in Fig. 17.6 (*a*). At lower frequencies, X_C is large but X_L is small. Hence, most of the input voltage drops across X_C and very little across X_L. However, when the frequency is increased, X_C becomes less but X_L is increased thereby causing the output voltage to increase. Consequently, high frequencies are passed while lower frequencies are attenuated.

(*ii*) **T-Type.** It uses two capacitors and a choke as shown in Fig. 17.6 (*b*). The additional capacitor improves the filtering action.

(*iii*) **π -Type.** It uses two inductors which shunt out the lower frequencies as shown in Fig. 17.6 (*c*).

It would be seen that in all high-pass filter circuits, capacitors are in series between the input and output and the coils are grounded. In fact, capacitors can be viewed as shorts to high frequencies but as open to low frequencies. Opposite is the case with chokes.

17.11. High-Pass RL Filter

It is shown in Fig. 17.7 and can be obtained by 'swapping' position of *R* and *L* in the low-pass *RL* circuit of Fig. 17.4 (*a*). Its response curves are the same as for high-pass *RC* circuit and are shown in Fig. 17.5 (*b*) and (*c*).

As usual, its output voltage equals the voltage which drops across X_L. It is given by

$$V_0 = V_i \frac{jX_L}{R + jX_L} \quad \text{and} \quad f_c = \frac{R}{2\pi L}$$

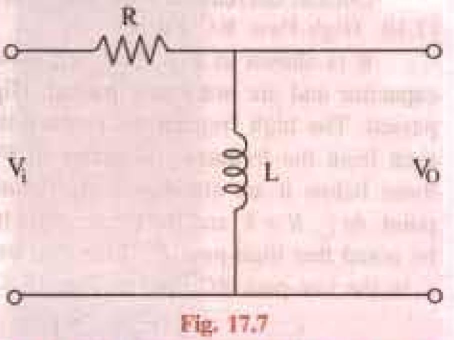


Fig. 17.7

Example 17.6. *Design a high-pass RL filter that has a cutoff frequency of 4 kHz when R = 3 kΩ. It is connected to a 10 ∠0° V variable frequency supply. Calculate the following :*

(a) Inductor of inductance L but of negligible resistance (b) output voltage V_o and its decibel decrease at

(i) f = 0 (ii) f = f_c (iii) 8 kHz and (iv) 40 kHz

Solution. (*a*) $L = R/2\pi f_c = 3/2\ \pi\ \times 4 = 119.4$ mH

(*b*) (*i*) At $f = 0$; $X_L = 0$ *i.e.* inductro acts as a short-circuit across which no voltage develops. Hence, $V_0 = 0$ V as shown in Fig. 17.74.

(*ii*) $f = f_c = 4$ kHz ; $X_L = R$. $\therefore\ jX_L = j3 = 3\angle 90°$ kΩ

$$\therefore\ V_0 = V_i \frac{jXL}{R + jX_L} = 10\angle 0° \frac{3\angle 90°}{3 + j3} = \frac{30\angle 90°}{4.24\angle 45°} = 7.07 - 45°\text{ V}$$

Decibel decrease = – 20 $\log_{10}$(10/7.07) = – 3 dB

(*iii*) $f = 2f_c = 8$ kHz. ere, $X_{L2} = 2 \times j3 = j6$ k Ω

$$\therefore\ V_0 = 10\angle 0° \frac{6\angle 90°}{3 + j6} = \frac{60\angle 90°}{6.7\angle 63.4°} = 8.95\angle 26.6°\text{ V}$$

Decibel decrease = – 20 $\log_{10}$(10/8.95) = – 0.96 dB

(iv) $f = 10f_c = 40$ kHz; $X_{L3} = 10 \times j3 = j30$ kΩ

$$\therefore \qquad B_0 = 10\frac{j30}{3 + j30} = \frac{300\angle 90^\circ}{30.15\angle 84.3^\circ} = 19.95\angle 5.7^\circ \text{ V}$$

Decibel decrease = $-20 \log_{10}(10/9.95) = 0.04$ dB

As seen from Fig. 17.7, as frequency is increased, V_0 is also increased.

17.12. R-C bandpass Filter

It is a filter that allows a certain band of frequencies to pass through and attenuates all other frequencies below and above the passband. This passband is known as the bandwidth of the filter. As seen, it is obtained by cascading a high-pass *RC* filter to a low-pass *RC* filter. It is shown in Fig. 17.8 alongwith its response curve. The passband of this filter is given by the band of frequencies lying between f_{c1} and f_{c2}. Their values are given by

$$f_{c1} = 1/2\pi\, C_1 R_1 \quad \text{and} \quad fc_2 = 1/2\pi C_2 R_2$$

The ratio of the output and input voltages is given by

$$\frac{V_0}{V_i} = \frac{R_1}{R_1 - jX_{C1}} \qquad \text{... from } f_1 \text{ to } f_{C1};$$

$$= \frac{-jX_{C2}}{R_2 - jX_{C2}} \qquad \text{... from } f_{C2} \text{ to } f_2$$

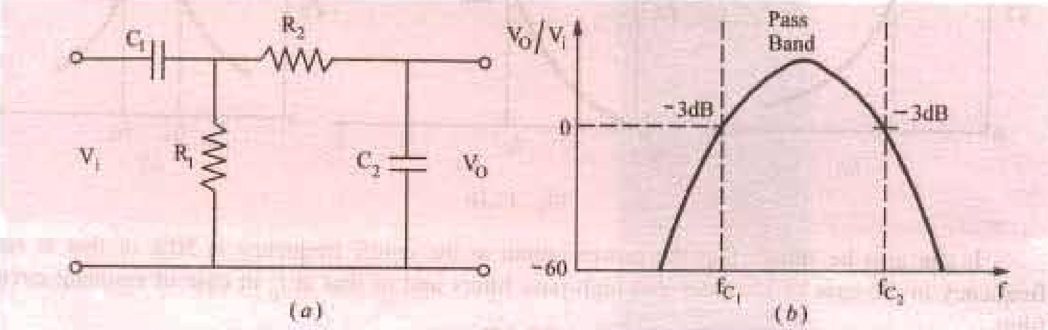


Fig. 17.8

17.13. R-C Bandstop Filter

It is a series combination of low-pass and high-pass RC filters as shown in Fig. 17.9 (*a*). In fact, it can be obtained by reversing the cascaded sequence of the RC bandpass filter. As stated earlier, this filter attenuates a single band of frequencies and allows those on either side to pass through. The stopband is represented by the group of frequencies that lie between f_1 and f_2 where response is below – 60 dB.

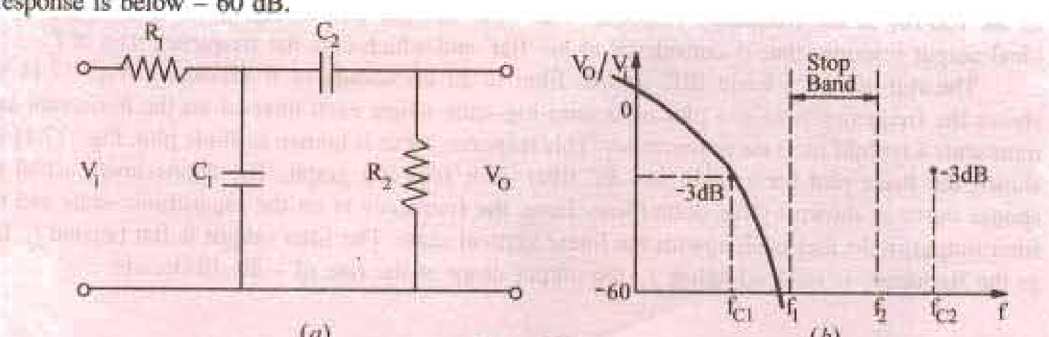


Fig. 17.9

For frequencies from f_{C1} to f_1, the following relationships hold good :

$$\frac{V_0}{V_i} = \frac{-j\,X_{C1}}{(R_1 - j\,X_{C1})} \quad \text{and} \quad f_{C1} = \frac{1}{2\pi C_1 R_1}$$

For frequencies from f_2 to f_{C2}, the relationships are as under :

$$\frac{V_0}{V_i} = \frac{R_2}{(R_2 - j\,X_{C2})} \text{ and } f_{C2} = \frac{1}{2\pi C_2 R_2}$$

In practices, several low-pass *RC* filter circuits cascaded with several high-pass *RC* filter circuits which provide almost vertical roll-offs and rises. Moreover, unlike *RL* filters, *RC* filters can be produced in the form of large-scale integrated circuits. Hence, cascading is rarely done with *RL* circuits.

17.14. The – 3 dB Frequencies

The output of an a.c. filter is said to be down 3 dB or –3 dB at the cutoff frequencies. Actually at this frequency, the output voltage of the circuit is 70.7% of the maximum input voltage as shown in Fig. 17.10 (*a*) for low-pass filter and in Fig. 17.10 (*b*) and (*c*) for high-pass and bandpass filters respectively. Here, maximum voltage is taken as the 0 dB reference.

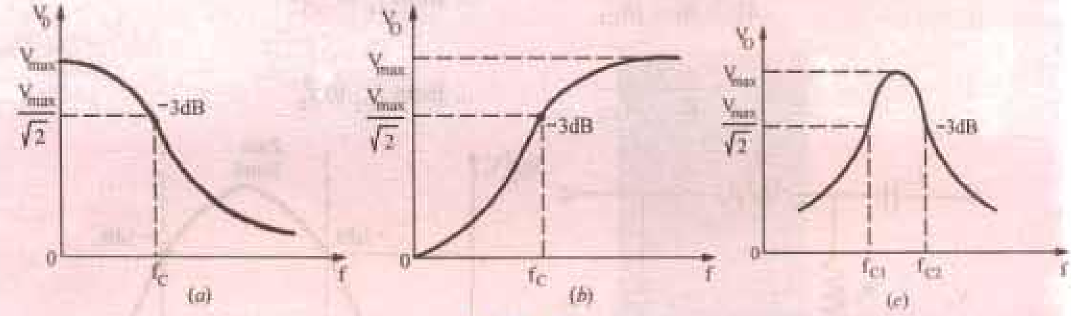


Fig. 17.10

It can also be shown that the power output at the cutoff frequency is 50% of that at zero frequency in the case of low-pass and high-pass filters and of that at f_0 in case of resonant-circuit filter.

17.15. Roll-off of the Response Curve

Gradual decreasing of the output of an a.c. filter is called roll-off. The dotted curve in Fig. 17.11 (*a*) shown an actual response curve of a low-pass *RC* filter. The maximum output is defined to be zero dB as a reference. In other words, 0 dB corresponds to the condition when $V_0 = v_i$ because $20 \log_{10} V_0/V_i = 20 \log 1 = $ dB. As seen, the output drops from 0 dB to – 3 dB at the cutoff frequency and then continues to decrease at a fixed rate. This pattern of decrease is known as the roll-off of the frequency response. The solid straight line in Fig. 17.11 (*a*) represents an ideal output response that is considered to be 'flat' and which cuts the frequency axis at f_c.

The roll-off for a basic IRC *or IRL* filter is 20 dB/decade or 6 dB/octave. Fig. 17.11 (*b*) shows the frequency response plot on a semi-log-scale where each interval on the horizontal axis represents a tenfold increase in frequency. This response curve is known as Bode plot. Fig. 17.11 (*c*) shown the Bode plot for a high-pass *RC* filter on a semi-log graph. The approximate actual response curve is shown by the dotted line. Here, the frequency is on the logarithmic scale and the filter output in decibel is alongwith the linear vertical scale. The filter output is flat beyond f_c. But as the frequency is reduced below f_c, the output drops at the rate of – 20 dB/decade.

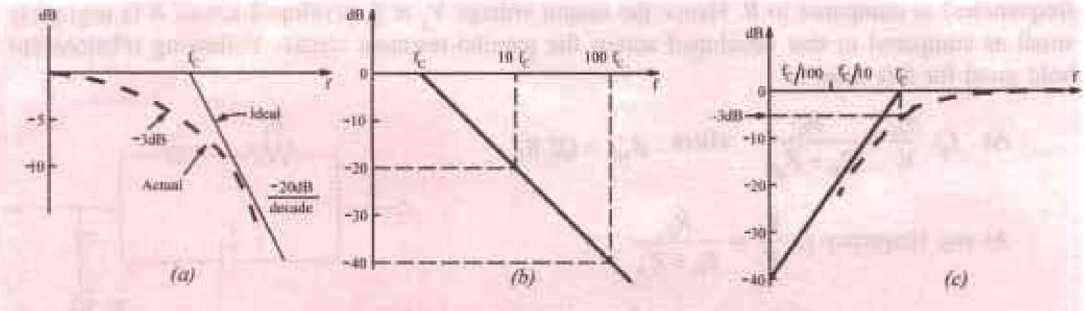


Fig. 17.11

17.16. Bandstop and Bandpass Resonant Filter Circuits

Frequency resonant circuits are used in electronic system to make either bandstop or bandpass filters because of their characteristic Q-rise to either current or voltage at the resonant frequency. Both series and parallel resonant circuits are used for the purpose. It has already been discussed in Chap. No. 7 that

(*i*) a series resonant circuit offers minimum impedance to input signal and provides maximum current. Minimum impedance equals *R* because $X_L = X_C$ and maximum current $I = V/R$.

(*ii*) a parallel circuit offers maximum impedance to the input signal and provides minimum current. Maximum impedance offered is = L/CR and minimum current $I = V/(L/CR)$.

17.17. Series-and Parallel-Resonant Bandstop Filters

The series resonant bandstop filter is shown in Fig. 17.12 (*a*) where the output is taken across the series resonant circuit. Hence, at resonant frequency f_0, the output circuit 'sees' a very low resistance *R* over which negligible output voltage V_0 is developed. That is why there is a shape resonant dip in the response curve of Fig. 17.12 (*b*). Such filters are commonly used to reject a particular frequency such as 50-cycle hum produced by transformers or inductors or turntable rumble in recording equipment.

For the series-resonant bandstop filter shown in Fig. 17.12 (*a*), the following relationships hold good :

$$\text{At } f_0, \frac{V_0}{V_i} = \frac{R_L}{(R_L + R_S)};\ Q_0 = \frac{\omega_0 L}{(R + R_S)} \text{ and } B_{ph} = \frac{1/2\pi\sqrt{LC}}{Q_0}$$

$$\text{At any other frequency } f,\ \frac{V_0}{V_i} = \frac{R_L + j(X_L - X_C)}{(R_L + R_S) + j(X_L - X_C)}$$

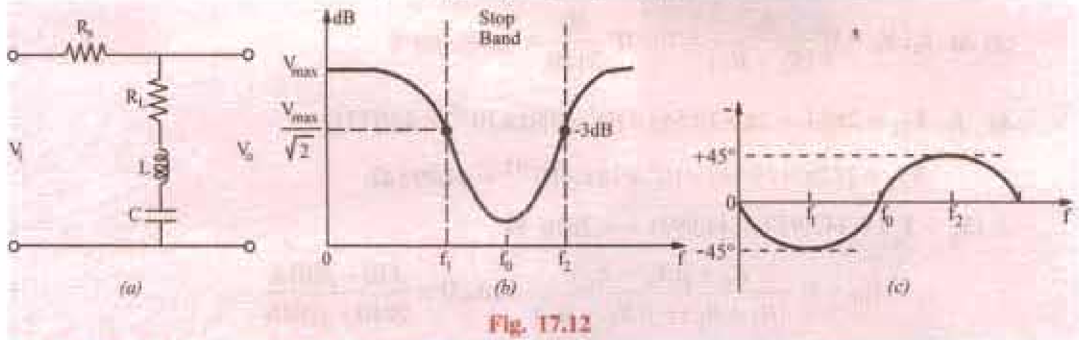


Fig. 17.12

17.18. Parallel-Resonant Bandstop Filter

In this filter, the parallel-resonant circuit is in series with the output resistor *R* as shown in Fig. 17.13. At resonance, the parallel circuit offers extremely high impedance to f_0 (and nearby frequencies) as compared to R. Hence the output voltage V_0 at f_0 developed across R is negligibly small as compared to that developed across the parallel-resonant circuit. Following relationships hold good for this filter :

At f_0; $\dfrac{V_0}{V_i} = \dfrac{R_0}{R_0 + Z_{p0}}$ where $Z_{p0} = Q_0^2 R_L$

At any frequency f, $\dfrac{V_0}{V_i} = \dfrac{R_0}{R_0 + Z_p}$

where $Z_p = \dfrac{Z_L Z_C}{R_L + j(X_L - X_C)}$

Also $Q_0 = \omega_0 L / R_L$ and $B_{hp} = (1/2\pi\sqrt{LC})/Q_0$

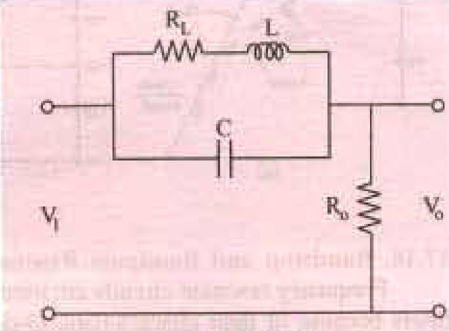


Fig. 17.13

It should be noted that the same amplitude phase response curves apply both to the series resonant and parallel-resonant bandstop filters. Since X_C *predominates at lower frequencies, phase angle* θ *is negative below* f_0, *above* f_0, X_L predominates and the phase current leads. At cutoff frequency f_1, $\theta = -45°$ and at other cutoff frequency f_2, $\theta = +45°$ as in the case of any resonant circuit.

Example 17.7. *A series-resonant bandstop filter consist of a series resistance of 2 kΩ across which is connected a series-resonant circuit consisting of a coil of resistance 10 Ω and inductance 350 mH and a capacitor of capacitance 181 pF. F if the applied signal voltage is* $10\angle 0°$ *of variable frequency, calculate*

(a) resonant frequency f_0 *; (b) half-power bandwidth* B_{hp} *; (c) edge frequencies* f_1 *and* f_2 *; (d) output voltage at frequencies* f_0, f_1 *and* f_2.

Solution. We are given that $R_S = 2$; $k\,\Omega\ R = 10\,\Omega$; $L = 350$ mH; $C = 181$ pF.

(*a*) $f_0 = 1/2\pi\sqrt{LC} = 1/2\pi\sqrt{350 \times 10^{-3} \times 181 \times 10^{-12}} = 20$ kHz

(*b*) $Q_0 = \omega_0 L/(R_S + R_L) = 2\pi \times 20 \times 10^3 \times 350 \times 10^{-3}/2010 = 21.88$

$B_{hp} = f_0/Q_0 = 0.914$ kHz

(*c*) $f_1 = f_0 - B_{hp}/2 = 20 - 0.457 =$ **19.453 kHz**; $f_2 = 20 + 0.457 =$ **20.457 kHz**

(*d*) At f_0, $V_0 = V_i \dfrac{R_L}{(R_L + R_S)} = 10\angle 0° \dfrac{10}{2110} =$ $\mathbf{0.05\angle 00° V}$

At f_1, $X_{L1} = 2\pi f_1 L = 2\pi \times 19.543 \times 10^3 \times 350 \times 10^{-3} = 42{,}977\,\Omega$

$X_{C1} = 1/2\pi \times 19.543 \times 10^3 \times 181 \times 10^{-12} = 44{,}993\,\Omega$

$\therefore (X_L - X_C) = (42{,}977 - 44{,}993) = -2016\ \Omega$

$$V_{01} = V_i \frac{R_L + j(X_L - X_C)}{(R_S + R_L) + j(X_L - X_C)} = 10\angle 0° \times \frac{f10 - j2016}{2010 - j2016}$$

$$= \frac{20160\angle -89.7°}{2847\angle -45°} = \mathbf{7.07\angle{-44.7}° V}$$

At f_2, $X_{L2} = X_{L2} = X_{L1}\ (f_2/f_1) = 42977 \times 20.457/19.453$
$= 44{,}987\ \Omega$: $X_{C2} = X_{C1}\ (f_1/f_2) = 44{,}993 \times 19.543/20.457$
$= 42{,}983\ \Omega$: $(X_{L2} - X_{C2}) = 44{,}987 - 42{,}983 = 2004\ \Omega$

$$\therefore\ V_0 = 10\angle 0^\circ \frac{10 + j2004}{2010 + j2004} = \frac{2004\angle 89.7^\circ}{2837\angle 44.9^\circ} = \mathbf{7.07\angle 44.8^\circ\ V}$$

17.19. Series-Resonant Bandpass Filter

As shown in Fig. 17.14 (*a*), it consists of a series-resonant circuit shunted by an output resistance R_0. It would be seen that this filter circuit can be produced by 'swapping' as series resonant bandstop filter. At f_0, the series resonant impedance is very small and equal R_L which is negligible as compared to R_0. Hence, output voltage is maximum at f_0 and falls to 70.7% at cutoff frequency f_1 and f_2 and shown in the response curve of Fig. 17.14 (*b*). The phase angle is positive for frequencies above f_0 and negative for frequencies below f_0 as shown in Fig 17.14 (*c*) by the solid curve.

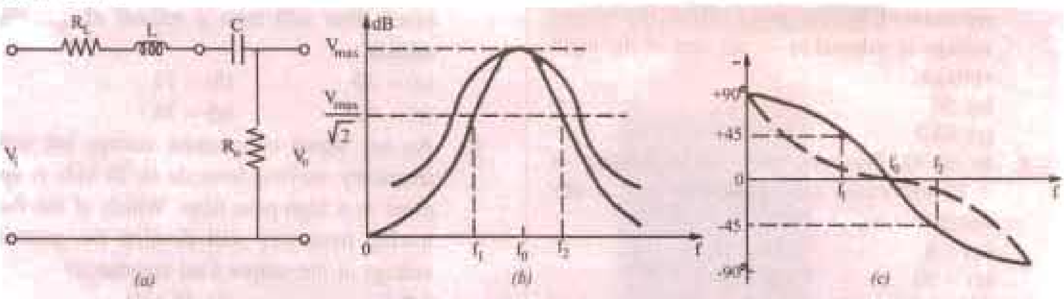


Fig. 17.14

Following relationships hold good for this filter circuit.

$$\text{At}\quad f_0,\ \frac{V_0}{V_i} = \frac{R_0}{(R_L + R_0)};\ Q_0 = \frac{\omega_0 L}{(R_L + R_0)}\ \text{and}\ B_{bp} = \frac{1/2\pi\sqrt{LC}}{Q_0}$$

17.20. Parallel-Resonant Bandpass Filter

It can be obtained by transposing the circuit elements of a bandstop a parallel-resonant filter. As shown in Fig. 17.15, the output is taken across the two-branch parallel-resonant circuit. Since this circuit offers maximum impedance at resonance, this filter produces maximum output voltage V_0 at f_0. The amplitude-response curve of this filter is similar to that of the series-resonant bandpass filter discussed above [Fig. 17.14 (*b*)]. The dotted curve in Fig. 17.14 (*c*) represents the phase relationship between the input and output voltages of this filter. The following relationships apply to this filter :

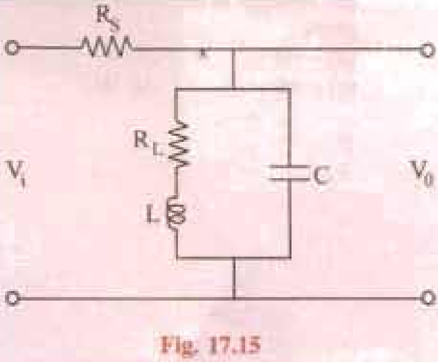


Fig. 17.15

$$\text{At}\ f_0,\ \frac{V_0}{V_i} = \frac{R_0}{(R_0 + Z_{p0})}\ \text{where}\ Z_{p0} = R_{p0} = Q_r^2\, R_L$$

$$\text{and}\quad Q_0 = \frac{R_{p0}}{X_{CO}}\ \text{and}\ B_{bp} = \frac{1/2\pi\sqrt{LC}}{Q_0}$$

$$\text{At any frequency } f,\ \frac{V_0}{V_i} = \frac{Z_p}{R_0 + Z_p}\ \text{where}\ Z_p = \frac{Z_L(-jX_C)}{R_L + j(X_L - X_C)}$$

OBJECTIVE TEST – 17

1. The decibel is a measure of
   (*a*) power (*b*) voltage
   (*c*) current (*d*) power level
2. When the output voltage level of a filter decreases by – 3 dB, its absolute value changes by a factor of
   (*a*) $\sqrt{2}$ (*b*) $1/\sqrt{2}$
   (*c*) 2 (*d*) 1/2
3. The frequency corresponding to half-power point on the response curve of a filter is known as —
   (*a*) cutoff (*b*) upper
   (*c*) lower (*d*) roll-off
4. In a low-pass filter, the cutoff frequency is represented by the point where the output voltage is reduced to — per cent of the input voltage.
   (*a*) 50 (*b*) 70.7
   (*c*) 63.2 (*d*) 33.3
5. In an *RL* low-pass filter, an attenuation of – 12 dB/octave corresponds to .... dB/decade.
   (*a*) – 6 (*b*) – 12
   (*c*) – 20 (*d*) – 40
6. A network which attenuate a single band of frequencies and allows those on either side to pass through is called .... filter.
   (*a*) low-pass (*b*) high-pass
   (*c*) bandstop (*d*) bandpass
7. In a simple high-pass RC filter, if the value of capacitance is doubled, the cutoff frequency is
   (*a*) ocubled (*b*) halved
   (*c*) tripled (*d*) quadrupled
8. In a simple high-pass RL filter circuit, the phase difference between the output and input voltages at the cutoff frequency is .... degrees.
   (*a*) – 90 (*b*) 45
   (*c*) – 45 (*d*) 90
9. In a simple low-pass RC filter, attenuation is – 3 dB at f_c. At $2 f_c$ attenuation is – 6 dB. At $10 f_c$ the attenuation would be .... dB.
   (*a*) – 30 (*b*) – 20
   (*c*) – 18 (*d*) – 12
10. An a.c. signal of constant voltage 10 V and variable frequency is applied to a simple high-pass RC filter. The output voltage at ten times the cutoff frequency would be .... volt.
    (*a*) 1 (*b*) 5
    (*c*) $10/\sqrt{2}$ (*d*) $10\sqrt{2}$
11. When two simple low-pass filters having same values of R and C are cascaded, the combined filter will have a roll-off of ...... dB/decade.
    (*a*) – 20 (*b*) – 12
    (*c*) – 40 (*d*) – 36
12. An a.c. signal of constant voltage but with frequency varying from dc to 25 kHz is applied to a high-pass filter. Which of the following frequency will develop the greatest voltage at the output load resistance?
    (*a*) d. (*b*) 15 kHz
    (*c*) 10 kHz (*d*) 25 kHz
13. A voltage signal source of constant amplitude with frequency varying from dc to 25 kHz is applied to a low-pass filter. Which frequency will develop greatest voltage across the output load resistance?
    (*a*) d.c. (*b*) 10 kHz
    (*c*) 15 kl (*d*) 25 kHz
14. The output of a filter drops from 10 to 5 V as the frequency is increased from 1 to 2 kHz. The dB change in the output voltage is
    (*a*) – 3 dB/decade (*b*) – 6 dB/octave
    (*c*) 6 dB/octave (*d*) – 3 dB/octave

ANSWERS

1. *d* 2. *b* 3. *a* 4. *b* 5. *d* 6. *c* 7. *b* 8. *b* 9. *b* 10. *a* 11. *c* 12. *d* 13. *a* 14. *b*

18 CIRCLE DIAGRAMS

18.1. Circle Diagram of a Series Circuit

Circle diagrams are helpful in analysing the operating characteristics of circuits, which, under some conditions, are used in representing transmission lines and a.c. machinery (like induction motor etc.)

Consider a circuit having a constant reactance but variable resistance varying from zero to infinity and supplied with a voltage of constant magnitude and frequency (Fig. 18.1).

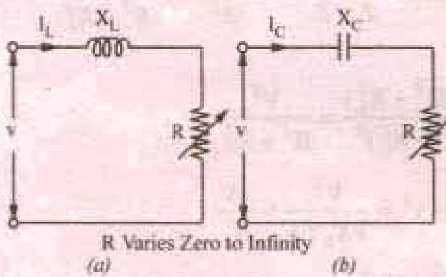


Fig. 18.1

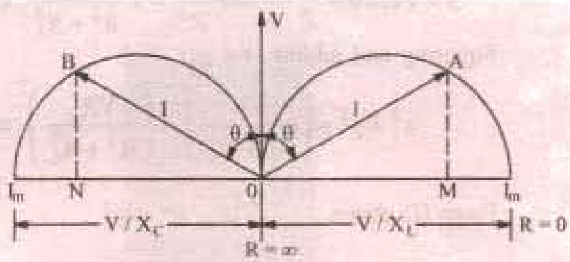


Fig. 18.2

If $R = 0$, then $I = V/X_L$ or V/X_C and has maximum value . It will lag or lead the voltage by 90° depending on whether the reactance is inductive or capacitive. In Fig. 18.2, angle θ represents the phase angle. If R is now increased from its zero value, then I and θ will both decrease. In the limiting case, when $R = \infty$, then $I = 0$ and $\theta = 0°$. It is found that the locus of end point of current vector OA or OB represents a semi-circuit with diameter equal to V/X as shown in Fig. 18.2. It can be proved thus :

$I = V/Z$ and $\sin\theta = X/Z$ or $Z = X/\sin\theta$ $\therefore$ $I = V \sin\theta / X$

For constant value of V and X, the above is the polar equation of a circle of diameter V/X. This equation is plotted in Fig. 18.2. Here, OV is taken as reference vector. It is also seen that for inductive circuit, the current semi-circle is on the right-hand side of reference vector OV so that current vector OA lags by $\theta°$. The current semi-circle for R-C circuit is drawn on the left hand side of OV so that current vector OB leads OV by $\theta°$. It is obvious that $AM = I \cos\theta$, hence AM represents, on a suitable scale, the power consumed by the R-L circuit, Similarly, BN represents the power consumed by the R-C circuit.

18.2. Rigorous Mathematical Treatment

We will again consider both R-L and R–C circuits. The voltage Drops across R and X_L (or X_C) will be 90° out of phase with each other. Hence, for any given value of resistance, the vector diagram for the two voltage drops (*i.e.* IR and IX) is a rightangled triangle having applied voltage as the hypotenuse.

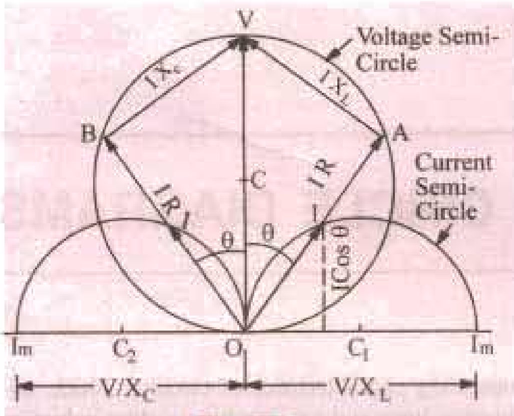


Fig. 18.3

For a constant applied voltage and reactance, the vector diagrams for different values of R are represented by a series of right-angled triangles having common hypotenuse as shown in Fig. 18.3. The locus of the apex of the right-angled voltage triangles is a semicircle described on the hypotenuse. The voltage semi–circle for R-L circuit circuit (OAV) is on the right and for R-C circuit (OBV) on the left of the reference vector OV as shown in Fig. 18.3.

The foci of end points of current vectors are also semi-circles as shown but their centres lie on the opposite sides of and in an axis perpendicular to the reference vector OV.

(*ii*) **R-L Circuit** [Fig. 18.1 (*a*)]

The co-ordinates of point A with respect to the origin O are

$$y = I\cos\theta = \frac{V}{Z}\cdot\frac{R}{Z} = V\frac{R}{Z^2} = V\frac{R}{R^2+X_L^2}\,;\, x = I\sin\theta = \frac{V}{Z}\cdot\frac{X_L}{Z} = V\cdot\frac{X_L}{Z^2} = V\frac{X_L}{R^2+X_L^2} \quad \dots (i)$$

Squaring and adding, we get

$$x^2+y^2 = \left(\frac{VR}{R^2+X_L^2}\right)^2 + \left(\frac{VX_L}{R^2+X_L^2}\right)^2 = \frac{V^2(R^2+X_L^2)}{(R^2+X_L^2)^2} = \frac{V^2}{R^2+X_L^2}$$

From (*i*) above, $\quad \dfrac{VX_L}{x} = R^2+X_L^2 \quad \therefore \quad x^2+y^2 = \dfrac{V^2}{VX_L/x} = \dfrac{xV}{X_L}$

$$\therefore \quad x^2+y^2 = \frac{xV}{X_L} \quad \text{or} \quad y^2 + \left(x - \frac{V}{2X_L}\right)^2 = \frac{V^2}{4X_L^2}$$

This is the equation of a circle, the co-ordinates of the centre of which are $y = 0$, $x = V/2\,X_L$ and whose radius is $V/2X_L$.

(*ii*) **R-C Circuit.** In this case it can be similarly proved that the locus of the end point of current vector is a semi-circle. The equation of this circle is

$$y^2 + \left(x + \frac{V}{2_C}\right)^2 = \frac{V^2}{4X_C^2}$$

The centre has co-ordinates of $y = 0$, $x = -V/2X_C$.

18.3. Constant Resistance But Variable Reactance

Fig. 18.4 shows two circuits having constant resistance but variable reactance X_L or X_C which vary from zero to infinity. When $X_L = 0$, current is maximum and equals V/R. For other

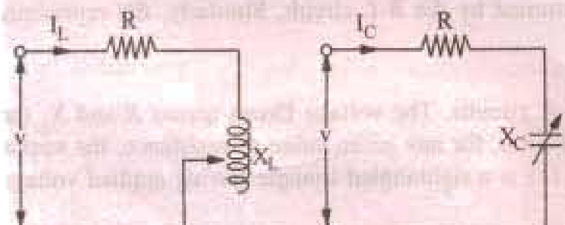


Fig. 18.4

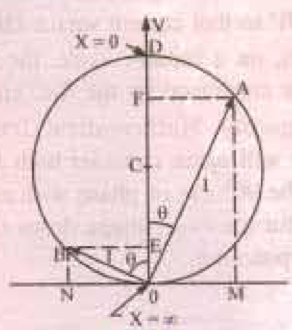


Fig. 18.5

values, $I = V / \sqrt{R^2 + X_L^2}$. Current becomes zero when $X_L = \infty$. As seen from Fig. 18.5, the end point of the current vector describes a semi-circle with redius $OC = V/2/R$ and centre lying in the reference sector *i.e.* voltage vector *OV*. For *R-C* circuit, the semi-circle lies to the left of *OV*. As before, it may be proved that the equation of the circle shown in Fig. 18.5 is

$$x^2 + \left(y - \frac{V}{2R}\right)^2 = \frac{V^2}{4R^2}$$

The co-ordinates of the centre are $x = 0$, $y + V/2R$ and radius $= V/2R$.

As before, power developed would be maximum when current vector makes an angle of 45° with the voltage vector *OV*. In that case, current is $I_m / \sqrt{2}$ and $P_m = VI_m / 2$.

18.4. Properties of Constant Reactance But Variable Resistance Circuit

From the circle diagram of Fig. 18.3, it is seen that circuits having a constant reactance but variable resistance or *vice-versa* have the following properties :

(*i*) the current has limiting value

(*ii*) the power supplied to the circuit has a limiting value also

(*iii*) the power factor corresponding to maximum power supply is 0.707 (= cos 45°)

Obviously, the maximum current in the circuit is obtained when $R = 0$.

$$\therefore \quad I_m = V / X_L = V / \omega L \quad \text{... for } R\text{-}L \text{ circuit}$$
$$= -V / X_C = -\omega VC \quad \text{... for } R\text{-}C \text{ circuit}$$

Now, power P taken by the circuit is $VI \cos \theta$ and if V is constant, then $P \propto I \cos\theta$. Hence, the ordinates of current semi-circles are proportional to $I \cos \theta$. The maximum ordinate possible in the semi-circle represents the maximum power taken by the circuit. The maximum ordinate passes through the centre of semi-circle so that current vector makes an angle of 45° with both the diameter and the voltage vector *OV*. Obviously, power factor corresponding to maximum power intake is cos 45° = 0.707.

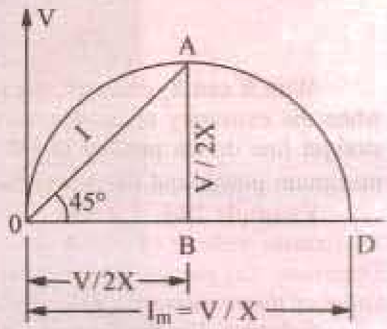


Fig. 18.6

Maximum power,

$$P_m = V \times AB = V \times \frac{I_m}{2} = \frac{1}{2} VI_m$$

Now, for *R-L* circuit, $I_m = V/X_L$

$$\therefore \quad P_m = \frac{V^2}{2X_l} = \frac{V^2}{2\omega L}$$

For *R-C* circuit $P_m = \dfrac{V^2}{2X_C} = \dfrac{V^2 \omega C}{2}$

As said above, at maximum power, $\theta = 45°$, hence vector triangle for voltages is an isosceles triangle which means that voltage drops across resistance and reactance are each equal to 0.707 of supply voltage *i.e.* $V / \sqrt{2}$. As current is the same, for maximum power, resistance equals reactance *i.e.* $R = X_L$ (or X_C).

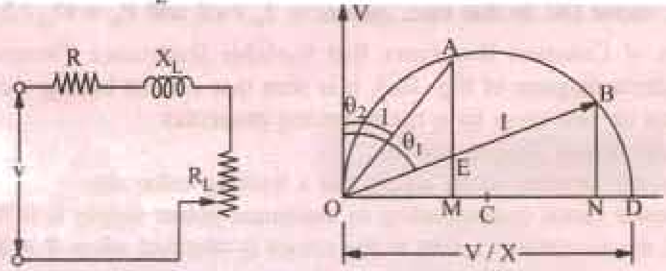
Hence, the expression representing maximum power may be written as $P_m = V^2/2R$.

18.5. Simple Transmission Line Circuit

In Fig. 18.7 (*a*) is shown a simple transmission circuit having negligible capacitance and reactance. R and X_L represent respectively the resistance and reactance of the line and R_L represents load resistance.

If R and X_L are constant, then as R_L is varied, the current AM follows the equation $I = (V/X) \sin\theta$ (Art. 18.1). The height AM in Fig. 18.7 (*b*) represents the power consumed by the circuit but, in the present case, this power is consumed both in R and R_L. The power absorbed by each resistance can be represented on the circle diagram.

In Fig. 18.7 (*b*), OB represents the line current when $R_L = 0$. The current $OB = V/\sqrt{(R^2+X_L^{2})}R^2 + X_L^2)$ and power factor is $\cos\theta_1$. The ordinate BN then represents on a different scale the power dissipated in R only. OA represents current when R_L has some finite value *i.e.* $OA = V/\sqrt{(R+R_L)^2+X_L^2}$. The ordinate AM represents total power dissipated, out of which ME is consumed in R and AE in R_L.



(*a*) (*b*)

Fig. 18.7

In fact, if $OA^2 \times R_L$ (= AE) is considered to be the output of the circuit (the power transmitted by the line), then

$$\eta = \frac{AE}{AM}$$

With R and X_L constant, the maximum power that can be transmitted by such a circuit occurs when the extremity of current vector OA coincides with the point of tangency to the circle of a straight line drawn parallel to OB. Obviously, V times AE under these conditions represents the maximum power and the power factor at that time is $\cos\theta_2$.

Example 18.1. *A circuit consists of a reactance of 5 Ω in series with a variable resistance. A constant voltage of 100 V is applied to the circuit. Show that the current locus is circular. Determine (a) the maximum power input to the circuit (b) the corresponding current, p.f. and value of the resistance.* **(Electrical Science II, Allahabad Univ. 1992)**

Solution. For the first part, please refer to Art. 18.1

(*a*) $I_m = V/X = 100/5 = 20\ A$; $P_m = \frac{1}{2}\ VI_m = \frac{1}{2} \times 100 \times 20 =$ **1000 W**

(*b*) At maximum power input, current is = OA (Fig. 18.6)

$\therefore\ OA = I_m/\sqrt{2} = 20/\sqrt{2} = 14.14\ A$; *p.f.* = cos 45° = **0.707** ; $R = X =$ **5 Ω**

Example 18.2. *If a coil of unknown resistance and reactance is connected in series with a 100-V, 50-HZ supply, the current locus diagram is found to have a diameter of 5 A and when the value of series resistor is 15 Ω, the power dissipated is maximum. Calculate the reactance and resistance of the coil and the value of the maximum power in the circuit and the maximum current.*

Solution. Let the unknown resistance and reactance of the coil be R and X respectively

Diameter = V/X $\quad\therefore\quad 5 = 100/X$ or $X =$ **20 Ω**

Power is maximum when total resistance = reactance

or $\quad 15 + R = 20 \quad\therefore\quad R = 5\ \Omega$

Maximum power $P_m = V^2/2X = 100^2/2 \times 20 =$ **250 W**

Maximum current $I_m = 100/\sqrt{(20^2+5^2)} =$ **4.85 A**

Example 18.3. *A constant alternating sinusoidal voltage at constant frequency is applied across a circuit consisting of an inductance and a variable resistance in series. Show that the locus diagram of the current vector is a semi-circle when the resistance is varied between zero and infinity.*

If the inductance has a value of 0.6 henry and the applied voltage is 100 V at 25 Hz, calculate (a) the radius of the arc (in amperes) and (b) the value of variable resistance for which the power taken from the mains is maximum and the power factor of the circuit at the value of this resistance.

Solution. $X_L = \omega L = 0.6 \times 2\pi \times 25 = 94.26\,\Omega$

(*a*) Radius = $V/2\,X_L$ = 100/2 × 4.26 = **0.531 A**

Example 18.4. *A resistor of 10 Ω is connected in series with an inductive reactor which is variable between 2 Ω and 20 Ω. Obtain the locus of the current vector when the circuit is connected to a 250-V supply. Determine the value of the current and the power factor when the reactance is (i) 5 Ω (ii) 10 Ω (iii) 15 Ω.* **(Basic Electricity, Bombay Univ. 1987)**

Solution. As discussed in Art. 18.3, the end point of current vector describes a semi-circle whose diameter (Fig. 18.8) equals V/R = 250/10 = 25 A and whose centre lies to right side of the vertical voltage vector OV.

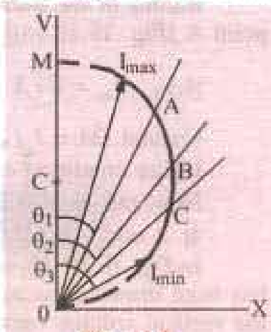


Fig. 18.8

$$I_{max} = 250/\sqrt{10^2+2^2} = 24\text{ A};\ \theta = \tan^{-1}(2/10) = 11.3°;$$

$$I_{min} = 250/\sqrt{10^2+20^2} = 11.2\text{ A};\ \theta = \tan^{-1}(20/10) = 63.5°$$

(*i*) $\theta_1 = \tan^{-1}(5/10) = 26.7°$, p.f. = cos 26.7° = 0.89
$I = OA$ = **22.4 A**

(*ii*) $\theta_2 = \tan^{-1}(10/10) = 45°$, p.f. = cos 45° = **1**;
I = OB = **17.7 A**

(*iii*) $\theta_3 = \tan^{-1}(15/10) = 56.3$; p.f. = cos 56.3° = **0.55**;
$I = OC$ = **13.9 A**.

Example 18.5. *A voltage of 100 sin 10,000 t is applied to a circuit consisting of a 1 μF capacitor in series with a resistance R. Determine the locus of the tip of the current phasor when R is varied from 0 to ∞. Take the applied voltage as the reference phasor.*
(Network Theory and Design, AMIETE 1990)

Solution. As seen from Art. 18.2 the locus of the tip of the current phasor is a circle whose equation is

$$y^2 + \left(x + \frac{V}{2X_C}\right)^2 = \frac{V}{4X_C^2}$$

We are given that $V = V_m/\sqrt{2} = 100/\sqrt{2} = 77.7$ V

$$\omega = 10{,}000 \text{ rad/s};\ X_C = 1/\omega \times C = 1/10{,}000 \times 1 \times 10^{-6} = 100\ \Omega\text{ C},$$

$$(V/2\,X_C)^2 = (77.7/2 \times 100)^2 = 0.151,\quad \therefore\quad y^2 + (x + 0.389)^2 = 0.151$$

Example 18.6. *Prove that polar locus of current drawn by a circuit of constant resistance and variable capacitive reactance is circular when the supply voltage and frequency are constant.*

If the constant resistance is 10 Ω and the voltage is 100 V, draw the current locus and find the values of the current and p.f. when the reactance is (i) 5.77 Ω (ii) 10 Ω and (iii) 17.32 Ω. Explain when the power will be maximum and find its value. **(Electromechanics, Allahabad Univ. 1992)**

Solution. For the first part, please refer to Art. 18.3. The current semicircle will be drawn on the vertical axis with a radius $OM = V/2\,R$ = 100/2 × 10 = 5 A as shown in Fig. 18.9 (*b*)

(*i*) $\theta_1 = \tan^{-1}(5.77/10) = 30°$; cos θ_1 = **0.866 (lead)** ; current = OA = **8.66 A**

(*ii*) $\theta_2 = \tan^{-1}(10/10) = 45°$; cos θ_2 = **0.707 (lead)** current = OB = **7.07 A**

(*iii*) $\theta_3 = \tan^{-1}(17.32/10) = 60°$; cos θ_3 = **0.5 (lead)** current = OC = **5 A**

Power would be maximum for point B when $\theta = 45°$; $I_m = V/R$ = 100/10 = 10 A

$P_m = V \times OB \times \cos 45° = V \times I_m \cos 45° \times \cos 45° = \frac{1}{2}\,VI_m = \frac{1}{2} \times 100 \times 10$ = **500 W**

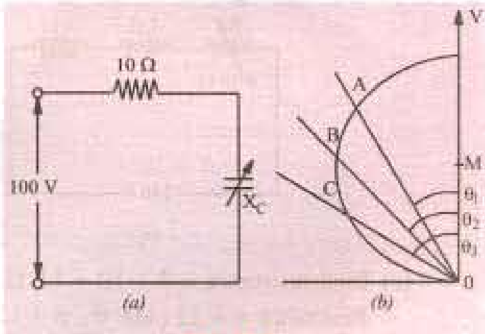


Fig. 18.9

Example 18.7. *Prove that the polar locus of the current drawn by a circuit of constant reactance and variable resistance is circular when the supply voltage and frequency are constant.*

If the reactance of such a circuit is 25 Ω and the voltage 250, draw the said locus and locate there on the point of maximum power and for this condition, find the power, current, power factor and resistance.

Locate also the point at which the power factor is 0.225 and for this condition, find the current, power and resistance. **(Basic Electricity, Bombay Univ. 1985)**

Solution. For the first part, please refer to Art. 18.3.

Radius of the current semi-circle is = $V/2X = 250/2 \times 25 = 5$ A. As discussed in Art. 18.3, point A [Fig. 18.10 (*a*)] corresponds to maximum power.

Now, $I_m = V/X = 250/25 = 10$ A; $P_m = \frac{1}{2}VI_m = \frac{1}{2} \times 250 \times 10 =$ **1250 W**

Current $OA = I_m/\sqrt{2} = 10/\sqrt{2} =$ **7.07 A** ; $p.f. = \cos 45° =$ **0.707**.

Under condition of maximum power, $R = X =$ **25** Ω.

Now, $\cos\theta = 0.225$;

$\theta = \cos^{-1}(0.225) = 77°$

In Fig. 18.10 (*b*), current vector *OA* has been drawn at an angle of 77° with the vertical voltage vector *OV*.

By measurement, current $OA =$ 9.74 A

By calculate, $OA, = I_m \cos 13° = 10 \times 0.974 =$ **9.74 A**

Power $= VI\cos\theta = 250 \times 9.74 \times 0.225 =$ **548 W**

$P = I^2R$; $R = P/I^2 = 548/9.74^2 =$ **5.775 Ω**

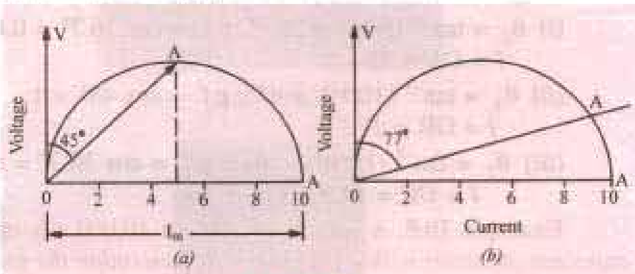


Fig. 18.10

Example 18.8. *A non-inductive resistance R, variable between 0 and 10 Ω, is connected in series with a coil of resistance 3 Ω and reactance 4 Ω and the circuit supplied from a 240-V a.c. supply. By means of a locus diagram, determine the current supplied to the circuit when R is (a) zero (b) 5 Ω and (c) 10 Ω. By means of the symbolic method, calculate the value of the current when R 5 Ω.*

Solution. The locus of the current vector is a semi-circle whose centre is (0, *V*/2*X*) and whose radius is obviously equal to *V*/2*X*. Now, $V/2X = 240/2 \times 4 = 30$ A.

Hence, the semi-circle is drawn as shown in Fig. 18.11 (*b*).

(*a*) Total resistance 3 Ω and $X = 4$ Ω $\therefore$ $\tan\theta_1 = 4/2$ $\therefore$ $\theta_1 = 53°8'$

Hence, current vector *OA* is drawn making an angle of 53°8′ with vector *OV*. Vector *OA* measures **49 A.**

(*b*) Total resistance $= 3 + 5 = 8$ Ω

Reactance = 4 Ω ; $\tan\theta_2 = 4/8 = 0.5$ $\therefore$ $\theta_2 = 26°34'$

Current vector *OB* is drawn at an angle of 26°34′ with *OV*. It measure **27 A (approx.)**

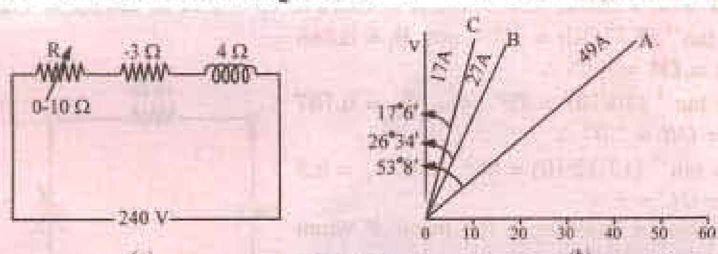


Fig. 18.11

(*c*) Total resistance $= 3 + 10 = 13$ Ω.

Reactance = 4 Ω ; $\tan\theta_3 = 4/13$ $\therefore$ $\theta_3 = 17°6'$

Current vector OC is drawn at an angle of 17°6′ with vector OA. It measures **17 A.**

Symbolic Method

$$\mathbf{I} = \frac{240 + j0}{(5+3)+j4} = \frac{240}{8+j4} = \frac{240}{8.96\angle 26.5^\circ} = 26.7\angle 26.5^\circ$$

Note. There is difference in the magnitudes of the currents and the angles as found by the two different methods. It is so because one has been found exactly by mathematical calculations, whereas the other has been measured from the graph.

Example 18.9. *A circuit consisting of a 50-Ω resistor in series with a variable reactor is shouted by a 100-Ω resistor. Draw the locus of the extremity of the total current vector to scale and determine the reactance and current corresponding to the minimum overall power factor, the supply voltage being 100 V.*

Solution. The parallel circuit is shown in Fig. 18.12 (*a*).

The resistive branch draws a fixed current I_2 100/100 = 1 A. T he current I_1 drawn by the reactive branch is maximum when $X_L = 0$ and its maximum value is = 100/50 = 2 A and is in phase with voltage.

In the locus diagram of Fig. 18.12 (*b*), the diameter OA of the reactive current semi-circle is = 2 A. OB is the value of I_1 for some finite value of X_L. $O'O$ represents I_2. Being in phase with voltage, it is drawn in phase with voltage vector OV. Obviously, $O'B$ represents total circuit current, being the vector sum of I_1 and I_2.

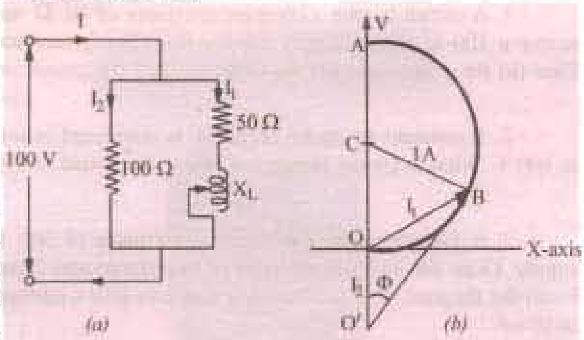


Fig. 18.12

The minimum power factor which corresponds to maximum phase difference between $O'B$ and $O'V$ occurs when $O'B$ is tangential to the semi-circle In that case, $O'B$ is perpendicular to BC. It means that $O'BC$ is a right-angled triangle.

Now, $\sin\phi = BC/O'C = 1/(1+1) = 0.5;\ \phi = 30^\circ$

∴ Minimum *p.f.* = cos 30° = 0.866 (lag)

Current corresponding to minimum *p.f.* is $O'B = O'C \cos\phi$ = 2 × 0.866 = **1.732 A.**

Now, Δ OBC is an equilateral triangle, hence $I_1 = OB$ = 1 A. Considering reactive branch, Z = 100/1 = 100 Ω, $X_L = \sqrt{100^2 - 50^2}$ = **88.6 Ω**

Example 18.10. *A coil of resistance 60 Ω and inductance 0.4 H is connected in series with a capacitor of 17.6 μF across a variable frequency source which is maintained at a fixed potential of 120 V. If the frequency is varied through a range of 40Hz to 80Hz, draw the complete current locus and calculate the following :*

(*i*) the resonance frequency, (*ii*) the current and power factor at 40 Hz and

(*iii*) *the current and power factor at 80 Hz.* **(Elect. Circuits, South Gujarat Univ. 1989)**

Solution. (*i*) $f_0 = 10^3 / 2\pi\sqrt{0.4 \times 17.6}$ = **60 Hz.**

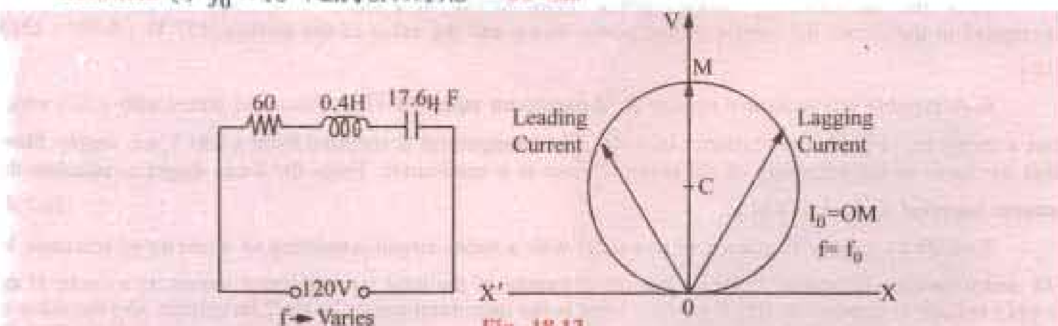


Fig. 18.13

(*ii*) f = **40 Hz**

$$X_L = 2\pi \times 40 \times 0.4 = 100\,\Omega;\quad X_C = 10^6 / 2\pi \times 40 \times 17.6 = 226\ \Omega$$

$X = 100 - 226 = -126\ \Omega$ (capacitive); $I = 120/\sqrt{60^2 + (-126)^2}$ = **0.86 A**

p.f. = cos θ = *R/Z* = 60/139.5 = **0.43 (lead)**

(*iii*) f = **80 Hz**

$X_L = 100 \times 2 = 200\ \Omega$; $X_C = 226/2 = 113\ \Omega$; $X = 200 - 113 = 87\ \Omega$ (inductive)

$$Z = \sqrt{R^2 + X^2} = \sqrt{60^2 + 87^2} = 105.3\,\Omega$$

I = 120/105.3 = **1.14 A** ; p.f. cos θ = 60/105.3 = **0.57 (lag)**

Tutorial Problems No. 18.1

1. A circuit having a constant resistance of 60 Ω and a variable inductance of 0 to 0.4 H is connected across a 100-V, 50-Hz supply. Derive from first principles the locus of the extremity of the current vector. Find (*a*) the power and (*b*) the inductance of the circuit when the power factor is 0.8.

[(*a*) 107 W (*v*) 0.143 H] (*App. Elect. London Univ.*)

2. A constant reactance of 10 Ω is connected in series with a variable resistor and the applied voltage is 100 V. What is (*i*) the maximum power dissipated and (*ii*) at what value of resistance does it occur ?

[(*a*) 500 W (*ii*) 10 Ω] (*City & Guilds London*)

3. A variable capacitance and a resistance of 300 Ω are connected in series across a 240-V; 50-Hz supply. Draw the complex or locus of impedance and current as the capacitance changes from 5μ*F* to 30 μ*F*. From the diagram, find (*a*) the capacitance to give a current of 0.7 A and (*b*) the current when the capacitance is 10 μ*F*. **[19.2 μF, 0.55A]** (*London Univ.*)

4. An a.c. circuit consists of a variable resistor in series with a coil, for which R = 20 Ω and L = 0.1 H. Show that when t618his circuit is supplied at constant voltage and frequency and the resistance is varied between zero and infinity, t he locus diagram of the current vector is a circular arc. Calculate when the supply voltage is 100 V and the frequency 50 Hz (*i*) the radius (in amperes) of the arc (*ii*) the value of the variable resistor in order that the power taken from the mains may be a maximum.

[(*i*) 1.592 A (*ii*) 11.4 Ω] (*London Univ.*)

5. A circuit consists fo an inductive coil (L = 0.2 *H*, R = 20 Ω) in series with a variable resistor (0 – 200 Ω). Draw to scale the locus of the current vector when the circuit is connected to 230-V, 50-Hz supply mains and the resistor is varied between 0 and 200 Ω . Determine (*i*) the value of the resistor which will give maximum power in the circuit, (*ii*) the power when the resistor is 150 Ω .

[(*i*) 42.8 Ω (*ii*) 275 W] (*Londown Univ.*)

6. A 15 μF capacitor, an inductive coil (L = 0.135 H, R = 50 Ω) and a variable resistor are in series and connected to a 230-V, 50-Hz supply.

Draw to scale the vector locus of the current when the variable resistor is varied between 0 and 500 Ω .

Calculate (*i*) the value of the variable resistor when the power is a maximum (*ii*) the power under these conditions. **[(*i*) 120 Ω (*ii*) 155.5 W]** (*Londown Univ.*)

7. As a.c. circuit supplied at 100 V, 50-Hz consists of a variable resistor in series with a fixed 100 μF capacitor.

Show that the extremity of the current vector moves on a circle. Determine the maximum power dissipated in the circuit the corresponding power factor and the value of the resistor.**[157 W ; 0.707 : 131.8 Ω]**

8. A variable non-inductive resistor R of maximum value 10 Ω is placed in series with a coil which has a resistance of 3 Ω and reactance of 4 Ω . The arrangement is supplied from a 240-V a.c. supply. Show that the locus of the extremity of the current vector is a semi-circle. From the locus diagram, calculate the current supplied when R = 5 Ω . **[26.7 A]**

9. A 20- Ω reactor is connected in parallel with a series circuit consisting of a reactor of reactance 10 Ω and a variable resistance R. Prove that the extremity of the total current vector moves on a circle. If the supply voltage is constant at 100 V (r.m.s.), what is the maximum power factor ? Determine also the value of R when the p.f. has its maximum value. **[0.5 : 17.3 Ω]**

19 POLYPHASE CIRCUITS

19.1. Generation of Polyphase Voltage

The kind of alternating currents and voltages discussed in chapter 12 to 15 are known as single-phase voltage and current, because they consist of a single alternating current and voltage wave. A single-phase alternator was diagrammatically depicted in Fig. 11.1 (*b*) and it was shown to have one armature winding only. But if the number of armature windings is increased, then it becomes polyphase alternator and it produces as many independent voltage waves as the number of windings or phases. These windings are displaced from one another by equal angles, the values of these angles being determined by the number of phases or windings. In fact, the word 'polyphase' means poly (*i.e.* many or numerous) and phases (*i.e.* winding or circuit).

In a two-phase alternator, the armature windings are displaced 90 electrical degrees apart. A 3-phase alternator, as the name shows, has three independent armature windings which are 120 electrical degrees apart. Hence, the voltages induced in the three windings are 120° apart in time-phase. With the exception of two-phase windings, it can be stated that, in general, the electrical displacement between different phases is $360/n$ where n is the number of phases or windings.

Three-phase systems are the most common, although, for certain special jobs, greater number of phases is also used. For example, almost all mercury-arc rectifiers for power purposes are either six-phase or twelve-phase and most of the rotary converters in use are six-phase. All modern generators are practically three-phase. For transmitting large amounts of power, three-phase is invariably used. The reasons for the immense popularity of three-phase apparatus are that (*i*) it is more efficient (*ii*) it uses less material for a given capacity and (*iii*) it costs less than single-phase apparatus etc.

In Fig. 19.1 is shown a two-pole, stationary-armature, rotating-field type three-phase alternator. It has three armature coils aa', bb' and cc' displaced 120° apart from one another. With the position and clockwise rotation of the poles as indicated in Fig. 19.1, it is found that the e.m.f. induced in conductor '*a*' for coil aa' is maximum and its direction* is away from the reader. The e.m.f. in conductor '*b*' of coil bb' would be maximum and away from the reader when the N-pole has turned through 120° *i.e.* when *N-S* axis lies along bb'. It is clear that the induced e.m.f. in conductor '*b*' reaches its maximum value 120° later than the maximum value in conductor '*a*'. In the like manner, the maximum e.m.f. induced (in the direction away from the reader) in conductor '*c*' would occur 120° later than that in '*b*' or 240° later than that in '*a*'.

Thus the three coils have three e.m.fs. induced in them which are similar in all respects except that they are 120° out of time phase with one another as pictured in Fig. 193. Each voltage wave is assumed to be sinusoidal and having maximum value of E_m.

In practice, the space on the armature is completely covered and there are many slots per phase per pole.

*The direction is found with the help of Flaming's Right-hand rule. But while applying this rule, it should be remembered that the relative motion of the conductor with respect to the field it anticlockwise although the motion of the field with respect to the conductor is clockwise as shown. Hence, thumb should point to the left.

Fig. 19.2 illustrates the relative positions of the windings of a 3-phase, 4-pole alternator and Fig. 19.4 shows the developed diagram of its armature windings. Assuming full-pitched winding and the direction of rotation as shown, phase '*a*' occupies the position under the centres of *N* and *S*-poles. It starts at S_a and ends or finishes at F_a.

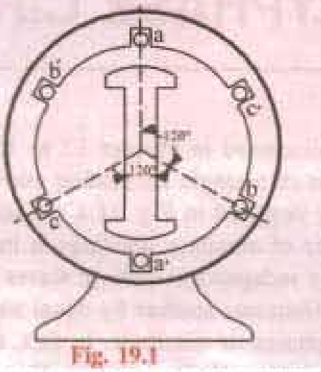


Fig. 19.1

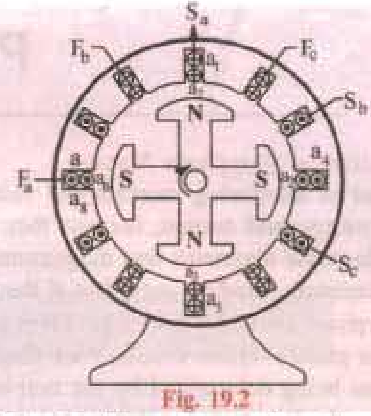


Fig. 19.2

The second phase '*b*' start at S_b which is 120 electrical degrees apart from the start of phase '*a*', progresses round the armature clockwise (as does '*a*') and finishes at F_b. Similarly, phase '*c*' starts at S_c, which is 120 electrical degrees away from S_b, progresses round the armature and finishes at F_c. As the three circuits are exactly similar but are 120 electrical degrees apart, the e.m.f. waves generated in them (when the field rotates) are displaced from each other by 120°. Assuming these waves to be sinusoidal and counting the time from the instant when the e.m.f. in phase '*a*' is zero, the instantaneous values of the three e.m.fs. will be given by curves of Fig. 193.

Their equations are :

$$e_a = E_m \sin \omega t \qquad \text{... (i)}$$

$$e_b = E_m \sin(\omega t - 120°) \qquad \text{... (ii)}$$

$$e_c = E_m \sin(\omega t - 240°) \qquad \text{... (iii)}$$

As shown in Art. 11.23. alternating voltages may be represented by revolving vectors which indicate their maximum values (or r.m.s. values if desired). The actual values of these voltages vary from peak positive to zero and to peak negative values in one revolution of the vectors. In Fig. 19.5 are shown the three vectors representing the r.m.s. voltages of the three phases E_a, E_b and E_c (in the present case $E_a = E_b = E_c = E$, say).

It can be shown that the sum of the three phase e.m.fs. is zero in the following three ways :

(*i*) The sum of the above three equations (*i*), (*ii*) and (*iii*) is zero as shown below :

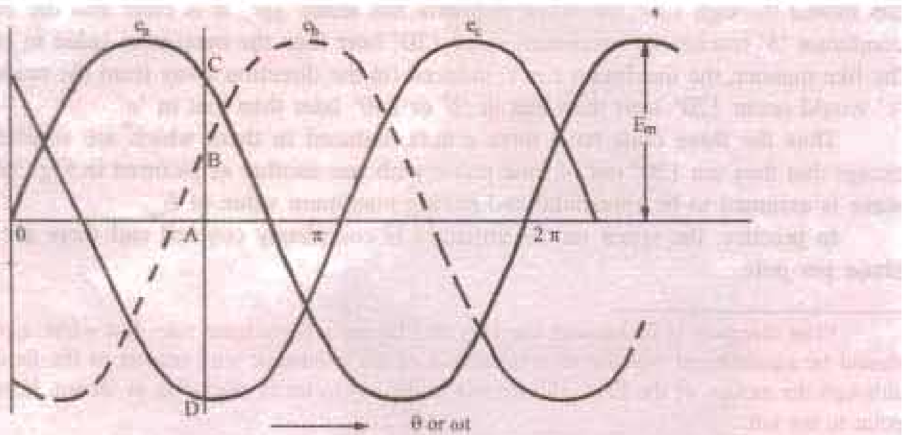


Fig. 19.3

Resultant instantaneous e.m.f. $= e_a + e_b + e_c$

$$= E_m \sin\omega t + E_m \sin(\omega t - 120°) + E_m (\omega - 240)°$$

$$= E_m [\sin\omega t + 2\sin(\omega t - 180°)\cos 60°]$$

$$= E_m [\sin\omega t - 2\sin\omega t \cos 60] = 0$$

(*ii*) The sum of ordinates of three e.m.f. curves of Fig. 19.3 is zero. For example, taking ordinates *AB* and *AC* as positive and *AD* as negative, it can be shown by actual measurement that

$$AB + AC + (-AD) = 0$$

(*iii*) If we add the three vectors of Fig. 19.5 either vectorially or by calculation, the result is zero

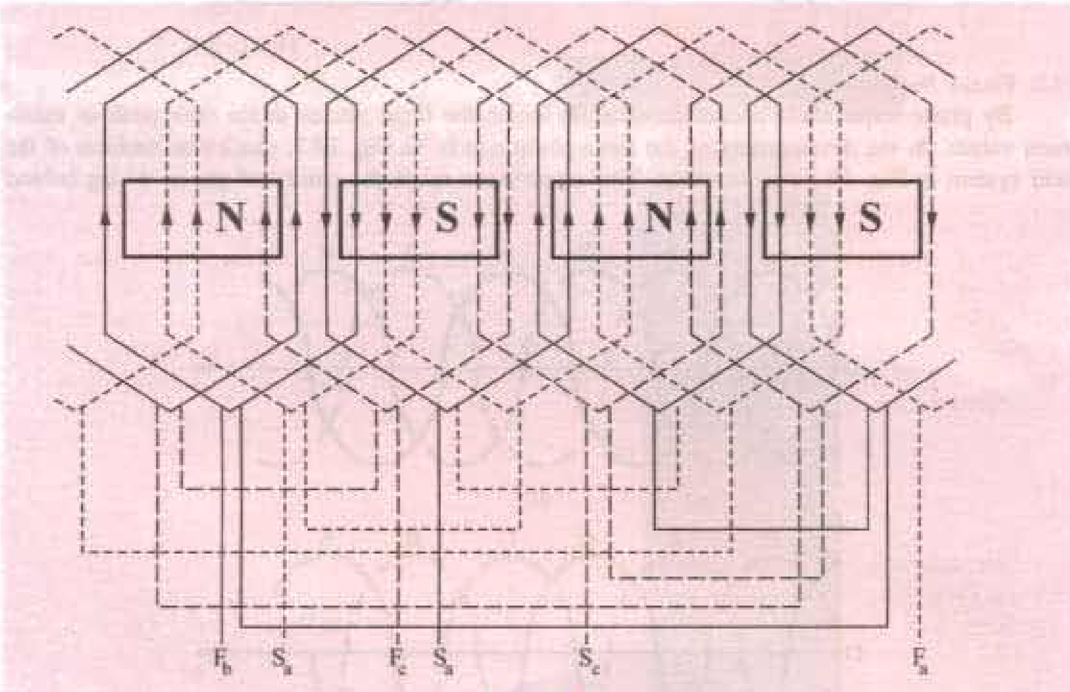


Fig. 19.4

Vector Addition

As shown in Fig. 19.6, the resultant of E_a and E_b is E_r and its magnitude is $2E \cos 60° = E$ where $E_a = E_b = E_c = E$.

This resultant E_r is equal and opposite to E_c. Hence, their resultant is zero.

By Calculation

Let us take E_a as reference voltage and assuming clockwise phase sequence

$$\mathbf{E}_a = E\angle 0° = E + j0$$

$$E_b = E\angle -120° = E(-0.5 - j0.866)$$

$$\mathbf{E}_c = E\angle -240° = E\angle 120° = E(-0.05 + j0.866)$$

$$\therefore\ \mathbf{E}_a + \mathbf{E}_b + \mathbf{E}_c = (E + j0) + E(-0.5 - 0.866) + E(-0.05 + j0.866) = 0$$

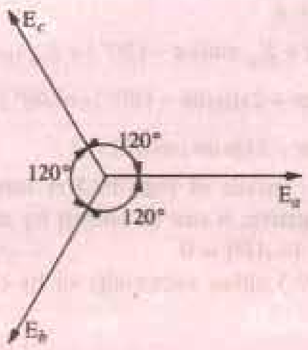


Fig. 19.5

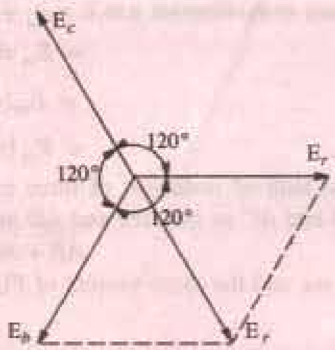


Fig. 19.6

19.2. Phase Sequence

By phase sequence is meant the order in which the three phases attain their peak or maximum values. In the development of the three-phase e.m.fs. in Fig. 19.7, clockwise rotation of the field system in Fig. 19.1 was assumed. This assumption made the e.m.fs. of phase '*b*' lag behind

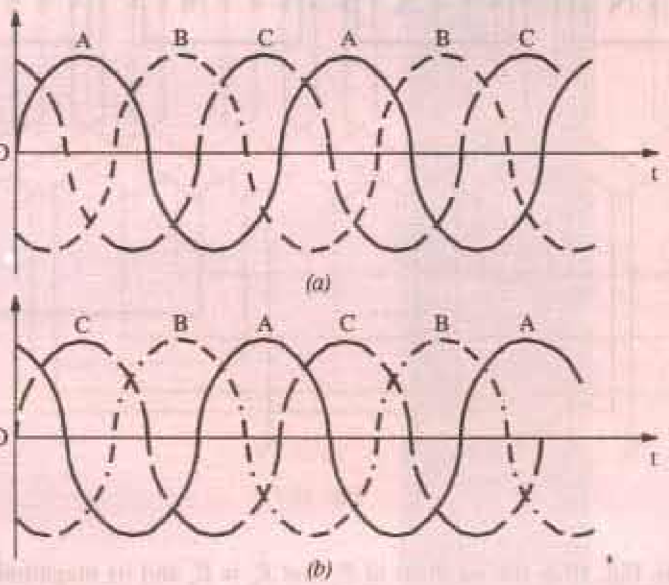


Fig. 19.7

that of '*a*' by 120° and in a similar way, made that of '*c*' lag behind that of '*b*' by 120° (or that of '*a*' by 240°). Hence, the order in which the e.m.fs. of phases *a*, *b* and *c* attain their maximum values is *a b c*. *It is called the phase order* or *phase sequence* $a \rightarrow b \rightarrow c$ as illustrated in Fig. 19.7 (*a*).

If, now, the rotation of the field structure of Fig. 19.1 is reversed *i.e.* made anticlockwise, then the order in which the three phases would attain their corresponding maximum voltages would also be reversed. The phase sequence would become $a \rightarrow b \rightarrow c$. This means that e.m.f. of phase '*c*' would now lag behind that of phase '*a*' by 120° instead of 240° as in the previous case as shown in Fig. 19.7 (*b*). By repeating the letters, this phase sequence can be written as *acbacba* which is the same thing as *cba*. Obviously, a three-phase system has only two possible sequences : *abc* and *cba* (*i.e. abc* read in the reverse direction).

19.3. Phase Sequence At Load

In general, the phase sequence of the voltages applied to load is determined by the order in which the 3-phase lines are connected. The phase sequence can be reversed by interchanging any pair of lines. In the case of an induction motor, reversal of sequence results in the reversed direction of motor rotation. In the case of 3-phase unbalanced loads, the effect of sequence reversal is, in general, to cause a completely different set of values of the currents. Hence, when working on such systems, it is essential that phase sequence be clearly specified otherwise unnecessary confusion will arise. Incidentally, reversing the phase sequence of a 3-phase generator which is to be paralleled with a similar generator can cause extensive damage to both the machines.

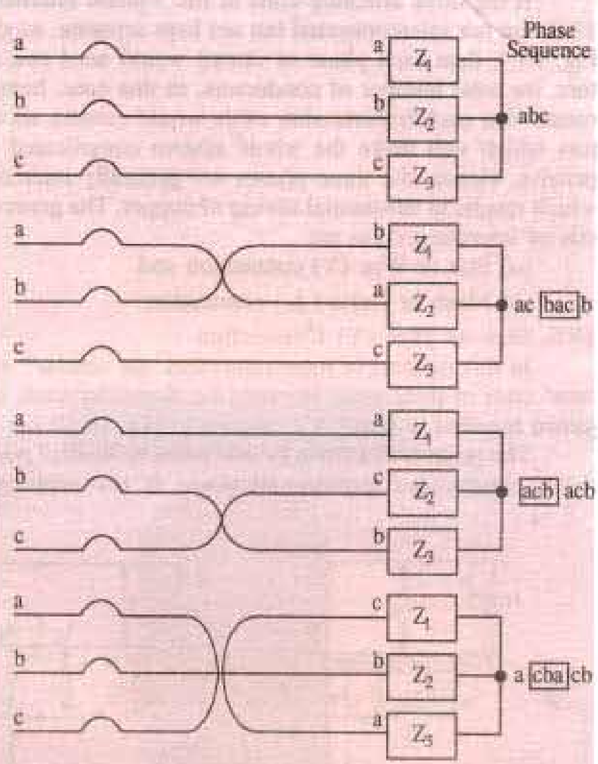


Fig. 19.8

Fig. 19.8 illustrates the fact that by interchanging any two of the three cables the phase sequence *at the load* can be reversed though sequence of 3-phase supply remains the same *i.e. abc*. It is customary to define phase sequence at the load by reading repetitively from top to bottom. For example, load phase sequence in Fig. 19.8 (*a*) would be read as *abcabcabc*– or simply *abc*. The changes are as tabulated below :

Cables Interchanged	*Phase Sequence*
a and *b*	*b a* $\boxed{c\ b\ a\ c\ b\ a}$ *c* – or *c b a*
b and *c*	*a* $\boxed{c\ b\ a\ c\ b\ a}$ *c b* – or *c b a*
c and *a*	$\boxed{c\ b\ a\ c\ b\ a}$ *c b a* – or *c b a*

19.4. Numbering of Phases

The three phases may be numbered 1, 2, 3 or *a, b, c* or as is customary, they may be given three colours. The colours used commercially are red, yellow (or sometimes white) and blue. In this case, the sequence is *RYB*.

Obviously, in any three-phase system, there are two possible sequences in which the three coil or phase voltages may pass through their maximum values *i.e.* re → yellow → blue (*RYB*) or red → blue → yellow (*RBY*). By convention, sequence *RYB* is taken as positive and *RBY* as negative.

19.5. Interconnection of Three Phases

If the three armature coils of the 3-phase alternator (Fig. 19.8) are not interconnected but are kept separate, as shown in Fig. 19.9, then each phase or circuit would need two conductors, the total number of conductors, in that case, being six. It means that each transmission cable would contain six conductors which will make the whole system complicated and expensive. Hence, the three phases are generally interconnected which results in substantial saving of copper. The general methods of interconnection are

(*a*) Star or Wye (Y) connection and

(*b*) Mesh or Delta (Δ) connection.

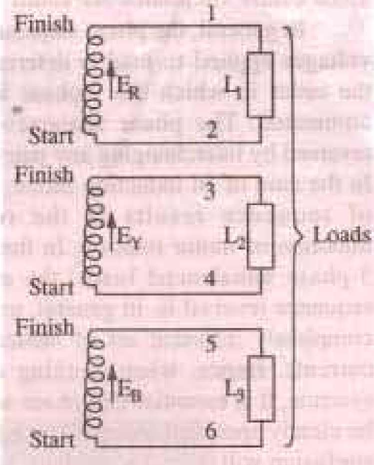


Fig. 19.9

19.6. Star or Wye (Y) Connection

In this method of interconnection, the *similar** ends say, 'star' ends of three coils (it could be 'finishing' ends also) are joined together at point *N* as shown in Fig. 19.10 (*a*).

The point *N* is known as *star point* or *neutral point.* The three conductors meeting at point *N* are replaced by a single conductor known as *neutral conductor* as shown in Fig. 19.10 (*b*). Such an interconnected system is known as four-wire, 3-phase system and is diagrammatically shown in Fig. 19.10 (*b*). If this three-phase voltage system is applied across a balanced symmetrical load, the neutral wire will be carrying three currents which are exactly equal in magnitude but are 120° out of phase with each other. Hence, their vector sum is zero.

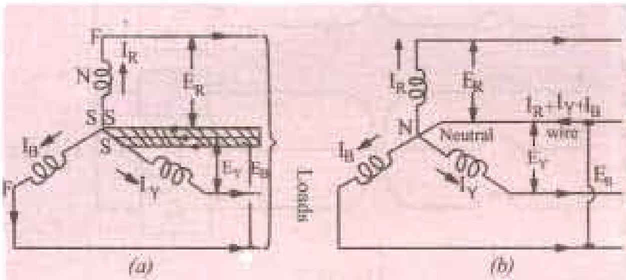


Fig. 19.10

i.e. $$\mathbf{I}_R + \mathbf{I}_Y + \mathbf{I}_B = 0 \quad \text{... vectorially}$$

The neutral wire, in that case, may be omitted although its retention is useful for supplying lighting loads at low voltages (Ex. 19.22). The p.d. between any terminal (or line) and neutral (or star) point gives the *phase* or *star* voltage. But the p.d. between any two lines gives the line-to-line voltage or simply line voltage.

19.7. Values of Phase Currents

When considering the distribution of current in a 3-phase system, it is extremely important to bear in mind that :

(*i*) the arrow placed alongside the currents I_R I_Y and I_B flowing in the three phases [Fig. 19.10 (*b*)] indicate the directions of currents when they are assumed to be *positive* and not the directions at a particular instant. It should be clearly understood *that at no instant will all the three currents flow in the same direction either outwards or inwards.* The three arrows indicate that first the current flows outwards in phase *R*, then after a phase-time of 120°, it will flow outwards from phase *Y* and after a further 120°, outwards from phase *B*.

(*ii*) the current flowing outwards in one or two conductors is always equal to that flowing inwards in the remaining conductor or conductors. In other words, *each conductor in turn, provides a return path for the currents of the other conductors.*

* As an aid to memory, remember that first letter *S* of Similar is the same as that of Star.

In Fig. 19.11 are shown the three phase currents, having the same peak value of 20 A but displaced from each other by 120°. At instant '*a*', the currents in phases *R* and *B* are each + 10 A (*i.e.* flowing outwards) whereas the current in phase *Y* is – 20A (*i.e.* flowing inwards). In other words, at the instant '*a*', phase *Y* is acting as return path for the currents in phases *R* and *B*. At instant *b*, I_R = +15 A and I_Y = +5 A but I_B = –20A which means that now phase *B* is providing the return path.

At instant *c*, I_Y = +15 *A* and I_B = +5A and I_R = – 20A.

Hence, now phase *R* carries current inwards whereas *Y* and *B* carry current outwards. Similarly at point *d*, I_R = 0, I_B = 17.3 A and I_Y = – 17.3 A. In other words, current is flowing outwards from phase *B* and returning via phase *Y*.

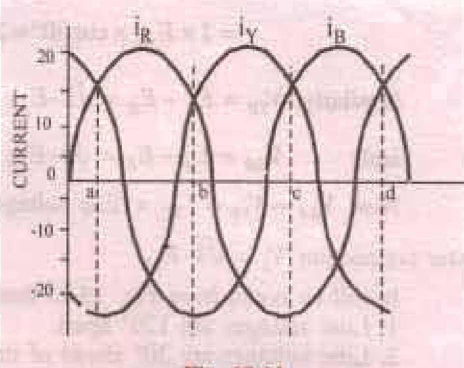


Fig. 19.11

In addition, it may be noted that although the distribution of currents between the three lines is continuously changing, yet at any instant the algebraic sum of the *instantaneous* values of the three currents is zero *i.e.* $i_R + i_Y + i_B = 0$ – algebraically.

19.8. Voltages and Currents in Y-Connection

The voltage induced in each winding is called the *phase* voltage and current in each winding is likewise known as *phase* current. However, the voltage available between any pair of terminals (or outers) is called *line* voltage (V_L) and the current flowing in each *line* is called *line* current (I_L).

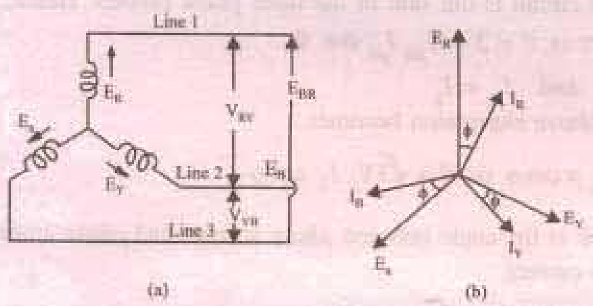


Fig. 19.12

As seen from Fig. 19.12 (*a*), in this form of interconnection, there are two phase windings between each pair of terminals but since their *similar* ends have been joined together, they are in opposition. Obviously, the *instantaneous* value of p.d. between any two terminals is the *arithmetic difference* of the two phase e.m.fs. concerned. However, the r.m.s. value of this p.d. is given by the *vector difference* of the two phase e.m.fs.

The vector diagram for phase voltages and currents in a star connection is shown in Fig. 19.12 (*b*) where a balanced system has been assumed.* It means that $E_R = E_Y = E_{ph}$ (phase e.m.f.).

Line voltage V_{RY} between line 1 and line 2 is the vector difference of E_R and E_Y.
Line voltage V_{YB} between line 2 and line 3 is the vector difference of E_Y and E_B.
Line voltage V_{BR} between line 3 and line 1 is the vector difference of E_B and E_R.

(*a*) Line Voltages and Phase Voltages

The p.d. between line 1 and 2 is $V_{RY} = E_R - E_Y$ ... vector difference.

Hence, V_{RY} is found by compounding E_R and E_Y reversed and its value is given by the diagonal of the parallelogram of Fig. 19.13. Obviously, the angle between E_R and E_Y reversed is 60°. Hence if $E_R = E_Y = E_B$ = say, E_{ph} – the phase e.m.f., then

* A balanced system is one in which (*i*) the voltages in all phases are equal in magnitude and offer in phase from one another by equal angles, in this case, the angle = 360/3 = 120°, (*ii*) the currents in the three phases are equal in magnitude and also differ in phase from one another by equal angles.

A 3-phase balanced load is that in which the loads connected across three phases are identical.

$$V_{RY} = 2 \times E_{ph} \times \cos(60°/2)$$

$$= 2 \times E_{ph} \times \cos 30° = 2 \times E_{ph} \times \frac{\sqrt{3}}{2} = \sqrt{3}\, E_{ph}$$

Similarly, $V_{YB} = E_Y - E_B = \sqrt{3} \cdot E_{ph}$ ... vector difference

and $V_{BR} = E_B - E_R = \sqrt{3} \cdot E_{ph}$

Now $V_{RY} = V_{YB} = Y_{BR}$ = line voltage, say V_L. Hence, in star connection $V_L = \sqrt{3} \cdot E_{ph}$

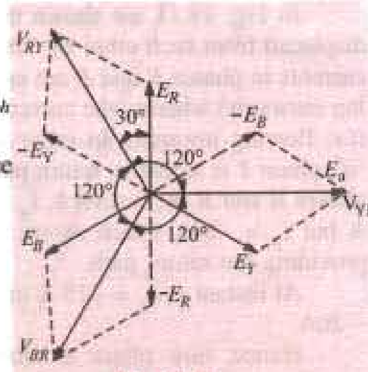


Fig. 19.13

It will be noted from Fig. 19.13 that

1. Line voltages are 120° apart.
2. Line voltages are 30° ahead of their respective *phase* voltages.
3. The angle between the line currents and the corresponding line voltages is (30 + ϕ) with current lagging.

(b) Line Currents and Phase Currents

It is seen from Fig. 19.12 (*a*) that each line is in series with its individual phase winding, hence the line current in each line is the same as the current in the phase winding to which the line is connected.

Current in line 1 = R_R ; Current in line 2 = R_Y; Current in line 3 = I_B

Since $I_R = I_Y = I_B$ = say, I_{ph} – the phase current

$\therefore$ line current $I_L = I_{ph}$

(c) Power

The total active or true power in the circuit is the sum of the three phase powers. Hence, total active power = 3 × phase power or $P = 3 \times V_{ph}\, I_{ph} \cos\phi$

Now $V_{ph} = V_L / \sqrt{3}$ and $I_{ph} = I_L$

Hence, in terms of line values, the above expression becomes

$$P = 3 \times \frac{V_L}{\sqrt{3}} \times I_L \times \cos\phi \quad \text{or} \quad P = \sqrt{3}\, V_L\, I_L \cos\phi$$

It should be particularly noted that ϕ is the angle between *phase* voltage and *phase* current and **not** between the line voltage and line current.

Similarly, total reactive power is given by $Q = \sqrt{3}\, V_L\, I_L \sin\phi$

By convention, reactive power of a coil is taken as positive and that of a capacitor as negative.

The total apparent power of the three phases is

$$S = \sqrt{3}\, V_L I_L \qquad \text{Obviously, } S = \sqrt{P^2 + Q^2} \qquad \text{– Art. 13.4}$$

Example 19.1. *A balanced star-connected load of (8 + j6) Ω per phase is connected to a balanced 3-phase 400-V supply. Find the line current, power factor, power and total volt-amperes.*

(Elect. Engg., Bhagalpur Univ. 1985)

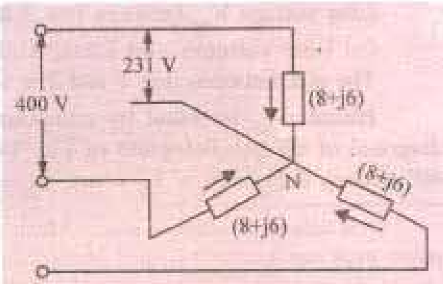


Fig. 19.14

Solution. $Z_{ph} = \sqrt{8^2 + 6^2} = 10\,\Omega$

$V_{ph} = 400 / \sqrt{3} = 231$ V

$I_{ph} = V_{ph} / Z_{ph} = 231 / 10 = 23.1$ A

(*i*) $I_L = I_{ph}$ = **23.1 A**

(*ii*) p.f. = $\cos\phi = R_{ph}/Z_{ph} = 8/10$ = **0.8 (lag)** (*iii*) Power $P\sqrt{3}\,V_L I_l \cos\phi$

$$= \sqrt{3}\times 400\times 23.1\times 0.8 = \mathbf{12{,}800\ W}\ [\text{Also, } P = 3I_{ph}^2\ R_{ph} = 3(23.1)^2\times 8 = 12{,}800\ W]$$

(*iv*) Total volt-amperes, $S = \sqrt{3}\,V_L I_L = \sqrt{3}\times 400\times 23.1 = \mathbf{16{,}000\ VA}$

Example 19.2. *Phase voltages of a star connected alternator are $E_R = 231\angle 0°$ V; $E_Y = 231\angle -120°$ V ; and $E_B = 231\angle +120°$V. What is the phase sequence of the system ? Compute the line voltages E_{RY} and E_{YB}.* **(Elect. Mechines AMIE Sec. B Winter 1990)**

Solution . The phase voltage $E_B = 231\angle -120°$ can be written as $E_B = 231\angle -240°$. Hence, the three voltages are: $E_R = 231\angle -0°$, $E_Y = 231\angle -120°$ and $E_B = 231\angle -240°$. It is seen that E_R is the reference voltage, E_Y lags behind it by 120° whereas E_B lags behind it by 240°. Hence, phase sequence is *RYB*. Moreover, it is a symmetrical 3-phase voltage system.

$$\therefore \qquad E_{RY} = E_{YB} = \sqrt{3}\times 231 = \mathbf{400\ V}$$

Example 19.3 *Three equal star-connected inductors take 8 kW at a power factor 0.8 when connected across a 460 V, 3-phase, 3-phase, 3-wire supply. Fnd the circuit constants of the load per phase.* **(Elect. Machines AMIE Sec. B 1992)**

Solution. $P = \sqrt{3}V_L I_L \cos\phi$ or

$8000 = \sqrt{3}\times 460\times I_L\times 0.8$

$\therefore\ I_L = 12.55$ A $\therefore\ I_{ph} = 12.55$ A;

$V_{ph} = V_L/\sqrt{3} = 460/\sqrt{3} = 460/\sqrt{3} = 265V$

$I_{ph} = V_{ph}/Z_{ph};\ \therefore Z_{ph} = V_{ph}/I_{ph} = 265/12.55 = 21.1\Omega$

$R_{ph} = Z_{ph}\cos\phi = 21.1\times 0.8 = 16.9\ \Omega$

$X_{ph} = Z_{ph}\sin\phi = 21.1\times 0.6 = 12.66\Omega$;

The circuit is shown in Fig. 19.15.

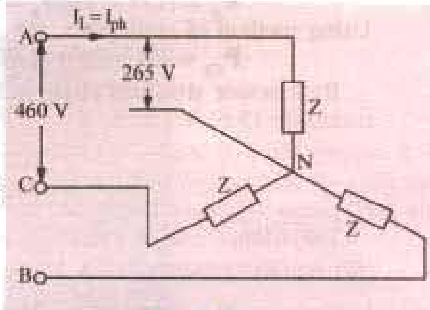


Fig. 19.15

Example 19.4. *Given a balanced 3−φ, 3-wire system with Y-connected load for which line voltage is 230 V and impedance of each phase is (6 + J8) ohm. Find the line current and power absorbed by each phase.* **(Elect. Engg - II Pune Univ. 1991)**

Solution. $Z_{ph} = \sqrt{6^2+8^2} = 10\Omega;\ V_{ph} = V_L/\sqrt{3} = 230/\sqrt{3} = 133\text{V}$

$$\cos\phi = R/Z = 6/10 = 0.6;\ I_{ph} = V_{ph}/Z_{ph} = 133/10 = 13.3\text{A}$$

$$\therefore \qquad I_L = I_{ph} = \mathbf{13.3\ A}$$

Power absorbed by each phase $= I_{ph}^{\ 2}R_{ph} = 13.3^2\times 6 = \mathbf{1061\ W}$

Solution by Symbolic Notation

In Fig. 19.16 (*b*), V_R, V_Y and V_B are the phase voltage whereas I_R, I_Y and I_B are phase currents. Taking V_R as the reference vector, we get

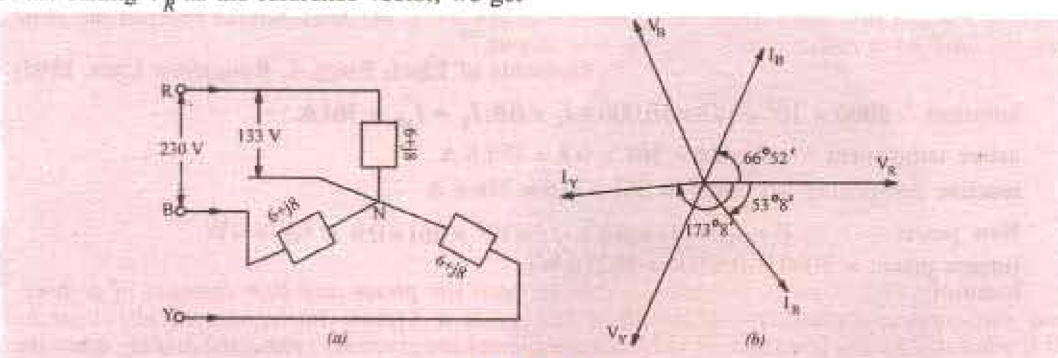


Fig 19.16

$$\mathbf{V}_R = 133\angle 0° = 133 + j0 \text{ volt}$$
$$V_Y = 133\angle -120° = 133(-0.5 - j0.866) = (-66.5 - j115) \text{ volt}$$
$$\mathbf{V}_B = 133 \angle 120° = 133\,(-0.5 + j\,0.866) = (-66.5 + j\,115) \text{ volt}$$
$$\mathbf{Z} = 6 + j8 = 10 \angle 53°\,8' \;;\; \mathbf{I}_R = \frac{\mathbf{V}_R}{\mathbf{Z}} = \frac{133\angle 0°}{10\angle 53°8'} = 13.3\angle -53°8'$$

This current lags behind the reference voltage by $53°8'$ [Fig. 19.16 (*b*)]

$$\mathbf{I}_Y = \frac{\mathbf{V}_Y}{\mathbf{Z}} = \frac{133\angle -120°}{10\angle 53°8'} = 13.3\angle -173°8'$$

It lags behind the reference vector *i.e.* V_R by $173°8'$ which amounts to lagging behind its phase voltage V_y by $53°8'$.

$$\mathbf{I}_B = \frac{\mathbf{V}_B}{\mathbf{Z}} = \frac{133\angle -120°}{10\angle 53°8'} = 13.3\angle 66°52'$$

This current leads V_R By $66°52'$ which is the same thing as *lagging* behind its phase voltage by $53°8'$. For calculation of power, consider *R*-phase

$$\mathbf{V}_R = (133 - j0);\; I_R = 13.3\,(0.6 - j0.8) = (7.98 - j10.64)$$

Using method of conjugates, we get

$$\mathbf{P}_{VA} = (133 - j0)\,(7.98 - j10.64) = 1067 - j1415$$

$\therefore$ Real power absorbed/phase = 1067 W – as before

Example 19.5. *When the three identical star-connected coils are supplied with 440 V, 50 Hz, 3-ϕ supply, the 1-ϕ wattmeter whose current coil is connected in line R and pressure coil across the phase R and neutral reads 6 kW and the ammeter connected in R-phase reads 30 Amp. Assuming RYB phase sequence find:*

(i) resistance and reactance of the coil. *(ii) the power factor, of the load*

(iii) reactive power of 3-ϕ load. **(Elect. Engg.-I, Nagpur Univ. 1993)**

Solution. $V_{ph} = 440/\sqrt{3} = 254$ V; $I_{ph} = 30$A (Fig. 19.17.)

Now, $V_{ph}\, I_{ph} \cos\phi = 6000$; $254 \times 30 \times \cos\phi = 6000$

$\therefore \cos\phi = 0.787$; $\phi = 38.06°$ and $\sin\phi = 0.616°$; $Z_{ph} = V_{ph} / I_{ph} = 254/30 = 8.47\,\Omega$

(*i*) Coil resistance $R = Z_{ph} \cos\phi = 8.47 \times 0.787 = 6.66\,\Omega$

$X_L = Z_{ph} \sin\phi = 8.47 \times 0.616 = 5.22\,\Omega$

(*ii*) p.f. = $\cos\phi$ = 0.787 (lag)

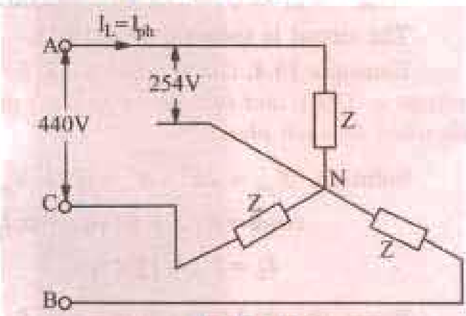


Fig. 19.17

(*iii*) Reactive power = $\sqrt{3}\,/\,V_L I_L \sin\phi = \sqrt{3} \times 440 \times 30 \times 0.616 = 14{,}083\text{VA} = 14.083\text{kVA}$.

Example 19.6 *Calculate the active and reactive components in each phase of Y-connected 10,000 V, 3-phase alternator supplying 5,000 kW at 0.8 p.f. If the total current remains the same when the load p.f is raised to 0.9, find the new output.*

(Elements of Elect. Engg.-I, Bangalore Univ. 1985)

Solution. $5000 \times 10^3 = \sqrt{3} \times 10{,}000 \times I_L \times 0.8; I_L = I_{ph} = 361\,\text{A}$

active component $= I_L \cos\phi = 361 \times 0.8 =$ **288.8 A**

reactive component $= I_L \sin\phi = 361 \times 0.6 =$ **216.6 A**

New power $\quad P = \sqrt{3} V_L I_L \cos\phi = \sqrt{3} \times 10^4 \times 361 \times 0.9 =$ **5,625 kW**

[ornew power = 5000 × 0.9/0.8 = 5625 kW]

Example 19.7. *Deduce the relationship between the phase and line voltages of a three-phase star-connected alternator. If the phase voltage of a 3-phase star-connected alternator be 200 V, what will be the line voltages (a) when the phases are correctly connected and (b) when the connections to one of the phases are reversed.*

Solution. (*a*) When phases are correctly connected, the vector diagram is as shown in Fig. 19.12. (*b*). As proved in Art. 19.7

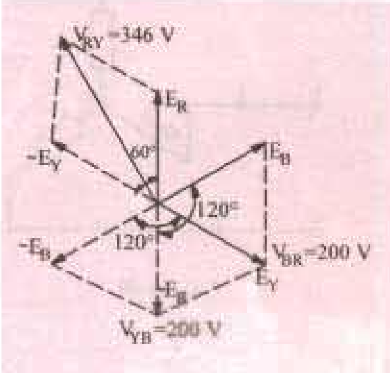


Fig. 19.18

$$V_{RY} = V_{YB} V_{BR} = \sqrt{3}.E_{Ph}$$

Each line voltage = $\sqrt{3} \times 200 =$ **346 V**

(*b*) Suppose connections to *B*-phase have been reversed. Then voltage vector diagram for such a case is shown in Fig. 19.18. It should be noted that E_B has been drawn in the reversed direction, so that angles between the three-phase voltages are 60° (instead of the usual 120°)

$V_{RY} = E_R - E_Y$ ... vector difference

$= 2 \times E_{ph} \times \cos 30° = \sqrt{3} \times 200 =$ **346 V**

$V_{YB} = E_Y - E_B$ ... vector difference

$= 2 \times E_{ph} \times \cos 60° = 2 \times 200 \times \frac{1}{2} =$ **200V**

$V_{BR} = E_B - E_R$ ... vector difference $= 2 \times E_{ph} \times \cos 60° = 2 \times 200 \times \frac{1}{2} =$ **200 V**

Example 19.8 *In a 4-wire, 3-phase system, two phases have currents of 10A and 6A at lagging power factors of 0.8 and 0.6 respectively while the third phase is open-circuited. Calculate the current in the neutral and sketch the vector diagram.*

Solution. The circuit is shown in Fig. 19.19 (*a*).

$$\phi_1 = \cos^{-1}(0.8) = 36°54'; \ \phi_2 = \cos^{-1}(0.6) = 53°6'$$

Let V_R be taken as the reference vector. Then

$\mathbf{I}_R = 10\angle -36°54' = (8 - j6)$ $\mathbf{I}_y = 6\angle -173°6' = (-6 - j0.72)$

The neutral current $\mathbf{I}_N$, as shown in Fig. 19.16 (*b*), is the sum of these two currents.

$\therefore$ $\mathbf{I}_N = (8 - j6) + (-6 - j0.72) = 2 - j6.72 = 7\angle -73°26'$

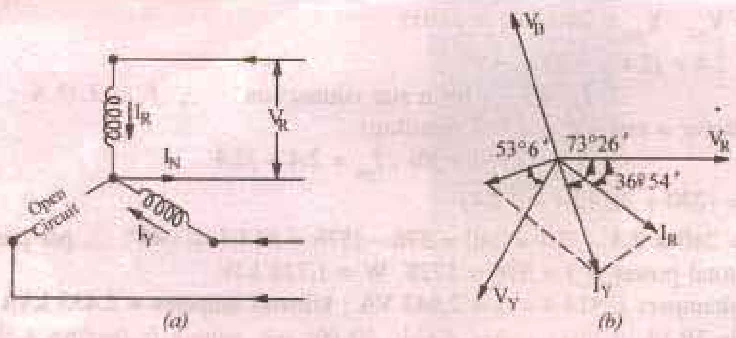


Fig. 19.19

Example 19.9 (*a*). *Three equal star-connected inductors take 8 kW at power factor 0.8 when connected a 460-V, 3-phase, 3-wire supply. Find the line currents if one inductor is short-circuited.*

Solution. Since the circuit is balanced, the three line voltages are represented by

$V_{ab} = 460\angle 0°; V_{bc} = 460\angle -120°$ and $V_{ca} = 460\angle 120°$

The phase impedance can be found from the given data :

$$8000 = \sqrt{3} \times 460 \times I_L \times 0.8 \qquad \therefore \quad I_L = I_{ph} = 12.55 \text{ A}$$

$$Z_{ph} = V_{ph} / I_{ph} = 460 / \sqrt{3} \times 12.55 = 21.2\ \Omega;$$

$$\therefore \quad Z_{ph} = 21.2\angle 36.9° \text{ because } \phi = \cos^{-1}(0.8) = 36.9°$$

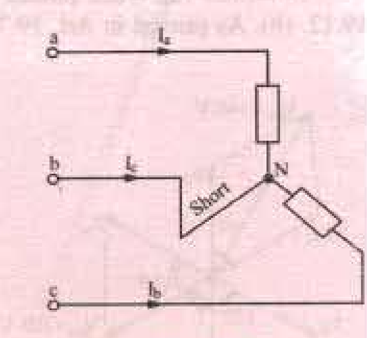


Fig. 19.20

As shown in the Fing. 19.20, the phase c has been short-circuited. The line current $I_a = V_{ac}/Z_{ph} = -V_{ca}/Z_{ph}$ because the current enters at point a and leaves from point c.

$$\therefore \ I_a = -460\angle 1200°/21.2\angle 36.9° = 21.7\angle 83.1°$$

Similarly, $I_b = V_{bc}/Z_{ph} = 460\ \angle 120°/21.2\ \ \angle 36.9° =$ $21.7\ \angle -156.9°$. The current I_c can be found by applying KVL to the neutral point N.

$\therefore I_a + I_b + I_c = 0 \quad$ or $\quad I_c = -I_a - I_b$

$$\therefore \ I_C = 21.7\angle 83.1° - 21.7\angle -156.9° = 37.3\angle 53.6°$$

Hence, the magnitudes of the three currents are : 21.7 A; 21.7 Al; 37.3 A.

Example 19.9 (*b*). *Each phase of a star-connected load consists of a non-reactive resistance of 100 Ω in parallel with a capacitance of 31.8 µF. Calculate the line current, the power absorbed, the total kVA and the power factor when connected to a 416-V, 3-phase, 50-Hz supply.*

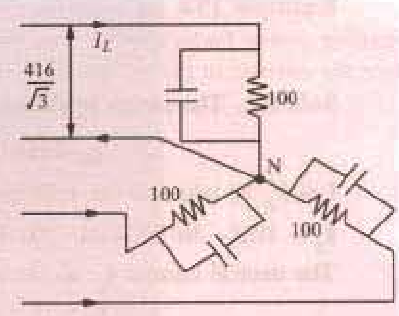


Fig. 19.21

Solution. The circuit is shown in Fig. 14.20.

$$V_{ph} = (416/\sqrt{3})\angle 0° = 240\angle 0° = (240 + j0)$$

Admittance of each phase is

$$\mathbf{Y}_{ph} = \frac{1}{R} + j\omega C = \frac{1}{100} + j314 \times 31.8 \times 10^{-6}$$

$$= 0.01 + j0.01$$

$$\therefore \ \mathbf{I}_{ph} = \mathbf{V}_{ph} \cdot \mathbf{Y}_{ph} = 240(0.01 + j0.01)$$

$$= 2.4 + j2.4 = 3.39\ \angle 45°$$

Since $\quad I_{ph} = I_L$ – for a star connection $\quad \therefore \ I_L =$ **3.39 A**

Power factor = cos 45° = **0.707 (leading)**

Now $\quad \mathbf{V}_{ph} = (240 + j0)$; $I_{ph} = 2.4 + j2.4$

$$\therefore \ \mathbf{P}_{VA} = (240 + j0)\ (2.4 + j\ 2.4)$$

$$= 240 \times 2.4 - j2.4 \times 240 = 576 - j576 = 814.4\ \angle -45° \quad \text{... per phase}$$

Hence, total power = 3 × 576 = 1728 W = **1.728 kW**

Total voltampers = 814.4 × 3 = 2,443 VA ; kilovolt amperes = **2.433 kVA**

Example 19.10. *A three pahse 400-V, 50 Hz, a.c. supply is feeding a three phase delta-connected load with each phase having a resistance of 25 ohms, an inductance of 0.15 H, and a capacitor of 120 microfarads in series. Determine the line current, volt-amp, active power and reactive volt-amp.* **[Nagpur University, November 1999]**

Solution. Impedance per phase $r + jX_L - jX_C$

$$X_L = 2\pi \times 50 \times 0.15 = 47.1\ \Omega$$

$$X_C = \frac{10^6}{32{\cdot}37} = 26.54\ \Omega$$

$\cos\phi = \dfrac{25}{32.37}$ Lagging, since inductive reactance is dominating.

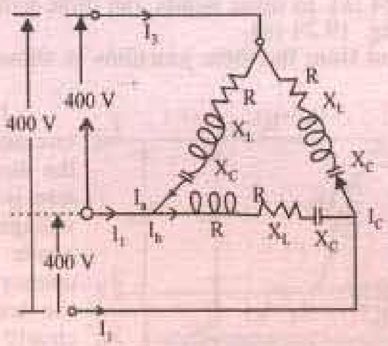


Fig. 19.22

$$\text{Phase Current} = \frac{400}{25 + j20.56} = 12.357$$

Line Current = $\sqrt{3} \times 12.357$ = 21.4 amp

Since the Power factor is 0.772 lagging.

P = total three phase power = $\sqrt{3}\, V_L\, I_L\, \cos\phi \times 10^{-3}$ kW

$= \sqrt{3} \times 400 \times 21.4 \times 0.772 \times 10^{-3} = 11.446$ kW

S = total 3 ph kVA $\dfrac{11.446}{0.772}$ = 14.83 kVA 14.83 kVA

Q = total 3 ph "reactive kilo-volt-amp" $\sqrt{3}$ = $(S^2 - P^2)^{0.50}$ = 9.43 kVAR lagging

Example 19.11. *Three phase star-connected load when supplied from a 400 V, 50 Hz source takes a line current of 10 A at 66.86° w.r. to its line voltage. Calculate : (i) Impedance-Parameters, (ii) P.f. and active-power consumed. Draw the phasor diagram.*

[Nagpur University, April 1998]

Solution. Draw three phasors for phase-voltages.

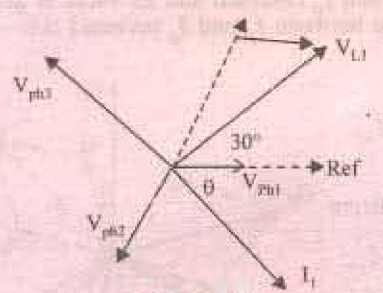


Fig. 19.23

These are V_{ph1}, V_{ph2}, V_{ph3} in Fig 19.23. As far as phase number 1 is concerned, its current is I_1 and the associated line voltage is V_{L1} . V_{L1} and V_{ph1} differ in phase by 30°. A current differing in phase with respect to line voltage by 66.86° and associated with V_{ph1} can only be lagging, as shown in Fig. 19.23. This means ϕ = 36.86°, and the corresponding load power factor is 0.80 lagging.

$Z = V_{ph}/I_{ph}$ = 231/10 = 23.1 ohms

$R = Z\cos\phi$ = 23.1 × 0.8 = 18.48 ohms

$X_L = Z\sin\phi$ = 23.1 × ?0.6 = 13.86 ohms

Total active power consumed = $3\, V_{ph}\, I_{ph}\, \cos\phi$

$= 3 \times 231 \times 10 \times 0.8 \times 10^{-3}$ kW = 5.544 kW

or total active power = $3 \times I^2R = 3 \times 10^2 \times 18.48$ = 5544 watts

For complete phasor diagram for three phases, the part of the diagram for Phase 1 in Fig 19.23 has to be suitably repeated for phase-numbers 2 and 3.

19.9 Delta (Δ) or Mesh Connection

In this form, of interconnection the *dissimilar* ends of the three phase winding are joined together *i.e.* the 'starting' end of one phase is joined to the 'finishing' end of the other phase and so on as showing in Fig. 19.24 (*a*). In other words, the three windings are joined in series to form a closed mesh as shown in Fig. 19.24 (*b*).

Three leads are taken out from the three junctions as shown as outward directions are taken as positive.

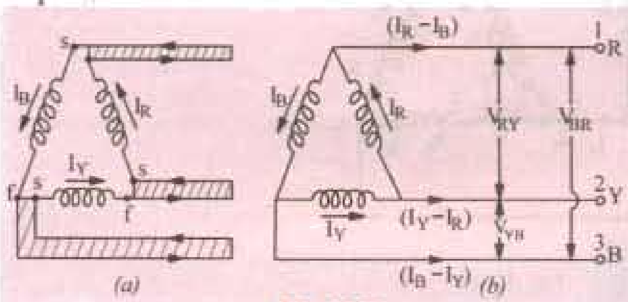


Fig. 19.24

It might look as if this sort of interconnection results in shortcircuiting the three windings. However, if the system is balanced then sum of the three voltages round the closed mesh is zero, hence no current of fundamental frequency can flow around the mesh when the terminals are open. It should be clearly understood that at any instant, the e.m.f. in one phase is equal and opposite to the resultant of those in the other two phases.

This type of connection is also referred to as 3-phase, 3-wire system.

(*i*) Line Voltages and Phase Voltages

It is seen from Fig. 19.24 (*b*) that there is only one phase winding completely included between any pair of terminals. Hence, in Δ-connection, the voltage between any pair of lines is equal to the phase voltage of the phase winding connected between the two lines considered. Since phase sequence is *R Y B*, the voltage having its positive direction from *R* to *Y* leads by 120° on that having its positive direction from *Y* to *B*. Calling the voltage between lines 1 and 2 as V_{RY} and that between lines 2 and 3 as V_{YB}, we find that V_{RY} lead V_{YB} by 120°. Similarly, V_{YB} leads V_{BR} by 120° as shown in Fig. 19.23. Let $V_{RY} = V_{YB} = V_{BR}$ = line voltage V_L. Then, it is seen that $V_L = V_{ph}$.

(*ii*) Line Currents and Phase Currents

It will be seen from Fig. 19.24 (*b*) that current in each line is the *vector difference* of the two phase currents flowing through that line. For example

$$\left.\begin{aligned} &\text{Current in line 1 is } I_1 = I_R = I_B \\ &\text{Current in line 2 is } I_2 = I_Y - I_R \\ &\text{Current in line 3 is } I_3 = I_B - I_Y \end{aligned}\right\} \text{ vector difference}$$

Current in line No. 1 is found by compounding I_R and I_B reversed and its value is given by the diagonal of the parallelogram of Fig. 19.25. The angle between I_R and I_B reversed (*i.e.* $-I_B$) is 60°. If $I_R = I_Y$ = phase current I_{ph} (say), then

Current in line No. 1 is

$$I_1 = 2 \times I_{ph} \times \cos(60°/2) = 2 \times I_{ph}\ \sqrt{3}/2 = \sqrt{3}\, I_{ph}$$

Current in line No. 2 is

$I_2 = I_B - I_Y$ ... vector difference = $\sqrt{3} I_{ph}$ and current in line No. 3 is $I_3 = I_B - I_Y$ ∴ Vector difference = $\sqrt{3} \cdot I_{ph}$

Since all the line currents are equal in magnitude *i.e.*

$$I_1 = I_2 - I_3 = I_L \qquad \therefore \qquad I_L = \sqrt{3}\, I_{ph}$$

With reference to Fig. 19.25, it should be noted that

1. line currents are 120° apart ;
2. line currents are 30° behind the respective phase currents ;
3. the angle between the line currents and the corresponding line voltages is (30 + ϕ) with the current lagging.

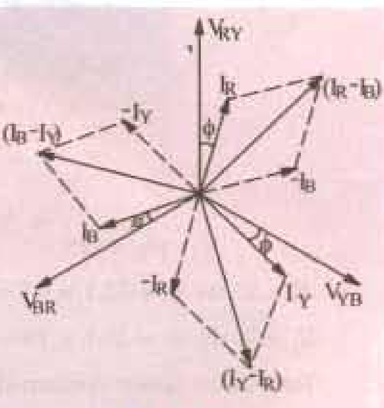


Fig. 19.25

* As an aid to memory, remember that first letter *D* of Dissimilar is the same as that of Delta.

(*iii*) **Power**

Power/phase = $V_{ph}I_{ph}\cos\phi$; Total power = $3 \times V_{ph}I_{ph}\cos\phi$. However, $V_{ph} = V_L$ and $I_{ph} = I_L/\sqrt{3}$

Hence, in terms of line values, the above expression for power becomes

$$P = 3\times V_L \times \frac{I_L}{\sqrt{3}} \times \cos\phi = \sqrt{3}\,V_L I_L \cos\phi$$

where ϕ is the phase power factor angle.

19.10. Balanced Y/Δ and Δ/Y Conversion

In view of the above relationship between line and phase currents and voltages, any balanced *Y*-connected system may be completely replaced by an equivalent Δ-connected system. For example, a 3-phase, *Y*-connected system having the voltage of V_L and line current I_L may be replaced by a Δ-connected system in which phase voltage is V_L and phase current is $I_L/\sqrt{3}$.

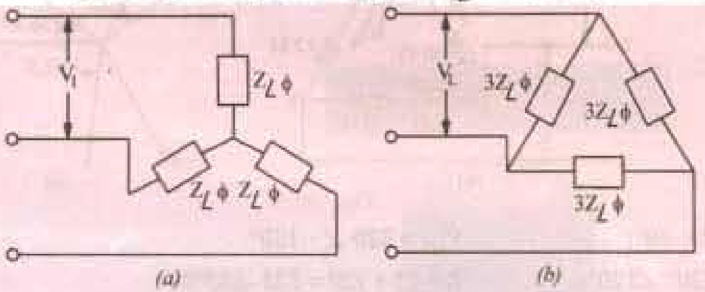


Fig. 19.26

Similarly, a balanced *Y*-connected load having equal branch impedances each of $Z\angle\phi$ may be replaced by an equivalent Δ-connected load whose each phase impedance is $3Z\angle\phi$. This equivalence is shown in Fig. 19.26.

For a balanced star-connected load, let

V_L = line voltage; I_L = line current ; Z_Y = impedance/phase

$$\therefore \qquad V_{ph} = V_L/\sqrt{3},\ \ I_{ph} = I_L;\ \ Z_Y = V_L/(\sqrt{3}I_L)$$

Now, in the equivalent Δ -connected system, the line voltages and currents must have the same values as in the *Y*-connected system, hence we must have

$$V_{ph} = V_L,\quad I_{ph} = I_L/\sqrt{3}\ \ \therefore\ \ Z_\Delta = V_L/(I_L/\sqrt{3}) = \sqrt{3}V_L/I_L = 3Z_Y$$

$$\therefore \qquad Z_\Delta\angle\phi = 3Z_Y\angle\phi \qquad (\because V_L/I_L = \sqrt{3}Z_Y)$$

or $\qquad \mathbf{Z}_\Delta = 3\mathbf{Z}_Y$ or $\mathbf{Z}_Y = \mathbf{Z}_\Delta/3$

The case of unbalanced load conversion is considered later. (Art. 19.34)

Example 19.12. *A star-connected alternator supplies a delta connected load. The impedance of the load branch is (8 + j6) ohm/phase. The line voltage is 230 V. Determine (a) current in the load branch, (b) power consumed by the load, (c) power factor of load, (d) reactive power of the load.* **(Elect. Engg. A.M.Ae. S.I. June 1991)**

Solution. Considering the Δ-connected load, we have $Z_{ph} = \sqrt{8^2+6^2} = 10\,\Omega$; $V_{ph} = V_L = 230$ V

(*a*) $I_{ph} = V_{ph}/Z_{ph} = 230/10 = 23$ A

(*b*) $I_L = \sqrt{3}\,I_{ph} = \sqrt{3}\times 23 = 39.8$ A; $\quad P = \sqrt{3}\,V_L I_L\cos\phi = \sqrt{3}\times 230\times 39.8\times 0.8 = 12{,}684$ W

(*c*) p.f. $\cos\phi = R/Z = 8/10 = 0.8$ (lag)

(*d*) Reactive power $Q = \sqrt{3}\,V_L I_L \sin\phi = \sqrt{3}\times 230\times 39.8\times 0.6 = 9513$ W

Example 19.13. *A 220-V, 3-ϕ voltage is applied to a balanced delta-connected 3-ϕ load of phase impedance (15 + j20)Ω.*

(a) Find the phasor current in each line. (b) What is the power consumed per phase ?
(c) What is the phasor sum of the three line currents ? Why does it have this value ?

(Elect. Circuits and Insturments, B.H.U. 1985)

Solution. The circuit is shown in Fig. 19.27 (*a*).

$V_{ph} = V_L = 220\ V;\ Z_{ph} = \sqrt{15^2 + 20^2} = 25\Omega,\ I_{ph} = V_{ph}/Z_{ph} = 220/25 = 8.8\,\text{A}$

(*a*) $I_L = \sqrt{3}\ I_{ph} = \sqrt{3} \times 8.8 =$ **15.24 A** (*b*) $P = I_{ph}{}^2 R_{ph} = 8.8^2 \times 15 =$ **462 W**

(*c*) Phasor sum would be zero because the three currents are equal in magnitude and have a mutual phase difference of 120°.

Solution by Symbolic Notation

Taking V_{RY} as the reference vector, we have [Fig. 19.27 (*b*)]

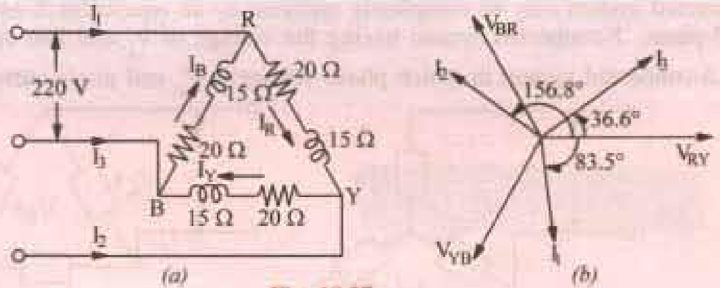


Fig. 19.27

$\mathbf{V}_{RY} = 20\ \angle 0°;$ $\mathbf{V}_{YB} = 220\ \angle -120°$

$\mathbf{V}_{BR} = 220°\ \angle 120°;$ $\mathbf{Z} = 15 + j20 = 125\ \angle 53°8'$

$$I_R = \frac{V_{RY}}{\mathbf{Z}} = \frac{220\angle 0°}{25\angle 53°8'} = 8.8\angle -53°8 = (5.28 - j7.04)\ A$$

$$\mathbf{I}_Y = \frac{V_{YB}}{\mathbf{Z}} = \frac{220\angle -120°}{25\angle 53°8'} = 8.8\angle -173°8' = (-8.75 - j1.05)\ A$$

$$\mathbf{I}_B = \frac{V_{BR}}{\mathbf{Z}} = \frac{220\angle -120°}{25\angle 53°8'} = 8.8\angle 66°55' = (3.56 + j8.1)$$

(*a*) Current in line No. 1 is

$\mathbf{I}_1 = \mathbf{I}_R - \mathbf{I}_B = (5.28 - j\ 7.04) - (4.56 + j8.1) = (1.72 - j15.14) = 15.23\ \angle\ -83.5°$

$\mathbf{I}_2 = \mathbf{I}_Y - \mathbf{I}_R = (-8.75 - j\ 1.05) - (5.28 - j7.04) = (-14.03 + j\ 6.0) = 15.47\ \angle\ -156.8°$

$\mathbf{I}_3 = \mathbf{I}_B - \mathbf{I}_Y = (3.56 - j\ 8.1) - (-8.75 - j1.05) = (12.31 + j\ 9.15) = 15.26\ \angle\ 36.8°$

(*b*) Using conjugate of voltage, we get for *R*-phase

$\mathbf{P}_{VA} = \mathbf{V}_{RY} \cdot \mathbf{I}_R = (220 - j0)\ (5.28 - 7.04) = (1162 - j\ 1550)$ voltamper

Real power per phase = **1162 W**

(*c*) Phasor sum of three line currents

$= \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 = (1.72 - j\ 15.14) + (-\ 14.03 + j\ 6.0) + (12.31\ j\ 9.15) = 0$

As expected, phasor sum of 3 line currents drawn by a balanced load is zero because these are equal in magnitude and have a phase difference of 120° amount themselves.

Example 19.14 *A 3-φ, Δ-connected alternator drives a balanced 3-φ load whose each phase current is 10 A in magnitude. At the time when $I_a = 10\angle 30°$, determine the following, for a phase sequence of abc.*

(i) Polar expression for I_b and I_c and (ii) polar expressions for the three line current. Show the phase and line currents on a phasor diagram.

Solution. (*i*) Since it is a balanced 3-phase system, I_b lags I_a by 120° and I_c lags I_a by 240° or leads it by 120°.

$$\therefore \quad I_b = I_a\ \angle -120° = 10\angle(30° - 120°) = 10\angle -90°$$

$$I_c = I_a \angle 120° = 10\angle(30° + 120°) = 10\angle 150°$$

The 3-phase currents have been represented on the phasor diagram of Fig. 19.28 (*b*).

As seen from Fig. 19.28 (*b*), the line currents lag behind their nearest phase currents by 30°.

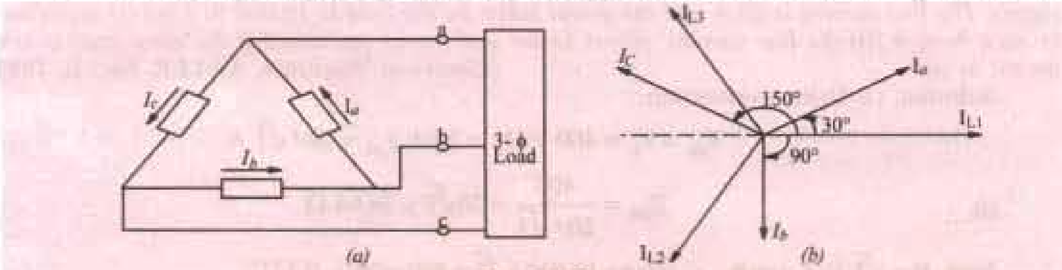


Fig. 19.28

$\therefore$ $$I_{L1} = \sqrt{3} \cdot I_a \angle (30° - 30°) = 17.3\angle 0°$$

$$I_{L2} = \sqrt{3} \cdot I_b \angle (-90° - 30°) = 17.3\angle -120°$$

$$I_{L3} = \sqrt{3} \cdot I_C \angle (150° - 30°) = 17.3\angle 120°$$

These line currents have also been shown in Fig. 19.28 (*b*).

Example 19.15. *Three similar coils, each having a resistance of 20 ohms and an inductance of 0.05 H are connected in (i) star (ii) mesh to a 3-phase, 50-Hz supply with 400-V between lines. Calculate the total power absorbed and the line current in each case. Draw the vector diagram of current and voltages in each case.* **(Elect. Technology, Punjab Univ. 1990)**

Solution. $X_L = 2\pi \times 50 \times 0.05 \cong 15\Omega$, $Z_{ph} = \sqrt{15^2 + 20^2} = 25\Omega$

(*i*) **Star Connection.** [Fig. 19.29 (*a*)]

$V_{ph} = 400/\sqrt{3} = 231$ V; $I_{ph} = V_{ph}/Z_{ph} = 231/25 = 9.24\Omega$

$I_L = I_{ph} = 9.24$ A; $P = \sqrt{3} \times 400 \times 9.24 \times (20/25) =$ 5120 W

(*ii*) **Delta Connection** [Fig. 19.29 (*b*)]

$V_{ph} = V_L = 400$ V; $I_{ph} = 400/25 = 16$ A; $I_L = \sqrt{3}\, I_{ph} = \sqrt{3} \times 16 =$ **27.7 A**

$P = \sqrt{3} \times 400 \times 27.7 \times (20/25) =$ **15,360 W**

Note. It may be noted that line current as well as power are three times the star values.

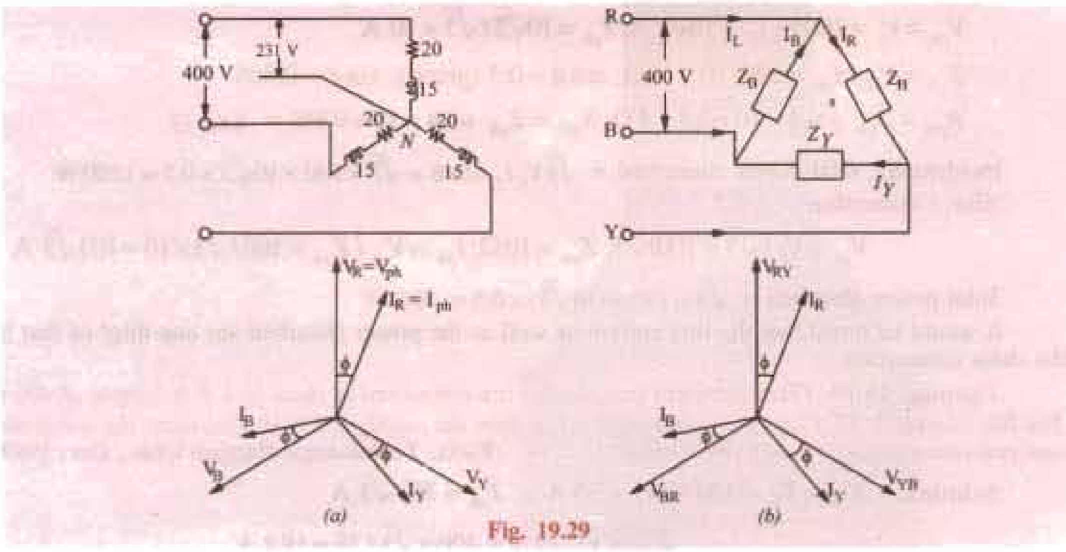


Fig. 19.29

Example 19.16. *A Δ -connected balanced 3-phase load is supplied from a 3-phase, 400-V supply. The line current is 20 A and the power taken by the load is 10,000 W. Find (i) impedance in each branch (ii) the line current, power factor and power consumed if the same load is connected in star.* **(Electrical Machines, A.M.I.E. Sec. B. 1992)**

Solution. (*i*) **Delta Connection.**

$$V_{ph} = V_L = 400\ V;\ I_L = 20\ A;\ I_{ph} = 20/\sqrt{3}\ \text{A}$$

(*i*) ∴ $$Z_{ph} = \frac{400}{20/\sqrt{3}} = 20\sqrt{3} = 34.64\ \Omega$$

Now $P = \sqrt{3}\ V_L I_L \cos\phi$ ∴ $\cos\phi = 10{,}000/\sqrt{3}\times 400\times 20 =$ **0.7217**

(*ii*) **Star Connection**

$$V_{ph} = \frac{400}{\sqrt{3}},\ I_{ph} = \frac{400/\sqrt{3}}{\sqrt{3}} = \frac{20}{3}\ \text{A},\ I_L = I_{ph} - = \frac{20}{3}\ \text{A}$$

Power factor remains the same since impedance is the same.

Power consumed $= \sqrt{3}\times 400\times(20/3)+0.7217 =$ **3,330 W**

Note. The power consumed is 1/3 of its value of Δ-connection.

Example 19.17. *Three similar resistors are connected in star across 400-V, 3-phase lines. The line current is 5 A. Calculate the value of each resistor. To what value should the line voltage be changed to obtain the same line current with the resistors delta-connected.*

Solution. Star Connection

$I_L = I_{ph} = 5\ A;\ V_{ph} = 400/\sqrt{3} = 231\text{V}\ \therefore R_{ph} = 231/5 =$ **46.2 Ω**

Delta Connection

$I_L = 5$ A ... (given) $I_{ph} = 5/\sqrt{3}$ A; $R_{ph} = 46.2\ \Omega$ ... found above

$V_{ph} = I_{ph}\times R_{ph} = 5\times 46.2/\sqrt{3} =$ **133.3 V**

Note. Voltage needed is 1/3rd the star value.

Example 19.18. *A balanced delta connected load, consisting of there coils, draws* $10\sqrt{3}$ *A at 0.5 power factor from 100 V, 3-phase supply. If the coils are re-connected in star across the same supply, find the line current and total power consumed.*

(Elect. Technology, Punjab Univ. Nov. 1988)

Solution. Delta Connection

$$V_{ph} = V_L = 100V;\ I_L = 10\sqrt{3}\ A;\ I_{ph} = 10\sqrt{3}/\sqrt{3} = 10\ \text{A}$$

$$Z_{ph} = V_{ph}/I_{ph} = 100/10 = 10\Omega;\ \cos\phi = 0.5\ \text{(given)};\ \sin\phi = 0.866$$

$$\therefore\ R_{ph} = Z_{ph}\cos\phi = 10\times 0.5 = 5\Omega;\ X_{ph} = Z_{ph}\sin\phi = 10\times 0.866 = \mathbf{8.66\ \Omega}$$

Incidentally, total power consumed $= \sqrt{3}\,V_L I_L\cos\phi = \sqrt{3}\times 100\times 10\sqrt{3}\times 0.5 = 1500\,\text{W}$

Star Connection

$$V_{ph} = V_L \mid \sqrt{3} = 100 \mid \sqrt{3};\ Z_{ph} = 10\Omega;\ I_{ph} = V_{ph}/Z_{ph} = 100/\sqrt{3}\times 10 = 10 1\sqrt{3}\ \text{A}$$

Total power absorbed $= \sqrt{3}\times 100\times(10\sqrt{3})\times 0.5 =$ **500 W**

It would be noted that the line current as well as the power absorbed are one-third of that in the delta connection.

Example 19.19. *Three identical impedances are connected in delta to a 3 ϕ supply of 400 V. The line current is 35 A and the total power taken from the supply is 15 kW. Calculate the resistance and reactance values of each impedance.* **(Elect. Technology, Punjab Univ., Dec. 1989)**

Solution. $V_{ph} = V_L = 400$ V; $I_L = 35$ A ∴ $I_{ph} = 35/\sqrt{3}$ A

$$Z_{ph} = V_{ph}/I_{ph} = 400\times\sqrt{3}/35 = 19.8\ \text{A}$$

Now, Power $P = \sqrt{3}V_L I_L \cos\phi \;\therefore\; \cos\phi = \dfrac{P}{\sqrt{3}V_L I_L} = \dfrac{15{,}000}{\sqrt{3}\times 400\times 35} = 0.619$; But $\sin\phi = 0.786$

$\therefore\; R_{ph} = Z_{ph}\cos\phi = 19.8\times 0.619 =$ **12.25Ω**; $X_{ph} = Z_{ph}\sin\phi$ and $X_{ph} = 19.8\times 0.786 =$ **15.5Ω**

Example 19.20. *Three 100Ω non-inductive resistances are connected in (a) star (b) delta across a 400-V, 50-Hz, 3-phase mains. Calculate the power taken from the supply system in each case. In the event of one of the three resistances getting open-circuited, what would be the value of total power taken from the mains in each of the two cases ?*

(Elect. Engg. A.M.Ae. S.I June, 1993)

Solution. (*i*) **Star Connection [Fig. 19.30 (*a*)]**

$V_{ph} = 400/\sqrt{3}\ V$

$P = \sqrt{3}\ V_L I_L\ \cos\phi$

$= \sqrt{3}\times 400\times 4\times 1/\sqrt{3} =$ **1600 W**

(*ii*) **Delta Connection Fig. 19.30 (*b*)**

$V_{ph} = 400\text{V};\ R_{ph} = 100\Omega$

$I_{ph} = 400/100 = 4\text{A}$

$I_L = 4\times\sqrt{3}\ \text{A}$

$P = \sqrt{3}\times 400\times 4\times\sqrt{3}\times 1 =$ **4800 W**

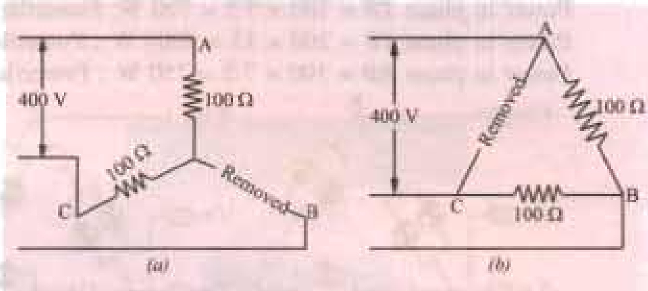


Fig. 19.30

When one of the resistors is disconnected

(*i*) **Star Connection** [Fig. 19.28 (*a*)]

The circuit no longer remains a 3-phase circuit but consists of two 100 Ω resistors in series across a 400-V supply. Current in lines *A* and *C* is = 400/200 = 2 A

Power absorbed in both = 400 × 2 = 800 W

Hence, by disconnecting one resistor, the power consumption is reduced by half.

(*ii*) **Delta Connection** [Fig. 19.28 (*b*)]

In this case, currents in *A* and *C* remain as usual 120° out of phase with each other.

Current in each phase = 400/100 = 4 A

Power consumption in both = $2\times 4^2\times 100 =$ **3200 W**

(or $P = 2\times 4\times 400 = 3200$ W)

In this case, when one resistor is disconnected, the power consumption is reduced by one-third.

Example 19.21. *A 200-V, 3-φ voltage is applied to a balanced Δ-connected load consisting of the groups of fifty 60-W, 200-V lamps. Calculate phase and line currents, phasges voltages, power consumption of all lamps and of a single lamp included in each phase for the following cases :*

(a) under normal conditions of operation

(b) after blowout in line R′R (c) after blowout in phase YB

Neglect impedances of the line and internal resistances of the sources of e.m.f.

Solution. The load circuit is shown in Fig. 19.31 where each lamp group is represented by two lamps only. It should be kept in mind that lamps remain at the line voltage of the supply irrespective of whether the Δ-connected load is balanced or not.

(*a*) **Normal operating conditions** [Fig. 19.31 (*a*)]

Since supply voltage equals the rated voltage of the bulbs, the power consumption of the lamps equals their rated wattage.

Power consumption/lamp = 60 W; Power consumption/phase = 50 × 60 = **3,000 W**

Phase current = 3000/200 = **15 A** ; Line current = $15\times\sqrt{3}$ = **26 A** (*b*) **Line** Blowout [Fig. 19.31 (*b*)]

When blowout occurs in line *R*, the lamp group of phase *Y-B* remains connected across line voltage $V_{YB} = V_{y'B'}$. However, the lamp groups of other two phases get connected in series across the same voltage V_{YB}. Assuming that lamp resistaces remain constant, voltage drop across YR = V_{YB} 200/2 = **100 V** and that across RB = **100 V**.

Hence, phase currents are as under :

$$I_{YB} = 3000/200 = \mathbf{15\ A},\ I_{YB} = I_{RB} = 15/2 = \mathbf{7.5\ A}$$

The line currents are :

$$I'_{RR} = 0,\ I'_{YY} = I'_{BB} = I_{YB} + I_{YR} - 15 + 7.5 = \mathbf{22.5\ A}$$

Power in phase YR = 100 × 7.5 = 750 W; Power/lamp = 750/50 = **15 W**

Power in phase YB = 200 × 15 = 3000 W ; Power/lamp = 3000/50 = **60 W**

Power in phase RB = 100 × 7.5 = 750 W ; Power/lamp = 750/50 = **15 W**

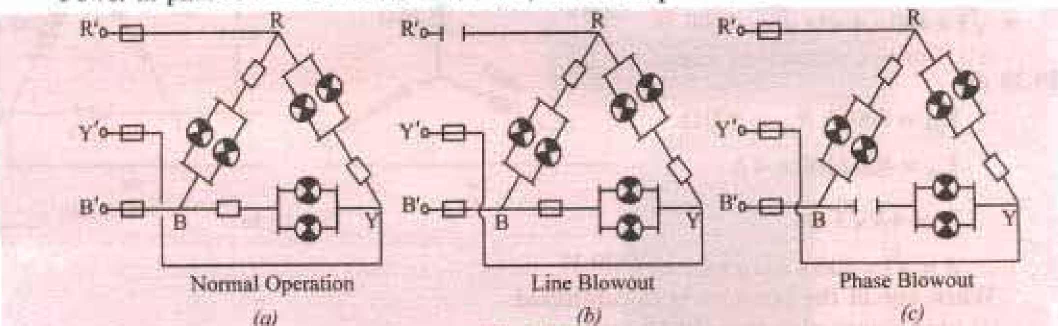


Fig. 19.31

(*c*) **Phase Blowout** [Fig. 19.31 (*c*)]

When fuse in phase *Y-B* blows out, the phase voltage becomes zero (though voltage across the open remains 200 V). However, the voltage across the other two phases remains the same as under normal operating conditions.

Hence, different phase currents are :

$$I_{RY} = \mathbf{15\ A},\ I_{BR} = \mathbf{15\ A},\ I_{YB} = \mathbf{0}$$

The line currents become

$$I'_{RR} = 15\sqrt{3} = 26\ A;\ I_{YY} = 15\ A,\ I'_{BB} = \mathbf{15\ A}$$

Power in phase RY = 200 × 15 = **3000 W**, Power/lamp = 3000/50 = **60 W**

Power in phase RB = 200 × 15 = **3000 W**, Power/lamp = 3000/50 = **60 W**

Power in phase YB = 0; power/lamp = 0.

Example 19.22. *The load connected to a 3-phase supply comprises three similar coils connected in star. The line currents are 25 A and the kVA and kW inputs are 20 and 11 respectively. Find the line and phase voltages, the kVAR input and the resistance and reactance of each coil.*

If the coils are now connected in delta to the same three-phase supply, calculate the line currents and the power taken.

Solution. Star Connection

$\cos\phi$ k W/kVA = 11/20 $\quad I_L = 25$ A $\quad P = 11$ kw = 11,000 W

Now $\quad P = \sqrt{3}\ V_L I_L \cos\phi \quad \therefore \quad 11{,}000 = \sqrt{3} \times V_L \times 25\ 0 + 11/20$

$\therefore\ V_L = \mathbf{462\ V};\quad V_{ph} = 462/\sqrt{3} = \mathbf{267\ V}$

$$\text{kVAR} = \sqrt{\text{kVA}^2 - \text{kW}^2} = \sqrt{20^2 - 11^2} = \mathbf{16.7}\ ;\ Z_{ph} = 267/2 = 10.68$$

$$\therefore\ R_{ph} = Z_{ph} \times \cos\phi = 10.68 \times 11/20 = \mathbf{5.87\ \Omega}$$

$$\therefore\ X_{ph} = Z_{ph} \times \sin\phi = 10.68 \times 0.838 = \mathbf{897\ \Omega}$$

Delta Connection

$$V_{ph} = V_L = 462\ V \text{ and } Z_{ph} = 10.68\ \Omega$$

$$\therefore \quad I_{ph} = 462/10.68\text{A} \quad I_L = \sqrt{3}\times 462/10.68 = 75\text{ A}$$

$$P = \sqrt{3}\times 462\times 75\times 11/20 = \mathbf{33{,}000\ W}$$

Example 19.23. *A 3-phase, star-connected system with 230 V between each phase and neutral has resistances of 4, 5 and 6 Ω respectively in the three phases. Estimate the current flowing in each phase and the neutral current. Find the total power absorbed.* **(I.E.E. London)**

Solution. Here, V_{ph} = 230 V [Fig. 19.32 (*a*)]

Current in 4-Ω resistor = 230/4 = 57.5 A

Current in 5-Ω resistor = 230/5 = 46 A

Current in 6-Ω resistor = 230/6 = 38.3 A

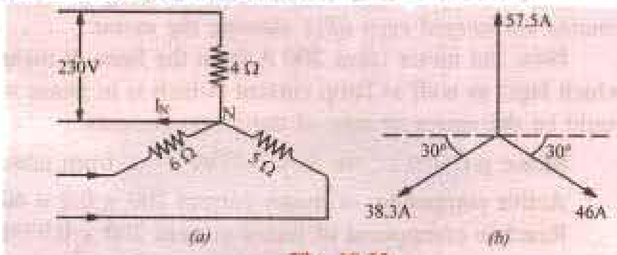


Fig. 19.32

These currents are mutually displaced by 120°. The neutral current I_N is the vector sum* of these three currents. I_N can be obtained by splitting up these three phase currents into their *X*-components and *Y*-components and then by combining them together, in diagram 19.32 (*b*).

X-component = 46 cos 30° − 38.3 cos 30° = 6.64 A

Y-component = 57.5 − 46 sin 30° − 38.3 sin 30° = 15.3 A $\therefore\ I_N = \sqrt{6.64^2 + 15.3^2} = \mathbf{16.71\ A}$

The power absorbed = 230 (57.5 + 46 + 38.3) = **32,610 W**

Example 19.24. *A 3-phase, 4-wire system supplies power at 400 V and lighting at 230 V. If the lamps in use require 70, 84 and 33 A in each of the three lines, what should be the current in the neutral wire ? If a 3-phase motor is now started, taking 200 A from the line at a power factor of 0.2, what would be the current in each line and the neutral current ? Find also the total power supplied to the lamps and the motor.* **(Elect. Technology, Aligarh Univ. 1985)**

Solution. The lamp and motor connections are shown in Fig. 19.33.

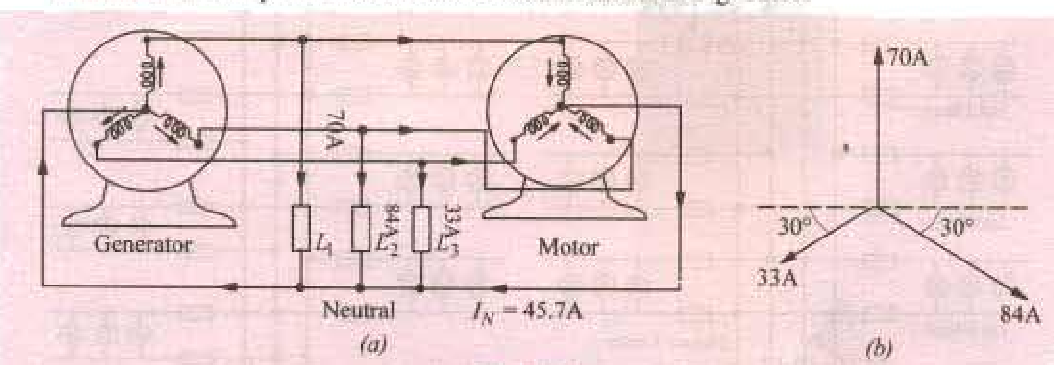


Fig. 19.33

* Some writers disagree with this statement on the ground that according to Kirchhoff's Current Law, at any junction, $\mathbf{I}_N + \mathbf{I}_R + \mathbf{I}_Y + \mathbf{I}_B = 0 \quad \therefore \quad \mathbf{I}_N = -(\mathbf{I}_R + \mathbf{I}_Y + \mathbf{I}_B)$

Hence, according to them, numerical value of $\mathbf{I}_N$ is the same but its phase is changed by 180°.

When motor is not started

The neutral current is the vector sum of lamp currents. Again, splitting up the currents into their X- and Y-components, we get

X-component = 84 cos 30° − 33 cos 30° = 44.2 A

Y-component = 70 − 84 sin 30° − 33 sin 30° = 11.5 A

$$\therefore \qquad I_N = \sqrt{44.2^2 + 11.5^2} = \mathbf{45.7\ A}$$

When motor is started

A 3-phase motor is a balanced load. Hence, when it is started, it will change the line currents but being a balanced load, it contributes nothing to the neutral current. Hence, *the neutral current remains unchanged even after starting the motor.*

Now, the motor takes 200 A from the lines. It means that each line will carry motor current (which lags) as well as lamp current (which is in phase with the voltage). The current in each line would be the vector of sum of these two currents.

Motor p.f. = 0.2 ; sin ϕ = 0.9799 ... from tables

Active component of motor current 200 × 0.2 = 40 A

Reactive component of motor current 200 × 0.9799 = 196 A

(*i*) Current in first line = $\sqrt{(40+70)^2 + 196^2}$ = **224.8 A**

(*ii*) Current in second line = $\sqrt{(40+84)^2 + 196^2}$ = **232 A**

(*iii*) Current in third line = $\sqrt{(40+33)^2 + 196^2}$ = **210.6 A**

Power supplied to lamps = 230 (33 + 84 + 70) = **43,000 W**

Power supplied to motor = $\sqrt{3} \times 200 \times 400 \times 0.2$ = **27,700 W**

19.11. Star and Delta connected Lighting Loads

In Fig. 19.34 (*a*) is shown a Y-connected lighting network in a three storey house. For such a load, it is essential to have neutral wire in order to ensure uniform distribution of load among the three phases despite random switching on and off or burning of lamps. It is seen from Fig. 1934 (*a*),

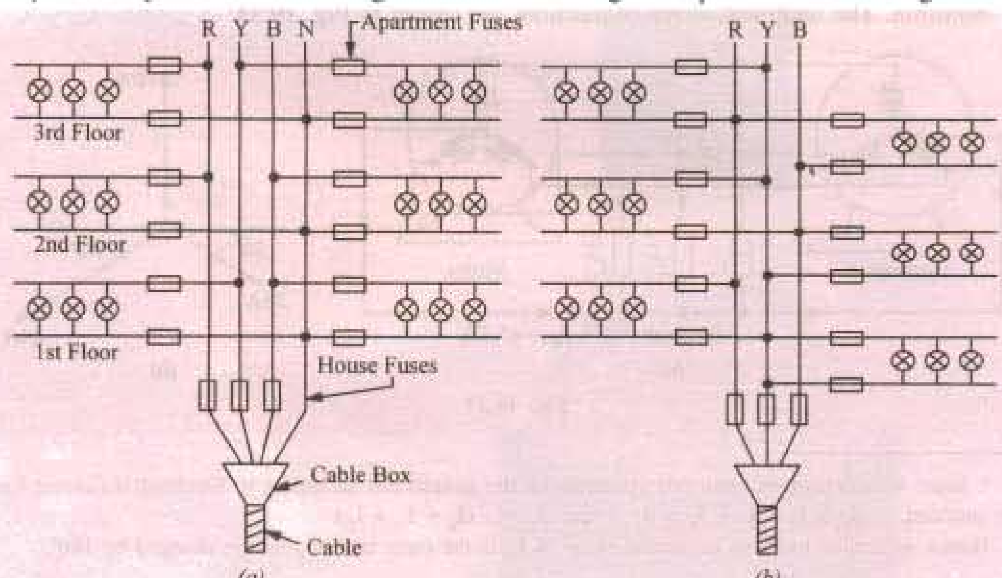


Fig. 19.34

that network supplies two flats on each floor of the three storey residence and there is balanced distribution of lamp load among the three phases. There are house fuses at the cable entry into the building which protect the two mains against short-circuits in the main cable. At the flat entry, there are apartment (or flat) fuses in the single-phase supply which protect the two mains and other flats in the same building from short-circuits in a given building. There is no fuse (or switch) on the neutral wire of the mains because blowing of such a fuse (or disconnection of such a switch) would mean a break in the neutral wire. This would result in unequal voltages across different groups of lamps in case they have different power ratings or number. Consequently, filaments in one group would burn dim whereas in other groups they would burn too bright resulting in their early burn-out.

The house-lighting wire circuit for Δ-connected lamps is shown in Fig. 19.34 (*b*).

19.12. Power Factor Improvement

The heating and lighting loads supplied from 3-phase supply have power factors, ranging from 0.95 to unity. But motor loads have usually low lagging power factors, ranging from 0.5 to 0.9. Single-phase motors may have as low a power factor as 0.4 and electric wedding units have even lower power factors of 0.2 or 0.3.

The power factor is given by $\cos\phi \dfrac{\text{kW}}{\text{kVA}}$ or $\text{kVA} = \dfrac{\text{kW}}{\cos\phi}$

In the case of single-phase supply, $\text{kVA} = \dfrac{VI}{1000}$ or $I = \dfrac{1000\ k\text{VA}}{\text{V}}$ $\quad\therefore\quad I \propto \text{kVA}$

In the case of 3-phase supply $\text{kVA} = \dfrac{\sqrt{3}\,V_L I_L}{1000}$ or $I_L = \dfrac{1000\ \text{kVA}}{\sqrt{3} \times V_L}$ $\quad\therefore\quad I \propto \text{kVA}$

In each case, the kVA is directly proportional to current. The chief disadvantage of a low p.f. is that the current required for a given power, is very high. This fact leads to the following undesirable results.

(*i*) Large kVA for given amount of power

All electric machinery, like alternators, transformers, switchgears and cables are limited in their current-carrying capacity by the permissible temperature rise, which is proportional to I^2. Hence, they may all be fully loaded with respect to their rated kVA, without delivering their full power. Obviously, it is possible for an existing plant of a given kVA rating to increase its earning capacity (which is proportional to the power supplied in kW) if the overall power factor is improved *i.e.* raised.

(*ii*) Poor voltage regulation

When a load, having allow lagging power factor, is switched on, there is a large voltage drop in the supply voltage because of the increased voltage drop in the supply lines and transformers. This drop in voltage adversely affects the starting torques of motors and necessitates expensive voltage stabilizing equipment for keeping the consumer's voltage fluctuations within the statutory limits. Moreover, due to this excessive drop, heaters take longer time to provide the desired heat energy, fluorescent lights flicker and incandescent lamps are not as bright as they should be. Hence, all supply undertakings try to encourage consumers to have a high power factor.

Example 19.25. *A 50-MVA, 11-kV, 3-ϕ alternator supplies full load at a lagging power factor of 0.7. What would be the percentage increase in earning capacity if the power factor is increased to 0.95 ?*

Solution. The earning capacity is proportional to the power (in MW or kW) supplied by the alternator.

MW supplied at 0.7 lagging = 50 × 0.7 = 35

MW supplied at 0.95 lagging = 50 × 0.95 = 47.5

increase in MW = 12.5

The increase in earning capacity is proportional to 12.5

∴ Percentage increase in earning capacity = (12.5/35) × 100 = **35.7**

19.13. Power Correction Equipment

The following equipment is generally used for improving or correcting the power factor :

(i) Synchronous Motors (or capacitors)

These machines draw leading kVAR when they are over-excited and, especially, when they are running idle. They are employed for correcting the power factor in bulk and have the special advantage that the amount of correction can be varied by changing their excitation.

(ii) Static Capacitors

They are installed to improve the power factor of a group of a.c. motors and are practically loss-free (*i.e.* they draw a current leading in phase by 90°). Since their capacitances are not variable, they tend to over-compensate on light loads, unless arrangements for automatic switching off the capacitor bank are made.

(iii) Phase Advancers

They are fitted with individual machines.

However, it may be noted that the economical degree of correction to be applied in each case, depends upon the tariff arrangement between the consumers and the supply authorities.

Example 19.26. *A 3-phase, 37.3 kW, 440-V, 50-Hz induction motor operates on full load with an efficiency of 89% and at a power factor of 0.85 lagging. Calculate the total kVA rating of capacitors required to raise the full-load power factor at 0.95 lagging. What will be the capacitance per phase if the capacitors are (a) delta-connected and (b) star-connected ?*

Solution. It is helpful to approach such problems from the 'power triangle' rather than from vector diagram viewpoint.

Motor power input $P = 37.3/0.89 = 41.191$ kW

Power Factor 0.85 (lag)

$$\cos\phi_1 = 0.85 : \phi_1 = \cos^{-1}(0.85) = 31.8°; \tan\phi_1 = \tan 31.8° = 0.62$$

Motor $\text{kVAR}_1 = P \tan\phi_1 = 41.91 \times 0.62 = 25.98$

Power Factor 0.95 (lag)

Motor power input $P = 41.91$ kW ... as before

It is the same as before because capacitors are loss-free *i.e.* they do not absorb any power.

$$\cos\phi_2 = 0.95 \quad \therefore \phi_2 = 18.2°; \ \tan 18.2° = 0.3288$$

Motor $\text{kVAR}_2 = P \tan\phi_2 = 41.91 \times 0.3288 = 13.79$

The difference in the values of kVAR is due to the capacitors which supply *leading* kVAR to partially neutralize the *lagging* kVAR of the motor.

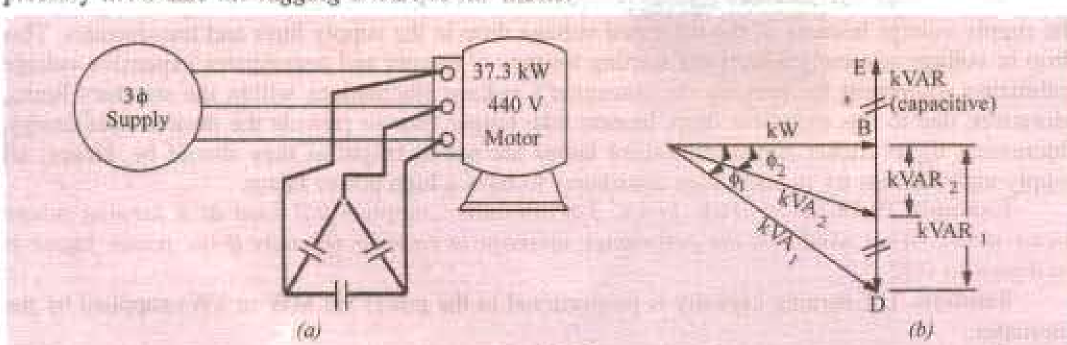


Fig. 19.35

$\therefore$ leading kVAR supplied by capacitors is

$= \text{kVAR}_1 - \text{kVAR}_2 = 25.98 - 13.79 =$ **12.19** ... *CD* in Fing. 19.35 (*b*)

Since capacitors are loss-free, their kVAR is the same as kVA

$\therefore$ kVA/capacitor = 12.19/3 = 4.063 $\therefore$ VAR/capacitor = 4,063

(*a*) In Δ-connection, voltage across each capacitor is 440 V

Current drawn by each capacitor $I_C = 4063/440 = 9.23$ A

Now, $$I_c = \frac{V}{X_c} = \frac{V}{1/\omega C} = \omega VC$$

$\therefore$ $$C = I_c/\omega V = 9.23/2\pi \times 50 \times 440 = 66.8 \times 10^{-6}\,\text{F} = \mathbf{66.8\ \mu F}$$

(*b*) In star connection, voltage across each capacitor is = $440/\sqrt{3}$ volt

Current drawn by each capacitor, $I_c - \dfrac{4063}{440/\sqrt{3}} = 16.0$ A

$$I_C = \frac{V}{X_c} = \omega VC \qquad \text{or} \qquad 16 = \frac{440}{\sqrt{3}} \times 2\pi \times 50 \times C$$

$\therefore$ $$C = 200.4 \times 10^{-6}\text{F} = \mathbf{200.4\ \mu F}$$

Note. Star value is three times the delta value.

Example 19.27. *If the motor of Example 19.24 is supplied through a cable of resistance 0.04 Ω per core, calculate*

(i) the percentage reduction in cable Cu loss and

(ii) the additional balanced lighting load which the cable can supply when the capacitors are connected.

Solution. Original motor kVA$_1$ = P/cos ϕ_1 = 41.91/0.85 = 49.3

Original line current, $I_{L1} = \dfrac{kVA_1 \times 1000}{\sqrt{3} \times 440} = \dfrac{49.3 \times 1000}{\sqrt{3} \times 440} = 64.49$ A

$\therefore$ Original Cu loss/conductor = $64.69^2 \times 0.04$ = 167.4 W

From Fig . 19.34, it is seen that the new kVA *i.e.*kVA$_2$ when capacitors are connected is given by kVA$_2$ = kW/cos ϕ_2 = 41.91/0.95 = 44.12

New line current $I_{L2} = \dfrac{44{,}120}{\sqrt{3} \times 440} = 57.89$ A

New Cu loss $= 57.89^2 \times 0.04 = 134.1$ W

(*i*) $\therefore$ percentage reduction $= \dfrac{167.4 - 134.1}{167.4} \times 100 =$ **19.9**

The total kVA which the cable can supply is 49.3 kVA. When the capacitors are connected, the kVA supplied is 44.12 at a power factor of 0.94 lagging. The lighting load will be assumed at unity power factor. The kVA diagram is shown in Fig. 19.34. We will tabulate the different loads as follows. Let the additional lighting load be x kW.

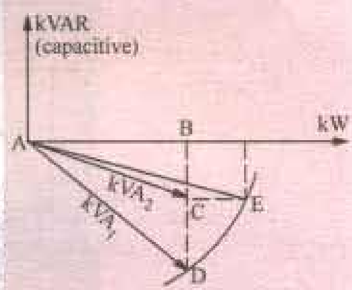


Fig. 19.36

Load	*kVA*	cos ϕ	*kW*	sin ϕ	*kVAR*
Motor	49.3	0.85 lag	41.91	0.527	–25.98
Capacitors	12.19	0 lead	0	1.0	+12.19
Lighting	–	1.0	x	0	0
			1.91 + x)		–13.79

From Fig. 19.36 it is seen that

$AF = 41.91 + x$ and $EF = 13.79$ AE = resultant kVA = 49.3

Also $AF^2 + EF^2 = AE^2$ or $(41.91 + x)^2 + 13.79^2 = 49.3^2$ $\therefore$ x = **5.42 kW**

Example 19.28. *Three impedance coils, each having a resistance of 20 Ω and a reactance of 15 Ω, are connected in star to a 400-V, 3-ϕ, 50-Hz supply. Calculate (i) the line current (ii) power supplied and (iii) the power factor.*

If three capacitors, each of the same capacitance, are connected in delta to the same supply so as to form parallel circuit with the above impedance coils, calculate the capacitance of each capacitor to obtain a resultant power factor of 0.95 lagging.

Solution. $V_{ph} = 100/\sqrt{3}V, Z_{ph} = \sqrt{20^2+15^2} = 25\Omega$

$\cos\phi_1 = R_{ph}/Z_{ph} = 20/25 = 0.8$ lag; $\phi_1 = 0.6$ lag

where ϕ_1 is the power factor angle of the coils.

When capacitors are not connected

(*i*) $I_{ph} = 400/25\times\sqrt{3} = 9.24$ A $\quad\therefore I_L =$ **9.24 A**

(*ii*) $P = \sqrt{3}\,V_L I_L \cos\phi_1 = \sqrt{3}\times 400\times 9.24\times 0.8 =$ **5.120 W**

(*iii*) Power factor = **0.8** (lag)

$\therefore$ Motor $VAR_1 = \sqrt{3}V_L I_L \sin\phi_1 = \sqrt{3}\times 400\times 9.24\times 0.6 = 3{,}840$

When capacitors are connected

Power factor, $\cos\phi_2 = 0.95, \phi_2 = 18.2°$; $\tan 18.2° = 0.3288$

Since capacitors themselves do not absorb any power, power remains the same *i.e.* 5,120 W even whin capacitors are connected. The only thing that changes is the VAR.

Now $VAR_2 = P\tan\phi_2 = 5120\times 0.3288 = 1684$

Leading VAR supplied by the three capacitors is

$= VAR_1 - VAR_2 = 3840 - 1684 = 2156$ BD or CE in Fig 19.37 (*b*)

VAR/ Capacitor = 2156/3 = 719

For delta connection, voltage across each capacitor is 400 V $\therefore I_c = 719/400 = 1.798$ A

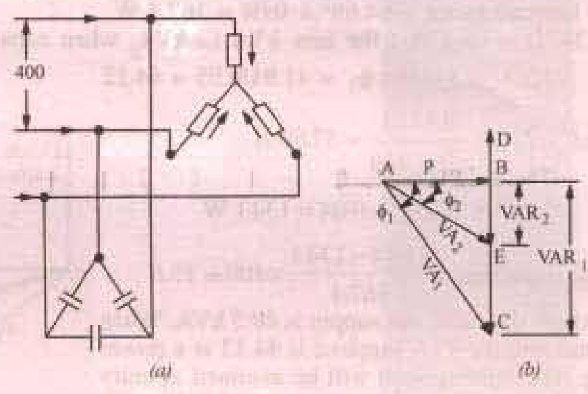


Fig. 19.37

Also $I_c = \dfrac{V}{1/\omega C} = \omega VC \therefore C = 1.798/\pi\times 50\times 400 = 14.32\times 10^{-6}$ F = **14.32 μF**

19.14. Parallel Loads

A combination of balanced 3-phase loads connected in parallel may be solved by any one of the following three methods :

1. All the given loads may be converted into equivalent Δ-loads and then combined together according to the law governing parallel circuits.

2. All the given loads may be converted into equivalent *Y*-loads and treated as in (1) above.

3. The third method, which requires less work, is to work in terms of volt-amperes. The special advantage of this approach is that voltameters can be added regardless of the kind of connection involved. The real power of various loads can be added arithmetically and VARs my be added algebraically so that total voltamperes are given by

$$VA = \sqrt{W^2 + VAR^2} \quad \text{or} \quad S = \sqrt{P^2 + Q^2}$$

where P is the power in water and Q represents reactive voltamperes.

Example 19.29. *For the power distribution system shown in Fig. 19.38, find*

(a) total apparent power, power factor and magnitude of the total current I_T without the capacitor in the system

(b) the capacitivie kVARs that must be supplied by C to raise the power factor of the system to unity ;

(c) the capacitance C necessary to achieve the power correction in part (b) above

(d) total apparent power and supply current I_T after the power factor correction.

Solution. (*a*) We will take the inductive *i.e.* lagging kVARs as negative and capacitive *i.e.* leading kVARs as positive.

Total $Q = -16 + 6 - 12 = -22$ kVAR (lag); Total $P = 30 + 4 + 36 = 70$ kW

$\therefore$ apparent power $S = \sqrt{(-22)^2 + 70^2}$ = **73.4 kVA;** p.f. = cos ϕ = P/S = 70/73.4 = **0.95**

$$S = VI_T \text{ or } 73.4 \times 10^3 = 400 \times I_T \quad \therefore \; I_T = \mathbf{183.5\ A}$$

(*b*) Since total lagging kVARs are – **22**, hence, for making the power factor unity, 22 leading kVARs must be supplied by the capacitor to neutralize them. In that case, total $Q = 0$ and $S = P$ and p.f. is unity.

(*c*) If I_C is the current drawn by the capacitor, then $22 \times 10^3 = 400 \times I_C$

Now, $I_C = V/X_C = V\omega C$

$= 400 \times 2\pi \times 50 \times C$

$\therefore \; 20 \times 10^3 = 400 \times (400 \times 2\pi \times 50 \times C)$;

$\therefore \; C =$ **483 μF**

(*d*) Since $Q = 0$,

hence, $S = \sqrt{10^2 + 70^2}$ = **70 kVA**

Now, $VI_T = 70 \times 10^3$;

$I_T = 70 \times 10^3 / 400 = 175$ A.

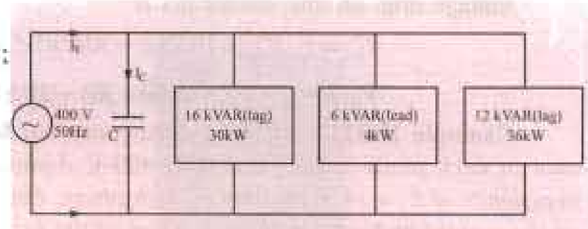


Fig. 19.38

It would be seen that after the power correction, lesser amount of current is required to deliver the same amount of real power to the system.

Example 19.30. *A symmetrical 3-phase, 3-wire supply with a line voltage of 173 V supplies two balanced 3-phase loads; one Y-connected with each branch impedance equal to (6 + j8) ohm and the other Δ-connected with each branch impedance equal to (18 + j24) ohm. Calculate*

(i) the magnitudes of branch currents taken by each 3-phase load

(ii) the magnitude of the total line current and

(iii) the power factor of the entire load circuit

Draw the phasor diagram of the voltages and currents for the two loads.

(Elect. Engineering-I, Bombay Univ. 1987)

Solution. The equivalent *Y*-load of the given Δ-load (Art.19.10) is = $(18 + j24)/3 = (6 + j8)\ \Omega$.

With this, the problem now reduces to one of solving two equal *Y*-loads connected in parallel across the 3-phase supply as shown in Fig. 19.39 (*a*). Phasor diagram for the combined load for one phase only is given in Fig. 19.39 (*b*).

Combined load impedance

$$= (6 + j8)/2 = 3 + j4$$

$$= 5\angle 53.1° \text{ ohm}$$

$$V_{ph} = 173/\sqrt{3} = 100 \text{ V}$$

Let $\quad V_{ph} = 100\angle 0°$

$$\therefore \quad I_{ph} = \frac{100\angle 0°}{5\angle 53.1°} = 20\angle -53.1°$$

Current in each load = $10\angle -53.1$ A

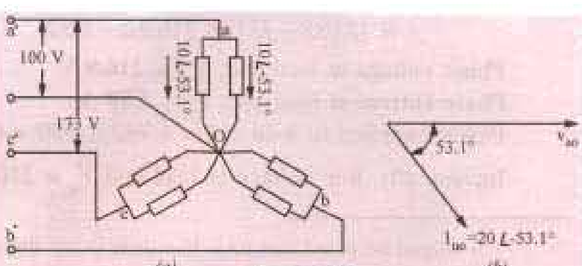


Fig. 19.39

(*i*) branch current taken by each load is **10 A**; (*ii*) line current is **20 A**;

(*iii*) combined power factor = cos 53.1° = **0.6 (lag).**

Example 19.31. *Three identical impedances of $30\angle 30°$ ohms are connected in delta to a 3-phase, 3-wire, 208 V volt abc system by conductors which have impedances of (0.8 + j 0.63) ohm. Find the magnitude of the line voltage at the load end.*

(Elect. Engg. Punjab Univ. may 1990)

Solution. The equivalent Z_Y, of the given Z_Δ is $30\angle 30/3 = 10\angle 30° = (8.86 + j5)$. Hence, the load connections become as shown in Fig. 19.40.

$$Z_{an} = (0.8 + j0.6) + (8.86 + j5)$$

$$= 9.66 + j5.6 = 11.16\angle 30.1°$$

$$V_{an} = V_{ph} = 208/\sqrt{3} = 120 \text{ V}$$

Let $V_{an} = 120\angle 0°$

$\therefore$ $I_{an} = 120\angle 0°/11.16\angle 30.1° = 10.75\angle -30.1°$

Now, $Z_{aa'} = 0.8 + j0.6 = 1\angle 36.9°$

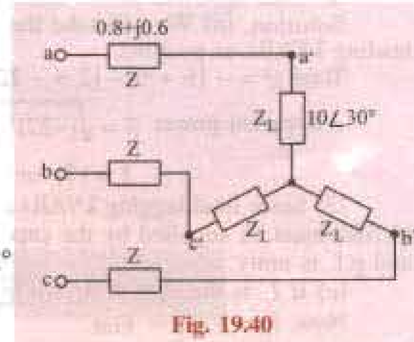


Fig. 19.40

Voltage drop on line conductors is

$$V'_{aa} = I_{an}Z'_{aa} = 10.75\angle -30.1° \times 1\angle 36.9° = 10.75\angle 6.8° = 10.67 + j1.27$$

$$\therefore \quad V'_{an} = V_{an} - V'_{aa} = (120 + j0) - (10.67 + j1.27) = \mathbf{109.3\angle 2.03°}$$

Example 19.32. *A balanced delta-connected load having an impedance Z_L = (300 + j210) ohm in each phase is supplied from 400-V, 3-phase supply through a 3-phase line having an impedance of Z_s = (4 + j8) ohm in each phase. Find the total power supplied to the load as well as the current and voltage in each phase of the load.*

(Elect. Circuit Theory, Kerala Univ. 1988)

Solution. The equivalent Y-load of the given Δ-load is

$$= (300 + j210)/3 = (100 + j70)\Omega$$

Hence, connections become as shown in Fig. 19.41

$$\mathbf{Z}'_{a0} = (4 + j8) + (100 + j70) = 104 + j78 = 130\angle 36.9°$$

$$V'_{a0} = 400/\sqrt{3} = 231V, \text{ Let } V'_{a0} = 231\angle 0°$$

$$\mathbf{I}'_{a0} = 231\angle 0°/130\angle 36.9° = 1.78\angle -36.9°$$

Now, $Z'_{a0} = (4 + j8) = 8.94\angle 63.4°$

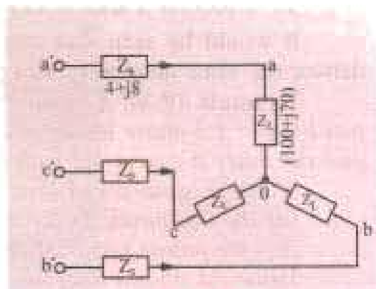


Fig. 19.41

Line drop $\mathbf{V}'_{aa} = \mathbf{I}'_{aa}\mathbf{Z}'_{aa} = 1.78\angle -36.9° \times 8.94\angle 63.4 = 15.9\angle 26.5° = 14.2 + j7.1$

$$\mathbf{V}_{a0} = \mathbf{V}'_{a0} - \mathbf{V}'_{aa} = (231 + j0) - (14.2 + j7.1)$$

$$= (216.8 - j7.1) = 216.9\angle -1°52'$$

Phase voltage at load end, V_{a0} = **216.9 V**

Phase current at load end, I_{a0} = **1.78 A**

Power supplied to load = 3 × 1.782 × 100 = **951 W**

Incidentally, line voltage at load end $V_{ac} = 216.9 \times \sqrt{3} = 375.7$ V*

---

* It should be noted that total line drop is not the numerical sum of the individual line drops because they are 120° out of phase with each other. By a laborious process $\mathbf{V}_{ac} = \mathbf{V}'_{ac} - \mathbf{V}'_{aa} - \mathbf{V}'_{ac}$.

Example 19.33. *A star connected load having R = 42.6 ohms/ph and X_L = 32 ohms/ph is connected across 400 V, 3 phase supply, calculate:*

(i) Line current, reactive power and power loss

(ii) Line current whenone of load becomes open circuited.

[Nagpur University, Summer 2001]

Solution.

(*i*) Z = 42.6 + j32

$$|Z| = 53.28 \text{ ohms, Impedance angle, } \theta = \cos^{-1}\left(\frac{42.6}{53.28}\right) = \cos^{-1} 0.80$$

$$\theta = 36.9°$$

Line Current = phase current, due to star-connection

$$= \frac{\text{Voltage / phase}}{\text{Impedance / phase}} = \frac{400/\sqrt{3}}{53.28} = 4.336 \text{ amp}$$

Due to the phase angle of 36.9° lagging.

Reactive Power for the three-phases

$$= 3\, V_{ph}\, I_{ph} \text{ Sin } \phi = 3 \times 231 \times 4.336 \times 0.6 = 1803 \text{ VAR}$$

Total Power-loss = $3\, V_{ph}\, I_{ph}$ Cos $\phi = 3 \times 231 \times 4.336 \times 0.8$ = 2404 watts

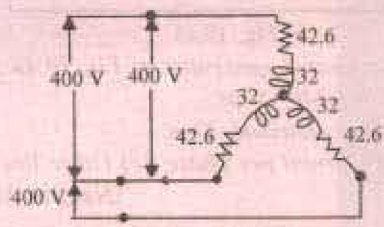


Fig. 19.42 (*a*)

(*ii*) One of the Loads is open-circuited.

The circuit is shown in Fig. 19.42 (*b*).

Between A and B, the Line voltage of 400 V drives a current through two "phase-impedances" in series.

Total Impedance between A and B = (42.6 + j 32) × 2 ohms

Hence, the line current I for the two Lines A and B

$$= \frac{400}{2 \times 53.25} = 3.754 \text{ amp}$$

Note : Third Line 'C' does not carry any current.

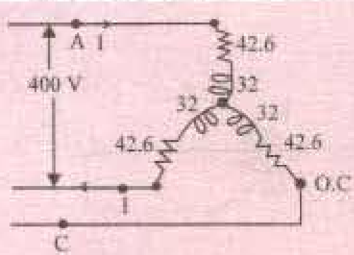


Fig. 19.42 (*b*) One phase open circuited

Example 19.34. *Three non-inductive resistances, each of 100 ohms, are connected in star to a three-phase, 440-V supply. Three inductive coils, each of reactance 100 ohms connected in delta are also connected to the supply. Calculate: (i) Line-currents, and (ii) power factor of the system*

[Nagpur University, November 1998]

Solution. (*a*) Three resistances are connected in star. Each resistance is of 100 ohms and 254 – V appears across it. Hence, a current of 2.54 A flows through the resistors and the concerned power-factor is unity. Due to star-connection,

Line-current = Phase-current = 2.54A

(*b*) Three inductive reactance are delta connected.

Line-Voltage = Phase – Voltage = 440 V

Phase Current = 440/100 = 4.4 A

Line current = 1.732 × 4.4 = 7.62 A

The current has a zero lagging power-factor.

Total Line Current = 2.54 - j 7.62 A

= 8.032 A, in each of the lines.

Power factor = 2.54/8.032 = 0.32 Lag.

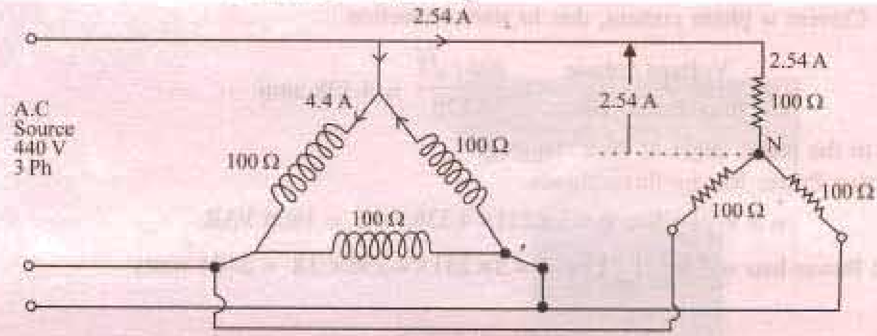


Fig. 19.43

Example 19.35. *The delta-connected generator of Fig 19.44 has the voltage: $V_{RY} = 220 \angle 0°$, $V_{YB} = 220 \angle -120°$ and $V_{BR} = 220 \angle -240°$ Volts.*

The load is balanced and delta-connected. Find:

(a) Impedance per phase, (b) Current per phase, (c) Other line – currents I_Y and I_B.

[Nagpur University, November 1997]

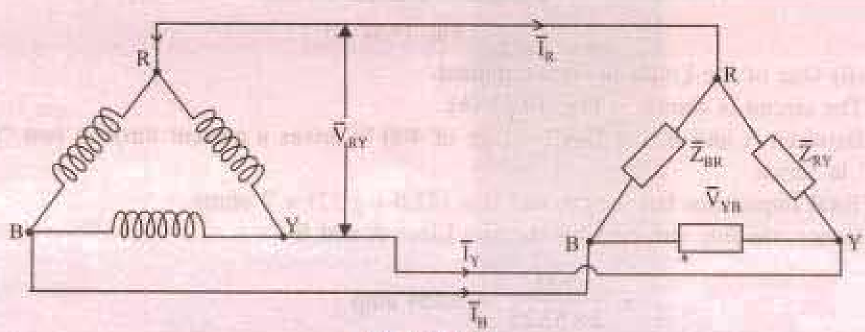


Fig 19.44 (*a*)

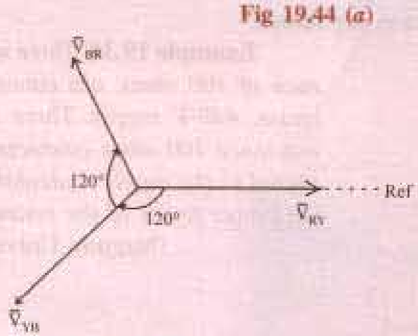


Fig. 19.44 (*b*)

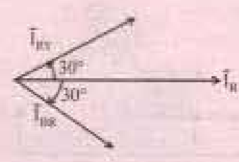


Fig. 19.44 (*c*)

Solution. Draw phasors for voltages as mentioned in the data. V_{RY} naturally becomes a reference-phasor, along which the phasor I_R also must lie, as shown in Fig. 19.44 (b) & (c). I_R is the line voltage which is related to the phase-currents I_{RY} and $-I_{BR}$. In terms of magnitudes,

$$|I_{RY}| = |I_{BR}| = |I_R|/\sqrt{3} = 10/\sqrt{3} = 5.8 \text{ Amp}$$

Thus, I_{RY} leads V_{RY} by 30°. This can take place only with a series combination of a Resistor and a capacitor, as the simplest impedance in each phase

(*a*) $|Z| = 220/5.8 = 38.1$ ohms

Resistance per phase = 38.1 × cos 30° = 33 ohms

Capacitive Reactance/phase = 38.1 × Sin 30° = 19.05 ohms

(*b*) Current per phase = 5.8 amp, as calculated above.

(*c*) Otherline currents: Since a symmetrical three phase system is being dealt with, three currents have a mutual phase-difference of 120°. Hence

$I_R = 10 \angle 0°$ as given, $I_Y = 10 \angle -120°$ amp $I_B = 10 \angle -240°$ amp.

Example 19.36. *A balanced 3-phase star-connected load of 8 + j 6 ohms per phase is connected to a three-phase 230 V supply. Find the Line-current, power-factor, active power, reactive-power, and total volt-amperes.*

[Rajiv Gandhi Technical University, Bhopal, April 2001]

Solution. When a statement is made about three–phase voltage, when not mentioned otherwise, the voltage is the Line-to-Line voltage. Thus, 230 V is the Line voltage, which means, in star-system, Phase-voltage is 230/1.732, which comes to 132.8 V.

$$|Z| = \sqrt{8^2 + 6^2} = 10 \text{ ohms}$$

Line current = Phase current

= 132.8/10 = 13.28 amp

Power – factor = R/Z = 0.8, Lagging

Total Active Power = P = 1.732 × Line Voltage × Line Current × P.f.

Or = 3. Phase Voltage × Phase-current × P.f

= 3 × 132.8 × 13.28 × 0.8 = 4232 watts

Total Reactive Power = Q

= 3 × Phase-voltage × Phase-current × Sin ϕ

= 3 × 132.8 × 13.28 × 0.60 = 3174 VAR

Total Volt-amps = S = $\sqrt{P^2 + Q^2}$ = 5290 VA

Or S = $\sqrt{3} \times 230 \times 13.28$ = 5290 VA

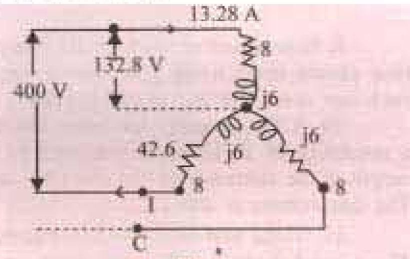


Fig. 19.45

Example 19.37. *A balanced three-phase star connected load of 100 kW takes a leading current of 80 amp, when connected across a three-phase 1100 V, 50 Hz, supply. Find the circuit constants of the load per phase.* **[Nagpur University, April 1996]**

Solution. Voltage per phase = 1100/1.732 = 635 V

Impedance = 635/80 = 7.94 ohms.

Due to the Leading current, a capacitor exists.

Resistance R can be evaluated from current and power consumed

$3 I^2 R = 100 \times 1000$, giving R = 5.21 ohms

$X_c = (7.94^2 - 5.21^2)^{0.5} = 6$ ohms

At 50 Hz, C = 1/(314 × 6) = 531 microfarads.

Tuutorial Problem No. 19.1

1. Each phase of a delta-connected load comprises a resistor of 50 Ω and capacitor of 50 μF in series. Calculate (*a*) the line and phase currents (*b*) the total power and (*c*) the kilovoltamperes when the load is connected to a 440-V, 3-phase, 50-Hz supply. **[(*a*) 9.46 A; 5.46 A (*b*) 4480 W (*c*) 7.24 kVA]**

2. Three similar-coils, *A*, *B* and *C* are available. Each coil has 9 Ω resistance and 12 Ω reactance. They are connected in delta to a 3-phase, 440-V, 50-Hz supply. Calculate for this load:

(*a*) the line current (*b*) the power factor
(*c*) the total kilovolt-amperes (*d*) the total kilowatts

If the coils are reconnected in star, calculate for the new load the quantities named at (*a*), (*b*); (*c*) and (*d*) above.

[50.7 A; 0.6; 38.6 kVA; 23.16 kW; 16.9 A; 0.6; 12.867 kVA; 7.72 kW]

3. Three similar choke coils are connected in star to a 3-phase supply. If the line currents are 15 A, the total power consumed is 11 kW and the volt-ampere input is 15 kVA, find the line and phase voltages, the VAR input and the reactance and resistance of each coil. **[577.3 V; 333.3 V; 10.2 kVAR; 15.1 Ω; 16.3 Ω]**

4. The load in each branch of a delta-connected balanced 3-ϕ circuit consists of an inductance of 0.0318 H in series with a resistance of 10 Ω. The line voltage is 400 V at 50 Hz. Calculate (*i*) the line current and (*ii*) the total power in the circuit. **[(*i*) 49 A (*ii*) 24 kW]** (London Univ.)

5. A 3-phase, delta-connected load, each phase of which has $R = 10\ \Omega$ and $X = 8\ \Omega$, is supplied from a star-connected secondary winding of a 3-phase transformer each phase of which gives 230 V. Calculate

(*a*) the current in each phase of the load and in the secondary windings of the transformer
(*b*) the total power taken by the load
(*c*) the power factor of the load. **[(a) 31.1 A; 54 A (b) 29 kW (c) 0.78]**

6. A 3-phase load consists of three similar inductive coils, each of resistance 50 Ω and inductance 0.3 H. The supply is 415 V, 50 Hz. Calculate (*a*) the line current (*b*) the power factor and (*c*) the total power when the load is (*i*) star-connected and (*ii*) delta-connected.

[(*i*) 2.25 A, 0.47 lag, 762 W (*ii*) 6.75 A, 0.47 lag, 2280 W] (*London Univ.*)

7. Three 20 Ω non-inductive resistors are connected in star across a three phase supply the line voltage of which is 480 V. Three other equal non-inductive resistors are connected in delta across the same supply so as to take the same-line current. What are the resistance values of these other resistors and what is the current- flowing through each of them? **[60 Ω; 8A]** (*Sheffield Univ. U.K.*)

8. A 415-V, 3-phase, 4-wire system supplies power to three non-inductive loads. The loads are 25 kW between red and neutral, 30 kW between yellow and neutral and 12 kW between blue and neutral.

Calculate (*a*) the current in each-line wire and (*b*) the current in the neutral conductor.

[(*a*) 104.2 A, 125 A, 50 A (*b*) 67 A] (*London Univ.*)

9. Non-inductive loads of 10, 6 and 4 kW are connected between the neutral and the red, yellow and blue phases respectively of a three-phase, four-wire system. The line voltage is 400 V. Find the current in each line conductor and in the neutral. **[(*a*) 43.3 A, 26A, 173. A, 22.9]** (*App. Elect. London Univ.*)

10. A three-phase, star-connected alternator supplies a delta-connected load, each phase of which has a resistance of 20 Ω and a reactance of 10 Ω. Calculate (*a*) the current supplied by the alternator (*b*) the output of the alternator in kW and kVA, neglecting the losses in the lines between the alternator and the load. The line voltage is 400 V. **[(*a*) 30.95 A (b) 19.2 kW, 21.45 kVA]**

11. Three non-inductive resistances, each of 100 Ω, are connected in star to 3-phase, 440-V supply. Three equal choking coils each of reactance 100 Ω are also connected in delta to the same supply. Calculate:

(*a*) line current (*b*) p.f. of the system. **[(*a*) 8.04 A (*b*) 0.3156]** (*I.E.E. London*)

12. In a 3-phase, 4-wire system, there is a balanced 3-phase motor load taking 200 kW at a power factor of 0.8 lagging, while lamps connected between phase conductors and the neutral take 50, 70 and 100 kW respectively. The voltage between phase conductors is 430 V. Calculate the current in each phase and in the neutral wire of the feeder supplying the load. **[512 A, 5.87 A, 699 A; 213.3 A]** (*Elect. Power, London Univ.*)

13. A 440-V, 50-Hz induction motor takes a line current of 45 A at a power factor of 0.8 (lagging). Three Δ-connected capacitors are installed to improve the power factor to 0.95 (lagging). Calculate the kVA of the capacitor bank and the capacitance of each capacitor. **[11.45 kVA, 62.7 μF]** (*I.E.E. London*)

14. Three resistances, each of 500 Ω, are connected in star to a 400-V, 50-Hz, 3-phase supply. If three capacitors, when connected in delta to the same supply, take the same line currents, calculate the capacitance of each capacitor and the line current. **[2.123 μ F, 0.653 A]** (*London Univ.*)

15. A factory takes the following balanced loads from a 440-V, 3-phase, 50-Hz supply:

(*a*) a lighting load of 20 kW (*b*) a continuous motor load of 30 kVA at 0.5 p.f. lagging.
(*c*) an intermittent welding load of 30 kVA at 0.5 p.f. lagging.

Calculate the kVA rating of the capacitor bank required to improve the power factor of loads (*a*) and (*b*) together to unity. Give also the value of capacitor required in each phase if a star-connected bank is employed.

What is the new overall p.f. if, after correction has been applied, the welding load is switched on.

[30 kVAR; 490μ F; 0.945 kg]

16. A three-wire, three-phase system, with 400 V between the line wires, supplies a balanced delta-connected load taking a total power of 30 kW at 0.8 power factor lagging. Calculate (*i*) the resistance and (*ii*) the reactance of each branch of the load and sketch a vector diagram showing the line voltages and line currents. If the power factor of the system is to be raised to 0.95 lagging by means of three delta-conected capacitors, calculate (*iii*) the capacitance of each branch assuming the supply frequency to be 50 Hz.

[(i) 10.24 A (ii) 7.68 Ω (*iii*) 83.2 μF] (*London Univ.*)

19.15. Power Measurement in 3-phase Circuits

Following methods are available for measuring power in a 3-phase load.

(*a*) Three Wattmeter Method

In this method, three wattmeters are inserted one in each phase and the algebraic sum of their readings gives the total power consumed by the 3-phase load.

(*b*) Two Wattmeter Method

(*i*) This method gives true power in the 3-phase circuit without regard to balance or wave form provided in the case of *Y*-connected load. The neutral of the load is isolated from the neutral of the source of power. Or if there is a neutral connection, the neutral wire should not carry any current. This is possible only if the load is perfectly balanced and there are no harmonics present of triple frequency or any other multiples of that frequency.

(*ii*) This method can also be used for 3-phase, 4-wire system in which the neutral wire carries the neutral current. In this method, the current coils of the wattmeters are supplied from current transformers inserted in the principal line wires in order to get the correct magnitude and phase differences of the currents in the current coils of the wattmeter, because in the 3-phase, 4-wire system, the sum of the instantaneous currents in the principal line wires is not necessarily equal to zero as in 3-phase 3-wire system.

(*c*) One Wattmeter Method

In this method, a single wattmeter is used to obtain the two readings which are obtained by two wattmeters by the two-wattmeter method. This method can, however, be used only when the load is balanced.

19.16. Three Wattmeter Method

A wattmeter consists of (*i*) a low resistance current coil which is inserted in series with the line carrying the current and (*ii*) a high resistance pressure coil which is connected across the two points whose potential difference is to be measured.

A wattmeter shows a reading which is proportional to the product of the current through its current coil, the p.d. across its potential or pressure coil and cosine of the angle between this voltage and current.

As shown in Fig. 19.46 in this method three wattmeters are inserted in each of the three phases of the load whether Δ-connected or *Y*-connected. The current coil of eaah wattmeter carries the current of one phase only and the pressure coil measures the phase-voltage of this phase. Hence, each wattmeter measures the power in a single phase. The algebraic sum of the readings of three wattmeters must give the total power in the load.

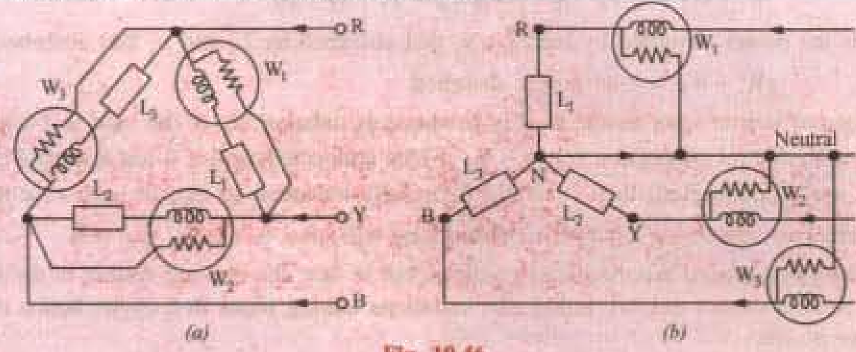


Fig. 19.46

The difficulty with this method is that under ordinary conditions it is not generally feasible to break into the phases of a delta-connected load nor is it always possible, in the case of a Y-connected load, to get at the neutral point which is required for connections as shown in Fig. 19.47 (*b*). However, it is not necessary to use three wattmeters to measure power, two wattmeters can be used for the purpose as shown below.

19.17. Two Wattmeter Method-Balanced or Unbalanced Load

As shown in Fig. 19.41, the current coils of the two wattmeters are inserted in *any two* lines and the potential coil of each joined to the third line. It can be proved that the sum of the instantaneous powers indicated by W_1 and W_2 gives the instantaneous power absorbed by the three loads L_1, L_2 and L_3. A star-connected load is considered in the following discussion although it can be equally applied to Δ-connected loads because a Δ-connected load can always be replaced by an equivalent Y-connected load.

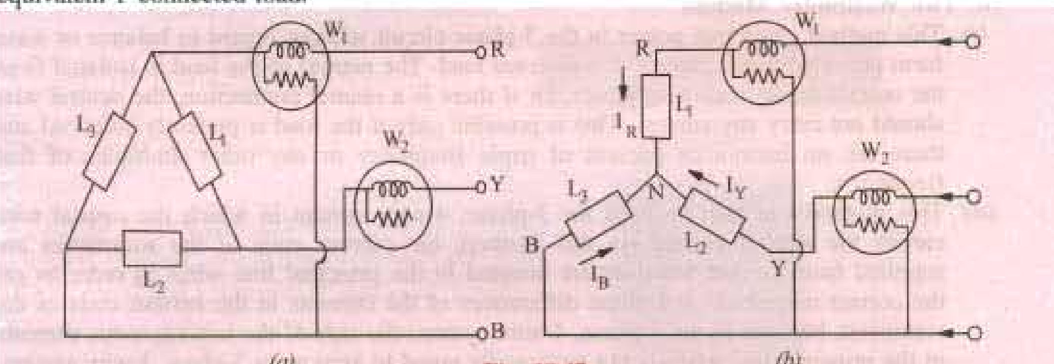


Fig. 19.47

Now, before we consider the currents through and p.d. across each wattmeter, it may be pointed out that ***it is important to take the direction of the voltage through the circuit the same as that taken for the current when establishing the readings of the two wattmeters.***

Instantaneous current through W_1 $= i_R$

p.d. across $W_1 = e_{RB}$ $= e_R - e_B$

p.d. across power read by W_1 $= i_R\,(e_R - e_B)$

Instantaneous current through W_2 $= i_Y$

Instantaneous p.d. across W_2 $= e_{YB} = (e_Y - e_B)$

Instantaneous power read by W_2 $= i_Y\,(e_Y - e_B)$

$$\therefore \quad W_1 + W_2 = e_R(e_R - e_B) + i_Y(e_Y - e_B) = i_R e_R + i_Y e_Y - e_B(i_R + i_Y)$$

Now, $\quad i_R + i_Y + i_B = 0$ ... Kirchhoff's Current Law

$$\therefore \quad i_R + i_Y = -i_B$$

or $\quad W_1 + W_2 = i_R . e_R + i_Y . e_y + i_B . e_B = p_1 + p_2 + p_3$

where p_1 is the power absorbed by load L_1, p_2 that absorbed by L_2 and p_3 that absorbed by L_3

$$\therefore \quad W_1 + W_2 = \text{total power absorbed}$$

The proof is true whether the load is balanced or unbalanced. If the load is Y-connected, it should have no neutral connection (*i.e.* $3-\phi$, 3-wire connected) and if it has a neutral connection (*i.e.* $3-\phi$, 4-wire connected) then it should be exactly balanced so that in each case there is no neutral current i_N otherwise Kirchoff's current Law will give $i_N + i_R + i_y + i_B = 0$.

We have considered *instantaneous* readings, but in fact, the moving system of the wattmeter, due to its inertia, cannot quickly follow the variations talking place in a cycle, hence it indicates the *average* power.

$$\therefore \quad W_1 + W_2 = \frac{1}{T}\int_0^T i_R e_{RB} dt + \frac{1}{T}\int_0^T i_Y e_{YB} dt$$

19.18. Two Wattmeter Method–*Balanced Load*

If the load is balanced, then power factor of the load can also be found from the two wattmeter readings. The *Y*-connected load in Fig. 19.47 (*b*) will be assumed inductive. The vector diagram for such a balanced *Y*-connected load is shown in Fig. 19.48. We will now consider the problem in terms of r.m.s. values instead of instantaneous values.

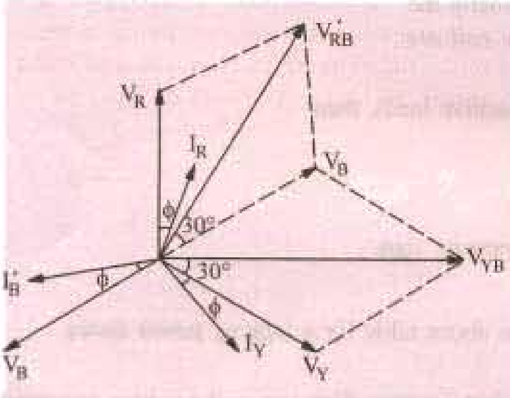


Fig. 19.48

Let V_R, V_Y and V_B be the r.m.s. values of the three phase voltages and I_R, I_Y and I_B the r.m.s. values of the currents. Since these voltages and currents are assumed sinusoidal, they can be represented by vectors, the currents lagging behind their respective phase voltages by ϕ.

Current through wattmeter W_1 [Fig. 19.47 (*b*)] is $= I_R$.

P.D. across voltage coil of W_1 is

$V_{RB} = V_R - V_B$ ... vectorially

This V_{RB} is foundby compounding V_R and V_B reversed as shown in Fig. 19.42. It is seen that phase difference between V_{RB} and $I_R = (30° - \phi)$.

$\therefore$ Reading of $W_1 = I_R V_{RB} \cos(30° - \phi)$

Similarly, as seen from Fig. 19.47 (*b*). Current through $W_2 = I_Y$

P.D. across $W_2 = V_{YB} = V_Y - V_B$ ... vectorially

Again, V_{YB} is found by compounding V_Y and V_B reversed as shown in Fig. 19.48. The angle between I_Y and V_{YB} is $(30° + \phi)$. Reading of $W_2 = I_Y V_{YB} \cos(30° + \phi)$

Since load is balanced, $V_{RB} = V_{YB}$ = line voltage V_L; $I_Y = I_R$ = line current, I_L

$\therefore \quad W_1 = V_L I_L \cos(30° - \Phi)$ and $W_2 = V_L I_L \cos(30° + \phi)$

$\therefore \quad W_1 + W_2 = V_L I_L \cos(30° - \phi) + V_L I_L (\cos(30° + \phi)$

$= V_L I_L [\cos 30° \cos\phi + \sin 30° \sin\phi + \cos 30° \cos\phi - \sin 30° \sin\phi]$

$= V_L I_L (2\cos 30° \cos\phi) = \sqrt{3} V_L I_L \cos\phi$ = total power in the 3-phase load

Hence, the sum of the two wattmeter readings gives the total power consumption in the 3-phase load.

It should benoted that phase sequence of *RYB* has been assumed in the above discussion. Reversal of phase sequence will interchange the readings of the two wattmeters.

19.19. Variations in Wattmeter Readings

It has been shown above that for a lagging power factor

$W_1 = V_L I_L \cos(30° - \varphi)$ and $W_2 = V_L I_L \cos(30° + \phi)$

From this it is clear that individual readings of the wattmeters not only depend *on the load but upon its power factor also*. We will consider the following cases:

(*a*) When $\phi = 0$ *i.e.* power factor is unity (*i.e.* resistive load) then,

$$W_1 = W_2 = V_L I_L \cos 30°$$

Both wattmeters indicate equal and positive *i.e.* up-scale readings.

(*b*) When $\phi = 60°$ *i.e.* power factor = 0.5 (lagging)

Then $W_2 = V_L I_L \cos(30° + 60°) = 0$. Hence, the power is measured by W_1 alone.

(*c*) When $90° > \phi > 60°$ *i.e.* $0.5 > \text{p.f.} > 0$, then W_1 is still positive but reading of W_2 is reversed because the phase angle between the current and voltage is more than 90°. For getting the total power, the reading of W_2 is to be subtracted from that of W_1.

ϕ	0°	60°	90°
$\cos\phi$	1	0.5	0
W_1	+ve	+ve	+ve
W_2^*	+ ve $W_1 = W_2$	0	− ve $W_1 = W_2$

Under this condition, W_2 will read 'down scale' *i.e.* backwards. Hence, to obtain a reading on W_2 it is necessary to reverse either its pressure coil or current coil, usually the ***All readings taken after reversal of pressure coil are to be taken as negative.***

(*d*) When $\phi = 90°$ (*i.e.* pure inductive or capacitive load), then

$$W_1 = V_L I_L \cos(30°-90°) = V_L I_L \sin 30°;$$

$$W_2 = V_L I_L \cos(30°+90°) = -V_L I_L \sin 30°$$

As seen, the two readings are equal but of opposite sign.

$$\therefore \ W_1 + W_2 = 0$$

The above facts have been summarised in the above table for a lagging power factor.

19.20. Leading Power Factor

In the above discussion, lagging angles are taken positive. Now, we will see how wattmeter readings are changed if the power factor becomes leading. For $\phi = +60°$ (lag), W_2 is zero. But for $\phi = -60°$ (lead), W_1 is zero. So we find that for angles of lead, the reading of the two wattmeters are interchanged. Hence, for a *leading* power factor.

$$W_1 = V_L I_L \cos(30°+\phi) \ \text{ and } \ W_2 = V_L I_L \cos(30°-\phi)$$

19.21. Power Factor–*Balanced Load*

In case the load is balanced (and currents and voltages are sinusoidal) and for a *lagging* power factor:

$$W_1 + W_2 = V_L I_L \cos(30°-\phi) + V_L I_L \cos(30°+\phi) = \sqrt{3} V_L I_L \cos\phi \qquad \text{... } (i)$$

Similarly $W_1 - W_2 = V_L I_L \cos(30°-\phi) - V_L I_L \cos(30°+\phi)$

$$= V_L I_L (2 \times \sin\phi \times 1/2) = V_L I_L \sin\phi \qquad \text{... } (ii)$$

Dividing (*ii*) by (*i*), we have $\tan\phi = \dfrac{\sqrt{3}(W_1 - W_2)^{**}}{(W_1 + W_2)}$ ... (*iii*)

Knowing tan ϕ and hence ϕ, the value of power factor cos ϕ can be found by consulting the trigonometrical tables. It should, however, be kept in mind that if W_2 reading has been taken after reversing the pressure coil *i.e.* if W_2 is negative, then the above relation becomes

---

* For a leading p.f., conditions are just the opposite of this. In that case, W_1 reads negative (Art. 19.22).

** For a leading power factor, this expression becomes

$$\tan\phi = -\sqrt{3}\left(\frac{W_1 - W_2}{W_1 + W_2}\right) \qquad \text{... Art 19.22}$$

$$\tan\phi = \sqrt{3}\frac{W_1-(-W_2)}{W_1+(-W_2)} = \sqrt{3}\frac{W_1+W_2}{W_1-W_2}$$

Obviously, inthis expression, only *numerical* values of W_1 and W_2 should be substituted. We may express power factor in terms of the ratio of the two wattmeters as under:

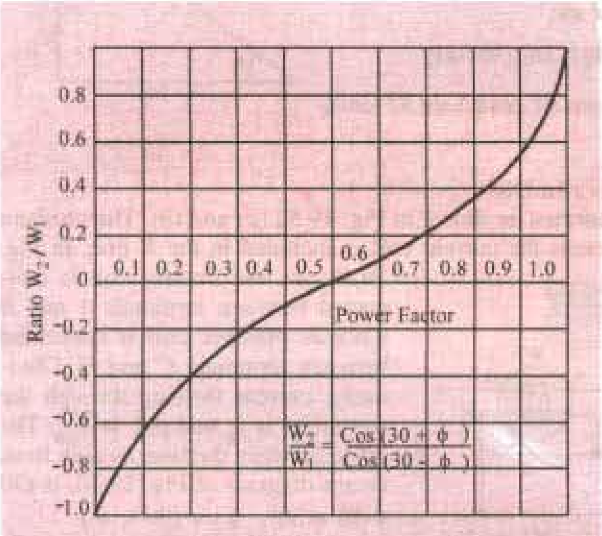


Fig. 19.49

Let $\dfrac{\text{smaller reading}}{\text{larger reading}} = \dfrac{W_2}{W_1} = r$

Then from equation (*iii*) above,

$$\tan\phi = \frac{\sqrt{3}[1-(W_2/W_1)]}{1+(W_2/W_1)} = \frac{\sqrt{3}(1-r)}{1+r}$$

Now $\sec^2\phi = 1+\tan^2\phi$

or $\dfrac{1}{\cos^2\phi} = 1+\tan^2\phi$

$$\therefore \quad \cos\phi = \frac{1}{\sqrt{1+\tan^2\phi}}$$

$$= \frac{1}{\sqrt{1+3\left(\frac{1-r}{1+r}\right)^2}}$$

$$= \frac{1+r}{2\sqrt{1-r+r^2}}$$

If r is plotted against $\cos\phi$, then a curve called watt-ratio curve is obtained as shown in Fig. 19.49.

19.22. Balanced Load – *leading power factor*

In this case, as seen from Fig. 19.50

$$W_1 = V_L I_L \cos(30+\phi)$$

and $\quad W_2 = V_L I_L \cos(30-\phi)$

$\therefore \quad W_1 + W_2 = \sqrt{3}V_L I_L \cos\phi$ – as found above

$$W_1 - W_2 = -V_L I_L \sin\phi$$

$$\therefore \quad \tan\phi = -\frac{\sqrt{3}(W_1-W_2)}{(W_1+W_2)}$$

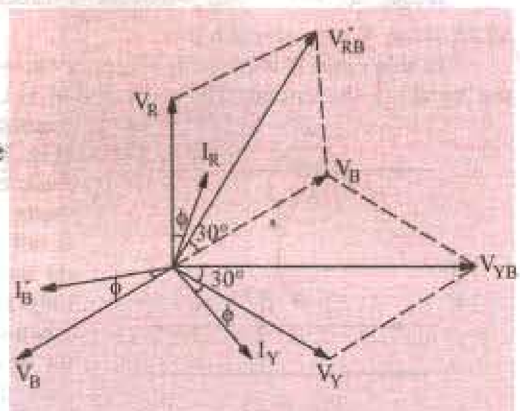


Fig. 19.50

Obviously, if $\phi > 60°$, then phase angle between V_{RB} and I_R becomes more than 90°. Hence, W_1 reads 'down-scale' *i.e.* it indicates negative reading. However, W_2 gives positive reading even in the extreme case when $\phi = 90°$.

19.23. Reactive Voltamperes with Two Wattmeters

We have seen that $\tan\phi = \dfrac{\sqrt{3}(W_1-W_2)}{(W_1+W_2)}$

Since the tangent of the angle of lag between phase current and phase voltage of a circuit is always equal to the ratio of the reactive power to the active power (in watts), it is clear that $\sqrt{3}\,(W_1 - W_2)$ represents the reactive power (Fig. 19.51). Hence, for a balanced load, the reactive power is given by $\sqrt{3}$ times the difference of the readings of the two wattmeters used to measure the power for a 3-phase circuit by the two wattmeter method. It may also be proved mathematically as follows:

$$= \sqrt{3}(W_1 - W_2) = \sqrt{3}[V_L I_L \cos(30^\circ - \phi) - V_L I_L \cos(30^\circ - \phi)]$$

$$= \sqrt{3} V_L I_L (\cos 30^\circ \cos\phi + \sin 30^\circ \sin\phi - \cos 30^\circ \cos\phi + \sin 30^\circ \sin\phi)$$

$$= \sqrt{3} V_L I_L \sin\phi$$

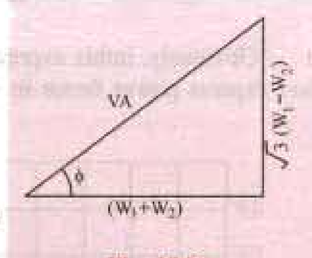


Fig. 19.51

19.24. Reactive Voltamperes with One Wattmeter

For this purpose, the wattmeter is connected as shown in Fig. 19.52 (*a*) and (*b*). The pressure coil is connected across *Y* and *B* lines whereas the current coil is included in the *R* line. In Fig. 19.48 (*a*), the current coil is connected between terminals *A* and *B* whereas pressure coil is connected between terminals *C* and *D*. Obviously, current flowing through the wattmeter is I_R and p.d. is V_{YB}. The angle between the two, as seen from vector diagram of Fig. 19.48, is $(30 + 30 + 30 - \phi) = (90 - \phi)$

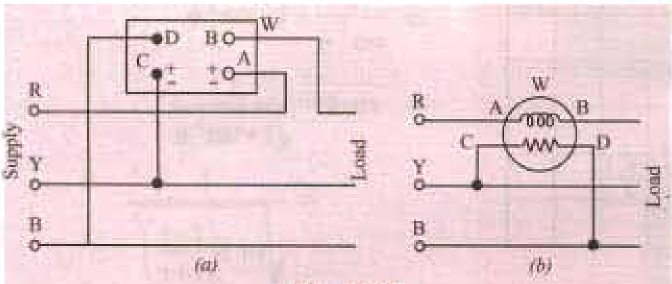


Fig. 19.52

Hence, reading of the wattmeter is

$$W = V_{YB} I_R \cos(90 - \phi) = V_{YB} I_R \sin\phi$$

For a balanced load, V_{YB} equals the line voltage V_L and I_R equals the line current I_L, hence

$$W = V_L I_L \sin\phi$$

We know that the total reactive voltamperes of the load are $Q = \sqrt{3} V_L I_L \sin\phi$.

Hence, to obtain total *VARs*, the wattmeter reading must be multiplied by a factor of $\sqrt{3}$.

19.25. One Wattmeter Method

In this case, it is possible to apply two-wattmeter method by means of one wattmeter without breaking the circuit. The current coil is connected in any one line and the pressure coil is connected alternately between this and the other two lines (Fig. 19.53). The two readings so obtained, for a balanced load, correspond to those obtained by normal two wattmeter method. It should be kept in mind that this method is not of as much universal application as the two wattmeter method because itis restricted to fairly balanced loads only. However, it may be conveniently applied, for instance, when it is desired to find the power input to a factory motor in order to check the load upon the motor.

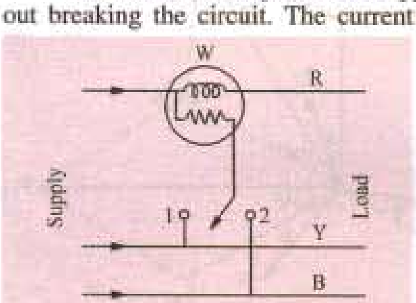


Fig. 19.53

It may be pointed out here that the two wattmeters used in the two-wattmeter method (Art. 19.17) are usually combined into a single instrument in the case of switchboard wattmeter which is then known as a polyphase wattmeter. The combination is affected by arranging the two sets of coils in such a way as to operate on a single moving system resulting in an indication of the total power on the scale.

19.26. Copper Required for Transmitting Power under Fixed Conditions

The comparison between 3-phase and single-phase systems will be done on the basis of a fixed amount of power transmitted to a fixed distance with the same amount of loss and at the same maximum voltage between conductors. In both cases, the weight of copper will be directly proportional to the number of wires (since the distance is fixed) and inversely proportional to the resistance of each wire. We will assume the same power factor and same voltage.

$$P_1 = VI_1 \cos\phi \text{ and } P_3 = \sqrt{3}VI_3 \cos\phi$$

where I_1 = r.m.s. value of current in 1-phase system

I_3 = r.m.s. value of line current in 3-phase system

$$\because \quad P_1 = P_2 \quad \therefore \quad VI_1 \cos\phi = \sqrt{3}VI_3 \cos\phi \quad \therefore \quad I_1 = \sqrt{3}I_3$$

also $I_1^2 R_1 \times 2 = I_2^3 R_3 \times 3$ or $\dfrac{R_1}{R_3} = \dfrac{3I_3^2}{2I_1^2}$

Substituting the value of I_1, we get $\dfrac{R_1}{R_3} = \dfrac{3I_3^2}{3I_3^2 \times 2} = \dfrac{1}{2}$

$$\therefore \quad \frac{\text{copper 3-phase}}{\text{copper 1-phase}} = \frac{\text{No. of wires 3-phase}}{\text{No. of wires 1-phase}} \times \frac{R_1}{R_3} = \frac{3}{2} \times \frac{1}{2} = \frac{3}{4}$$

Hence, we find that for transmitting the same amount of power over a fixed distance with a fixed line loss, we need only three-fourths of the amount of copper that would be required for a single phase or to put it in another way, one-thrid more copper is required for a 1-phase system than would be necessary for a three-phase system.

Example 19.33. *Phase voltage and current of a star-connected inductive load is 150 V and 25 A. Power factor of load is 0.707 (lag). Assuming that the system is 3-wire and power is measured using two wattmeters, find the readings of wattmeters.*

(Elect. Instrument & Measurements, Nagpur Univ. 1993)

Solution. $V_{ph} = 150V; V_L = 150 \times \sqrt{3}V; I_{ph} = I_L = 25$ A

$$\text{Total power} = \sqrt{3}\,V_L I_L \cos\phi = \sqrt{3} \times 150 \times \sqrt{3} \times 25 \times 0.707 = 7954 \text{ W}$$

$$\therefore \quad W_1 + W_2 = 7954 \text{ W} \qquad \text{... (i)}$$

$$\cos\phi = 0.707; \phi = \cos^{-1}(0.707) = 45°; \tan 45° = 1$$

Now, for a lagging power factor, $\tan\phi = \sqrt{3}(W_1 - W_2)/(W_1 + W_2)$ or $1 = \sqrt{3}(W_1 - W_2)/7954$

$$\therefore \quad (W_1 - W_2) = 4592 \text{ W} \qquad \text{... (ii)}$$

From (*i*) and (*ii*) above, we get, W_1 = **6273 W; W_2 = 1681 W.**

Example 19.34. *In a balanced 3-phase 400-V circuit, the line current is 115.5 A. When power is measured by two wattmeter method, one meter reads 40 kW and the other zero. What is the power factor of the load? If the power factor were unity and the line current the same, what would be the reading of each wattmeter?*

Solution. Since $W_2 = 0$, the whole power is measured by W_1. As per Art. 19.18, in such a situation, p.f. = 0.5. However, it can be calculated as under.

Since total power is 40 kW, $\therefore$ 40,000 = $\sqrt{3} \times 400 \times 115.5 \times \cos\phi; \cos\phi = 0.5$

If the power factor is unity with line currents remaining the same, we have

$$\tan\phi = \frac{\sqrt{3}(W_1 - W_2)}{(W_1 + W_2)} = 0 \text{ or } W_1 = W_2$$

Also, $(W_1 + W_2) = \sqrt{3} \times 400 \times 115.5 \times 1$ = 80000 W = 80 kW

As per Art. 19.19, at unity p.f., $W_1 = W_2$. Hence, each wattmeter reads = 80/2 = **40 kW.**

Example 19.35. *The input power to a three-phase motor was measured by two wattmeter method. The readings were 10.4 KW and – 3.4 KW and the voltage was 400 V. Calculate (a) the power factor (b) the line current.* **(Elect. Engg. A.M.Ae, S.I. June 1991)**

Solution: As given in Art. 19.21, when W_2 reads negative, then we have

$\tan\phi = \sqrt{3}(W_1 + W_2)/(W_1 - W_2)$. Substituting numerical values of W_1 and W_2, we get

$$\tan\phi = \sqrt{3}(10.4 + 3.4)/(10.4 - 3.4) = 1.97; \phi = \tan^{-1}(1.97) = 63.1°$$

(*a*) p.f. = $\cos\phi = \cos 63.1° = $ **0.45 (lag)**

(*b*) W = 10.4 – 3.4 = 7 KW = 7,000 W

$7000 = \sqrt{3} I_L \times 400 \times 0.45; I_L = $ **22.4 A**

Example 19.35 (A). *A three-phase, three-wire, 100-V, ABC system supplies a balanced delta connected load with impedance of* $20\angle 45°$ *ohm.*

(a) Determine the phase and line currents and draw the phase or diagram (b) Find the wattmeter readings when the two wattmeter method is applied to the system.

(Elect. Machines, A.M.I.E. Sec B, 1989)

Solution. (*a*) Thephasor diagram is shown in Fig. 19.54 (*b*).

Let $V_{AB} = 100\angle 0°$. Since phase sequence is *ABC*, $V_{BC} = 100\angle -120°$ and $V_{CA} = 100°\ 120\angle°$

Phase current $I_{AB} = \dfrac{V_{AB}}{Z_{AB}} = \dfrac{100\angle 0°}{20\angle 45°} = 5\angle -45°$

$$I_{BC} = \frac{V_{BC}}{Z_{BC}} = \frac{100\angle -120°}{20\angle 45°} = 5\angle -165°, I_{CA} = \frac{V_{CA}}{Z_{CA}} = \frac{100\angle 120°}{20\angle 45°} = 5\angle 75°$$

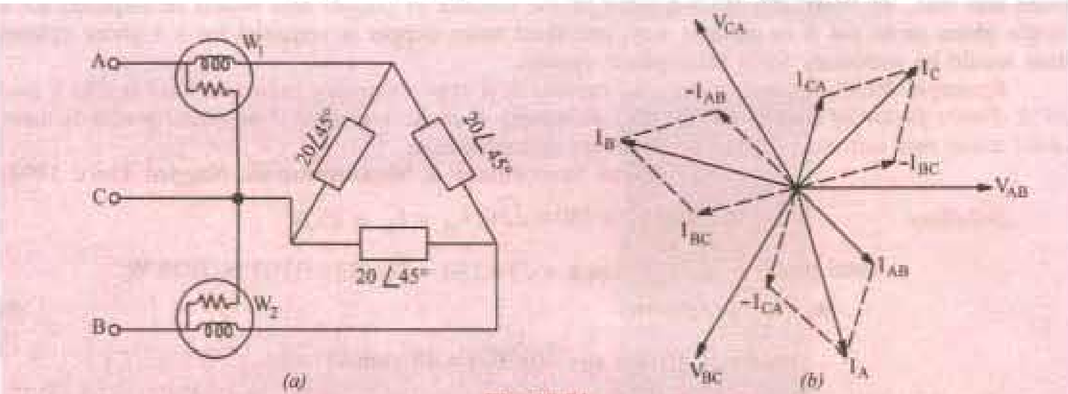


Fig. 19.54

Applying *KCL* to junction A, we have

$$I_A + I_{CA} - I_{AB} = 0 \text{ or } I_A = I_{AB} - I_{CA}$$

$\therefore$ Line current $I_A = 5\angle -45° - 5\angle 75° = 8.66\angle -75°$

Since the system is balanced, I_B will lag I_A by 120° and I_C will lag I_A by 240°.

$\because I_B = 8.66\angle(75° - 120°) = 8.66\angle -195°; I_C = 8.66\angle(-75° - 240°) = 8.66\angle -315° = 8.66\angle 45°$

(*b*) As shown in Fig. 19.54 (*b*), reading of wattmeter W_1 is $W_1 = V_{AC}\ I_C \cos\varphi$. Phasor V_{AC} is the reverse of phasor V_{CA}. Hence, V_{AC} is the reverse of phasor V_{CA}. Hence, V_{AC} lags the reference vector by 60° whereas I_A lags by 75°. Hence, phase difference between the two is (75° – 60°) = 15°

$\therefore W_1 = 100 \times 8.66 \times \cos 15° = 836.5$ W

Similarly $W_2 = V_{BC} I_B \cos\phi = 100 \times 8.66 \times \cos 75° = 224.1$ W

$\therefore W_1 + W_2 = 836.5 + 224.1 = 1060.6$ W

Resistance of each delta branch = 20 cos 45° = 14.14 Ω

Total power consumed = $3 I^2 R = 3 \times 5^2 \times 14.14 = 1060.6$ W

Hence, it proves that the sum of the two wattmeter readings gives the total power consumed.

Example 19.36. *A 3-phase, 500-V motor load has a power factor of 0.4 Two wattmeters connected to measure the power show the input to be 30 kW. Find the reading on each instrument.* **(Electrical Meas., Nagpur Univ. 1991)**

Solution. As seen from Art. 19.21

$$\tan\phi = \frac{\sqrt{3}(W_1 - W_2)}{W_1 + W_2} \quad \text{... (i)}$$

Now, $\cos\phi = 0.4; \phi = \cos^{-1}(0.4) = 66.6°$; $\tan 66.6° = 2.311$

$$W_1 + W_2 = 30 \qquad \text{... (ii)}$$

Substituting these values in equation (*i*) above, we get

$$2.311 = \frac{\sqrt{3}(W_1 - W_2)}{30} \quad \therefore \quad W_1 - W_2 = 40 \qquad \text{... (iii)}$$

From Eq. (*ii*) and (*iii*), we have W_1 = **45 kW** and W_2 = **− 5 kW**

Since W_2 comes out to be negative, second wattmeter reads 'down scale'. Even otherwise it is obvious that p.f. being less than 0.5, W_2 must be negative (Art. 19.19)

Example 19.36 (a). *The power in a 3-phase circuit is measured by two wattmeters. If the total power is 100 kW and power factor is 0.66 leading, what will be the reading of each wattmeter? Give the connection diagram for the wattmeter circuit. For what p.f. will one of the wattmeter read zero?*

Solution. $\phi = \cos^{-1}(0.66) = 48.7°; \tan\phi = 1.1383$

Since p.f. is leading,

$$\therefore \quad \tan\phi = -\frac{\sqrt{3}(W_1 - W_2)}{W_1 + W_2} \quad \therefore \; 1.1383 = \qquad -\sqrt{3}(W_1 - W_2)/100$$

$\therefore\; W_1 - W_2 = -65.7$ and $W_1 + W_2 = 100 \;\therefore\; W_1$ = **17.14 kW; W_2 = 82.85 kW**

Connection diagram is similar to that shown in Fig. 19.47 (*b*). One of the wattmeters will read zero when p.f. = **0.5**

Example 19.37. *Two wattmeters are used for measuring the power input and the power factor of an over-exicted synchronous motor. If the readings of the meters are (− 2.0 kW) and (+ 7.0 kW) respectively, calculate the input and power factor of the motor.*

(Elect. Technology, Punjab Univ., June, 1991)

Solution. Since an over-excited synchronous motor runs with a leading p.f., we should use the relationship derived in Art. 19.22.

$$\tan\phi = \frac{\sqrt{3}(W_1 - W_2)}{W_1 + W_2}$$

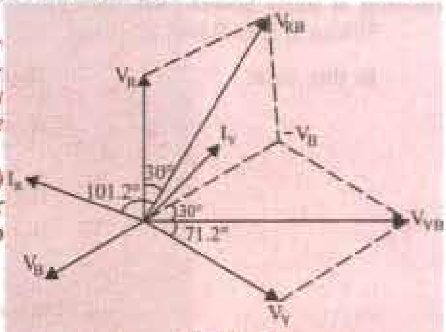


Fig. 19.55

Moreover, as explained in the same article, it is W_1 that gives negative reading and not W_2. Hence,

$$W_1 = -2\,\text{kW}$$

$$\therefore \qquad \tan\phi = -\frac{\sqrt{3}(-2-7)}{-2+7} = \sqrt{3} \times \frac{9}{5} = 3.1176$$

$$\therefore \qquad \phi = \tan^{-1}(3.1176) = 71.2° \;\text{(lead)}$$

$$\therefore \qquad \cos\phi = \cos 71.2° = \mathbf{0.3057\ (lead)} \text{ and}$$

$$\text{Input} = W_1 + W_2 = -2 + 7 = \mathbf{5\ kW}$$

Example 19.38. *A 440-V, 3-phase, delta-connected induction motor has an output of 14.92 kW at a p.f. of 0.82 and efficiency 85%. Calculate the readings on each of the two wattmeters connected to measure the input. Prove any formula used.*

If another star-connected load of 10 kW at 0.85 p.f. lagging is added in parallel to the motor, what will be the current draw from the line and the power taken from the line?

(Elect. Technology-I, Bombay Univ. 1986)

Solution. Motor input = 14,920/0.85 = 17,600 W $\therefore\; W_1 + W_2 = 17.6$ kW ... (*i*)

$$\cos\phi = 0.82; \phi = 34.9°, \;\; \tan 34.9° = 0.6976 \quad 0.6976 = \sqrt{3}\frac{W_1 - W_2}{17.6}$$

$$\therefore \quad W_1 - W_2 = 7.09\ \text{kW} \qquad \text{... (ii)}$$

From (*i*) and (*ii*) above, we get W_1 = **12.35 kW** and W_2 = **5.26 kW**

Motor **kVA**. $S_m = \dfrac{\text{motor kW}}{\cos\phi_m} = \dfrac{17.6}{0.82} = 21.46 \quad \therefore \mathbf{S_m} = 21.46 \angle -34.9° = (17.6 - j\,12.28)$ kVA

Load p.f. = 0.85 $\therefore\ \phi = \cos^{-1}(0.85) = 31.8°$ Load kVA, $S_Y = 10/0.85 = 11.76$

$\therefore \qquad S_Y = 11.76\angle -31.8° = (10 - j6.2)$ kVA

Combined kVA, $\quad \mathbf{S} = \mathbf{S_m} + \mathbf{S_Y} = (27.6 - j\,18.48) = 32.2 \angle -33.8°$ kVA

$$I = \frac{S}{\sqrt{3}.V} = \frac{33.2\times 10^3}{\sqrt{3}\times 440} = \mathbf{43.56\ A}$$

Power taken $\qquad$ = **27.6 kW**

Example 19.39. *The power input to a synchronous motor is measured by two wattmeters both of which indicate 50 kW. If the power factor of the motor be changed to 0.866 leading, determine the readings of the two wattmeters, the total input power remaining the same. Draw the vector diagram for the second condition of the load.* **(Elect. Technology, Nagpur Univ. 1992)**

Solution. In the first case both wattmeters read equal and positive. Hence motor must be running at unity power (Art. 19.22).

When p.f. is 0.866 leading

In this case; $\qquad W_1 = V_L I_L \cos(30°+\phi)$;

$$W_2 = V_L I_L \cos(30° - \phi)$$

$$\therefore \qquad W_1 + W_2 = \sqrt{3} V_L I_L \cos\phi$$

$$W_1 - W_2 = -V_L I_L \sin\phi$$

$$\therefore \qquad \tan\phi = \frac{\sqrt{3}(W_1 - W_2)}{(W_1 + W_2)}$$

$$\phi = \cos^{-1}(0.866) = 30°$$

$$\tan\phi = 1/\sqrt{3}$$

$$\therefore \qquad \frac{1}{\sqrt{3}} = \frac{-\sqrt{3}(W_1 - W_2)}{100}$$

$\therefore \qquad W_1 - W_2 = -100/3$

and $\qquad W_1 + W_2 = 100$

$\therefore \qquad 2W_1 = 200/3;\ W_1$ = **33.33 kW**: W_2 = **66.67 kW**

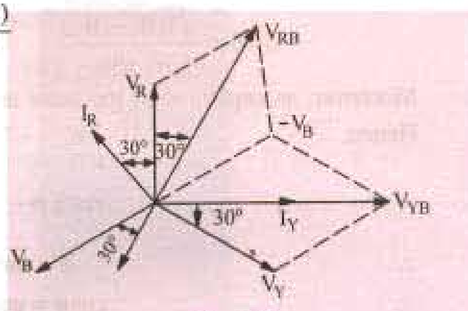


Fig. 19.56

For connection diagram, please refer to Fig. 19.47. The vector or phasor diagram is shown in Fig. 19.56.

Example 19.39 (a). *A star-connected balanced load is supplied from a* $3-\phi$ *balanced supply with a line voltage of 416 volts at a frequency of 50 Hz. Each phase of the load consists of a resistance and a capacitor joined in series and the reading on two wattmeters connected to measure the total power supplied are 782 W and 1980 W, both positive. Calculate*

(i) power factor of circuit, (ii) the line current, (iii) the capacitance of each capacitor.

(Elect. Engg. I, Nagpur Univ. 1993)

Solution. (*i*) As seen from Art. 19.21 $\tan\phi = -\dfrac{\sqrt{3}(W_1 - W_2)}{(W_1 + W_2)} = -\dfrac{\sqrt{3}(782-1980)}{(782+1980)} = 0.75;$ $\phi = 36.9°$, $\cos\phi = 0.8$

(*ii*) $\sqrt{3}\times 416\times I_L\times 0.8 = 2762, I_L = 4.8\,\text{A}$

(*ii*) $Z_{ph} = V_{ph}/I_{ph} = (416\sqrt{3})/4.8 = 50\Omega, X_C = Z_{ph}\sin\phi = 50\times 0.6 = 30\,\Omega$

Now, $X_C = 1/2\pi fC = 1/2\phi\times 50\times C = 106\times 10^{-6}\,\text{F}$

Example 19.40. *Each phase of a 3-phase, Δ-connected load consists of an impedance Z = 20 ∠60° ohm. The line voltage is 440 V at 50 Hz. Compute the power consumed by each phase impedance and the total power. What will be the readings of the two wattmeters connected?* **(Elect. and Mech. Technology, Osmania Univ. 1980)**

Solution. $Z_{ph} = 20\Omega; V_{ph} = V_L = 440\text{V}; I_{ph} = V_{ph}/Z_{ph} = 440/20 = 22\ A$

Since $\phi = 60°; \cos\phi = \cos 60° = 0.5°;\ R_{ph} = Z_{ph}\times\cos 60° = 20\times 0.5 = 10\,\Omega$

$\therefore$ Power/phase $= I_{ph}^2 R_{ph} = 22^2\times 10 =$ **4,840 W**

Total power $= 3\times 4{,}840 =$ **14,520 W** [or $P = \sqrt{3}\times 440\times(\sqrt{3}\times 22)\times 0.5 = 14{,}520$ W]

Now, $W_1 + W_2 = 14{,}520.$

Also $\tan\phi = \sqrt{3}.\dfrac{W_1 - W_2}{W_1 + W_2}$ $\quad\therefore\ \tan 60° = \sqrt{3} = \sqrt{3}.\dfrac{W_1 - W_2}{14{,}520}$

$\therefore$ $W_1 - W_2 = 14{,}520$. Obviously, $W_2 = 0$

Even otherwise it is obvious that W_2 should be zero because p.f. = cos 60° = 0.5 (Art. 19.19).

Example 19.41. *Three identical coils, each having a reactance of 20 Ω and resistance of 20 Ω are connected in (a) star (b) delta across a 440-V, 3-phase line. Calculate for each method of connection the line current and readings on each of the two wattmeters connected to measure the power.* **(Electro-mechanics, Allahabad Univ. 1992)**

Solution. (*a*) Star Connection

$$Z_{ph} = \sqrt{20^2 + 20^2} = 20\sqrt{2} = 28.3\Omega; V_{ph} = 440/\sqrt{3} = 254\,\text{V}$$

$$I_{ph} = 254/28.3 = 8.97\text{A}; I_L = \mathbf{8.97\ A};\ \cos\phi = R_{ph}/Z_{ph} = 20/28.3 = 0.707$$

Total power taken $= \sqrt{3}V_L I_L\cos\phi = \sqrt{3}\times 440\times 8.97\times 0.707 = 4830$ W

If W_1 and W_2 are wattmeter readings, then $W_1 + W_2 = 4830$ W ... (*i*)

Now, $\tan\phi = 20/20 = \sqrt{3}(W_1 - W_2)/(W_1 + W_2); (W_1 - W_2); = 2790\,\text{W}$ ... (*ii*)

From (*i*) and (*ii*) above, $W_1 =$ **3810 W**; $W_2 =$ **1020 W**

(*b*) Delta Connection

$$Z_{ph} = 28.3\Omega, V_{ph} = 440\,V, I_{ph} = 440/28.3 = 15.5A; I_L = 15.5\times\sqrt{3} = 28.8\,\text{A}$$

$$P = \sqrt{3}\times 440\times 28.8\times 0.707 = 14{,}490\text{ W}$$ (it is 3 times the *Y*-power)

$\therefore$ $W_1 + W_2 = 14{,}490$ W ... (*iii*)

$\tan\phi = 20/20 = \sqrt{3}(W_1 - W_2)/14{,}490; W_1 - W_2 = 8370$ ... (*iv*)

From Eq. (*iii*) and (*iv*), we get, $W_1 =$ **11,430 W**; $W_2 =$ **3060 W**

Note: These readings are 3-times the Y-readings.

Example 19.42. *Three identical coils are connected in star to a 200-V, three-phase supply and each takes 500 W. The power factor is 0.8 lagging. What will be the current and the total power if the same coils are connected in delta to the same supply? If the power is measured by two wattmeters, what will be their readings? Prove any formula used.* **(Elect. Engg. A.M. A. S.I. Dec. 1991)**

Solution. When connected in star as shown in Fig. 19.57 (*a*), $V_{ph} = 200/\sqrt{3} = 115.5$ V

Now, $V_{ph}I_{ph}\cos\phi$ = power per phase or $115.5\times I_{ph}\times 0.8 = 500$

$\therefore \quad I_{ph} = 5.41A; Z_{ph} = V_{ph} / I_{ph} = 115.5/5.41 = 21.34\,\Omega$

$R = Z_{ph} \cos\phi = 21.34 \times 0.8 = 17\Omega;\ X_L = Z_{ph} \sin\phi = 21.34 \times 0.6 = 12.8\,\Omega$

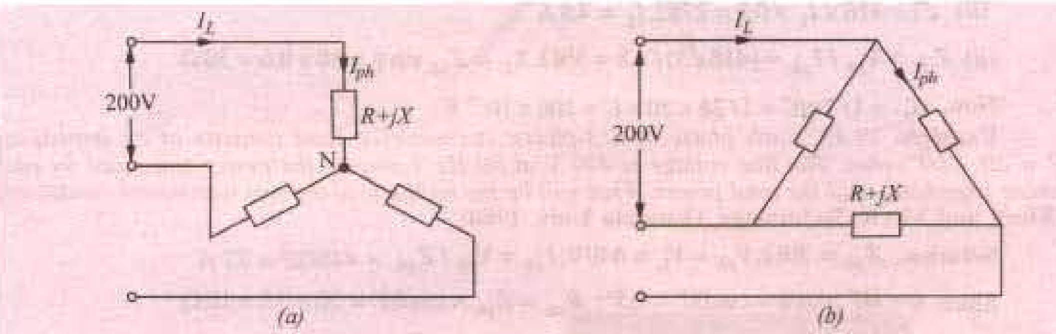


Fig. 19.57

The same three coils have been connected in delta in Fig. 19.57 (*b*). Here, $V_{ph} = V_L = 200\,\text{V}$.

$$I_{ph} = 200/21.34 = 9.37A; I_L = \sqrt{3} I_{ph} = 9.37 \times 1.732 = \mathbf{16.23\ A}$$

Total power consumed = $\sqrt{3} \times 200 \times 16.23 \times 0.8$ = **4500 W**

It would be seen that when the same coils are connected in delta, they consume three times more power than when connected in star.

Wattmeter Readings

Now, $$W_1 + W_2 = 4500; \tan\phi = \frac{\sqrt{3}(W_1 - W_2)}{(W_1 + W_2)}$$

$$\phi = \cos^{-1}(0.8) = 36.87°; \tan\phi = 0.75$$

$$0.75 = \frac{\sqrt{3}(W_1 - W_2)}{4500} \quad \therefore \ (W_1 - W_2) = 1950\,\text{W}$$

$\therefore$ $$W_1 = (4500 + 1950)/2 = 3225\ \text{W};\ W_2 = \mathbf{1275\ W.}$$

Example 19.43 (*a*). *A 3-phase, 3-wire, 415-V system supplies a balanced load of 20 A at a power factor 0.8 lag. The current coil of wattmeter 1 is in phase R and of wattmeter 2 in phase B. Calculate (i) the reading on 1 when its voltage coil is across R and Y (ii) the reading on 2 when its voltage coil is across B and Y and (iii) the reading on 1 when its voltage coil is across Y and B. Justify your answer with relevant phasor diagram.* **(Elect. Machines, A.M.I.E. Sec. B, 1991)**

Solution. (*i*) As seen from phasor diagram of Fig. 19.57 (*a*)

$$W_1 = V_{RY} I_A \cos(30 + \phi) = \sqrt{3} \times 415 \times 20 \times \cos(36.87° + 30°) = \mathbf{5647\ W}$$

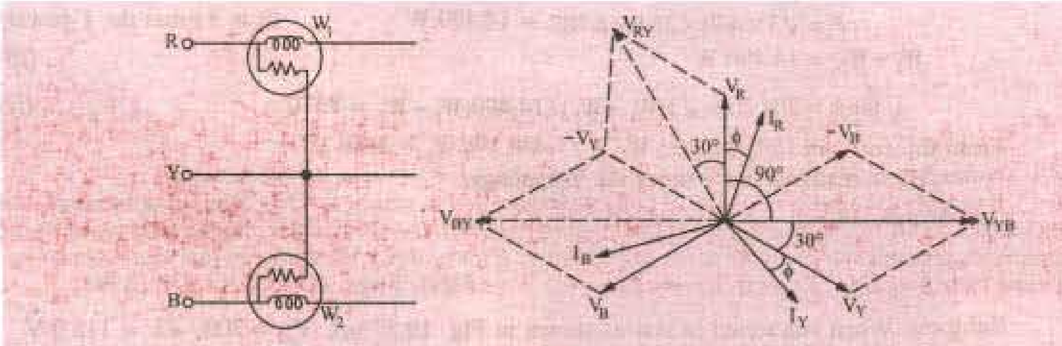


Fig. 19.57 (*a*)

(*ii*) Similarly, $W_2 = V_{BY} I_B \cos(30 - \phi)$

It should be noted that voltage across W_2 is V_{BY} and not V_{YB}. Moreover, $\phi = \cos^{-1}(0.8) = 36.87°$.

$$\therefore \qquad W_2 = \sqrt{3} \times 415 \times 20 \times \cos(30° - 36.87°) = \mathbf{14{,}275\ W}$$

(*iii*) Now, phase angle between I_R and V_{YB} is $(90° - \phi)$

$$\therefore \qquad W_2 = V_{YB} I_R \cos(90° - \phi) = \sqrt{3} \times 415 \times 20 \times \sin 36.87° = 8626 \text{ VAR}$$

Example 19.43 (*b*). *A wattmeter reads 5.54 kW when its current coil is connected in R phase and its voltage coil is connected between the neutral and the R phase of a symmetrical 3-phase system supplying a balanced load of 30 A at 400 V. What will be the reading on the instrument if the connections to the current coil remain unchanged and the voltage coil be connected between B and Y phases? Take phase sequence RYB. Draw the corresponding phasor diagram.*

(Elect. Machines, A.M.I.E., Sec. B, 1992)

Solution. As seen from Fig. 19.57 (*b*).

$W_1 = V_R I_R \cos\phi \text{ or } 5.54 \times 10^3 = (400/\sqrt{3}) \times 30 \times \cos\phi; \ \therefore \ \cos\phi = 0.8, \sin\phi = 0.6$

In the second case (Fig. 19.57 (*b*))

$W_2 = V_{YB} I_R \cos(90° - \phi) = 400 \times 30 \times \sin\phi = 400 \times 30 \times 0.6 = \mathbf{7.2\ kW}$

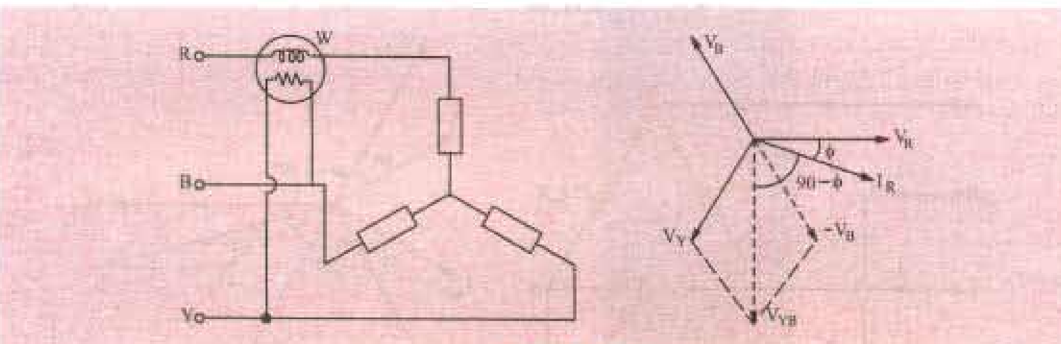


Fig. 19.57 (*b*)

Example 19.43 (*c*). *A 3-phase, 3-wire balanced load with a lagging power factor is supplied at 400 V (between lines). A 1-phase wattmeter (scaled in kW) when connected with its current coil in the R-line and voltage coil between R and Y lines gives a reading of 6 kW. When the same terminals of the voltage coil are switched over to Y- and B-lines, the current-coil connections remaining the same, the reading of the wattmeter remains unchanged. Calculate the line current and power factor of the load. Phase sequence is* $R \to Y \to B$.

(Elect. Engg-I, Bombay Univ. 1985)

Solution. The current through the wattmeter is I_R and p.d. across its pressure coil is V_{RY}. As seen from the phasor diagram of Fig. 19.58, the angle between the two is $(30° + \phi)$.

$$\therefore \qquad W_1 = V_{RY} I_R \cos(30° + \phi) = V_L I_L \cos(30° + \phi) \qquad \ldots (i)$$

In the second case, current is I_R but voltage is V_{YB}. The angle between the two is $(90° - \phi)$

$\therefore \ W_2 = V_{YB} I_R \cos(90° - \phi) = V_L I_L \cos(90° - \phi)$

Since $W_1 = W_2$ we have

$V_L I_L \cos(30° + \phi) = V_L I_L \cos(90° - \phi)$

$\therefore \ 30° + \phi = 90° - \phi$

or $2\phi = 60° \ \therefore \ \phi = 30°$

$\therefore$ load power factor $= \cos 30° =$ **0.866 (lag)**

Now $\quad W_1 = W_2 = 6$ kW.

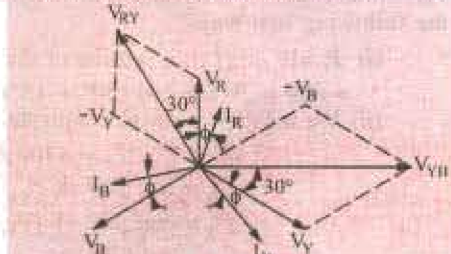


Fig. 19.58

Hence, from (*i*) above, we get

$$6000 = 400 \times I_L \cos 60°; I_L = \mathbf{30\ A}$$

Example 19.44. *A 3-phase, 400 V circuit supplies a Δ-connected load having phase impedances of $Z_{AB} = 25\angle 0°; Z_{BC} = 25\angle 30°$ and $V_{CA} = 25\angle -30°$.*

Two wattmeters are connected in the circuit to measure the load power. Determine the wattmeter readings if their current coils are in the lines(a) A and B ; (b) B and C; and (c) C and A. The phase sequence is ABC. Draw the connections of the wattmeter for the above three cases and check the sum of the two wattmeter readings against total power consumed.

Solution. Taking V_{AB} as the reference voltage, we have $Z_{AB} = 400\angle 0°; Z_{BC} = 400\angle -120°$ and $Z_{CA} = 400\angle 120°$.

The three phase currents can be found as follows:

$$I_{AB} = \frac{V_{AB}}{Z_{AB}} = \frac{400\angle 0°}{25\angle 0°} = 16\angle 0° = (16 + j0)$$

$$I_{BC} = \frac{V_{BC}}{Z_{BC}} = \frac{400\angle -120°}{25\angle 30°} = .16\angle -150° = (-13.8 - j8)$$

$$I_{CA} = \frac{V_{CA}}{Z_{CA}} = \frac{400\angle -120°}{25\angle 30°} = 16\angle -150° = (-13.8 - j8)$$

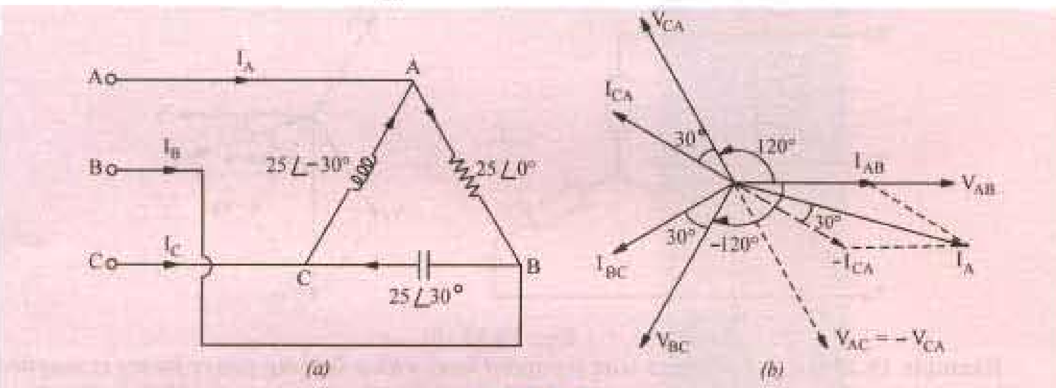


Fig. 19.59

The line currents I_A, I_B and I_C can be found by applying KCL at the three nodes *A*, *B* and *C* of the load.

$$I_A = I_{AB} + I_{AC} = I_{AB} - I_{CA} = (16 + j0) - (-13.8 + j8) = 29.8 - j8 = 30.8\angle -15°$$

$$I_B = I_{BC} - I_{AB} = (-13.8 - j8) - (16 + j0) = -29.8 - j8 = 30.8\angle -165°$$

$$I_C = I_{CA} - I_{BC} = (-13.8 + j8) - (-13.8 - j8) = 0 + j16 = 16\angle 90°$$

The phasor diagram for line and phase currentsis shown in Fig. 19.59 (*a*) and (*b*).

(*a*) As shown in Fig. 19.60 (*a*), the current coils of the wattmeters are in the line *A* and *B* and the voltage coil of W_1 is across the lines *A* and *C* and that of W_2 is across the lines *B* and *C*. Hence, current through W_1 is I_A and voltage across it is V_{AC}. The power indicated by W_1 may be found in the following two ways:

(*i*) $P_1 = |V_{AC}|.|I_A| \times$ (cosine of the angle between V_{AC} and I_A).

$= 400 \times 30.8 \times \cos(30° + 15°) = 8710$ W

(*ii*) We may use current conjugate (Art.) for finding the power

$$P_{VA} = V_{AC}.I_A = -400\angle 120° \times 30.8\angle 15°$$

$\therefore$ P_1 = real part of $P_{VA} = -400 \times 30.8 \times \cos 135° = 8710$ W

P_2 = real part of $[V_{BC} Z_B] = 400\angle 120° \times 30.8\angle -165°$

$= 400 \times 30.8 \times \cos(-45°) = 8710$ W

$\therefore\ P_1 + P_2 = 8710 + 8710 = 17{,}420\text{ W}.$

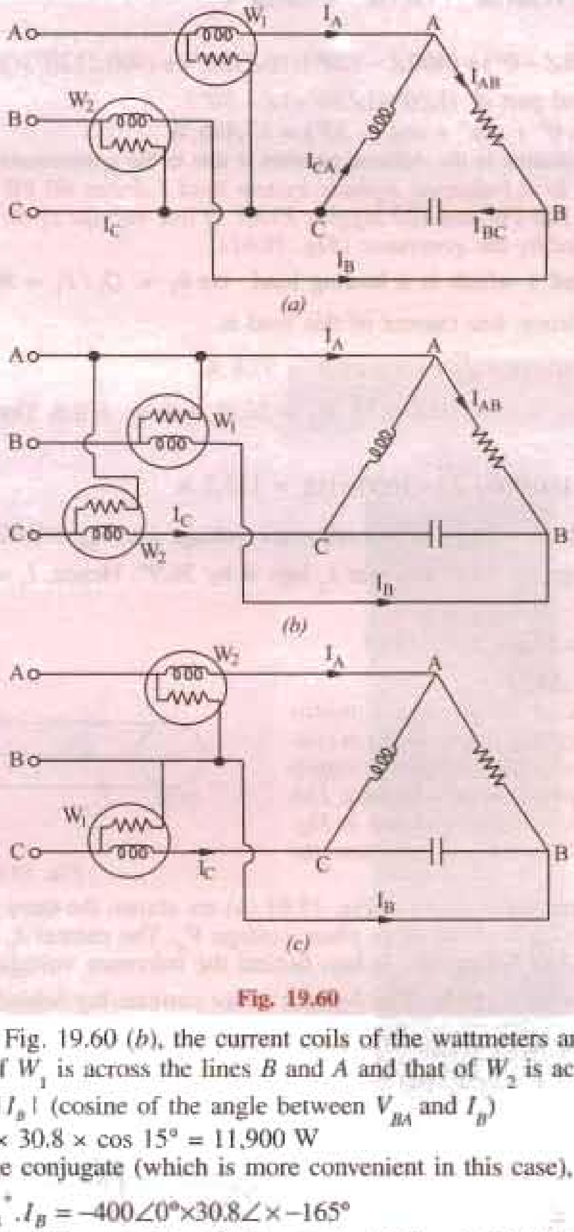


Fig. 19.60

(*b*) As shown in Fig. 19.60 (*b*), the current coils of the wattmeters are in the lines *B* and *C* whereas voltage coil of W_1 is across the lines *B* and *A* and that of W_2 is across lines *C* and *A*.

(*i*) $\therefore\ P_1 = |V_{BA}|.|I_B|$ (cosine of the angle between V_{BA} and I_B)

$= 400 \times 30.8 \times \cos 15° = 11{,}900$ W

(*ii*) Using voltage conjugate (which is more convenient in this case), we have

$$P_{VA} = V_{BA}^{\ *}.I_B = -400\angle 0° \times 30.8\angle \times -165°$$

$\therefore\ P_1$ = real part of $P_{VA} = -400 \times 30.8 \times \cos(-165°) = 11{,}900$ W

P_2 = real part of $[V_{CA} {}^* I_C] = [400\ \angle -120° \times 16\angle 90°] = 400 \times 16 \times \cos(-30°) = 5{,}540$ W.

$\therefore\ P_1 + P_2 = 11{,}900 + 5{,}540 = 17{,}440$ W.

(*c*) As shown in Fig. 19.60 (*c*), the current coils of the wattmeters are in the lines *C* and *A* whereas the voltage coil of W_1 is across the lines *C* and *B* and that of W_2 is across the lines *A* and *B*.

(*i*) P_1 = real part of $[V_{CB} {}^* I_C] = [(-400\angle 120°) \times 16\angle 90°]$

$= -400 \times 16 \times \cos\angle 210° = 5540$ W

P_2 = real part of $[V_{AB} {}^* I_A] = [400\angle°0 \times 30.8\angle -15° = 400 \times 30.8 \times \cos\angle -15° = 11.900$ W

$\therefore\ P_1 + P_2 = 5{,}540 + 11{,}900 = 17{,}440$ W

Total power consumed by the phase load can be found directly as under :-

P_T = real part of $[V_{AB}I_{AB}^* + V_{BC}I_{BC}^* + V_{CA}I_{CA}^*]$

= real part of

$$[(400\angle 0^\circ)(16\angle -0^\circ) + (400\angle -120^\circ)(16\angle 150^\circ) + (400\angle 120^\circ)(16\angle -150^\circ)]$$

$= 400 \times 16 \times$ real part of $(1\angle 0^\circ + 1\angle 30^\circ + 1\angle -30^\circ)$

$= 400 \times 16\ (\cos 0^\circ + \cos^\circ + \cos(-30^\circ)) = 17{,}485$ W

Note. The slight variation in the different answers is due to the approximation made.

Example 19.45. *In a balanced 3-phase system load 1 draws 60 kW and 80 leading kVAR whereas load 2 draws 160 kW and 120 lagging kVAR. If line voltage of the supply is 1000 V, find the line current supplied by the generator. (Fig. 19.61)*

Solution. For load 1 which is a leading load, $\tan\phi_1 = Q_1/P_1 = 80/60 = -1.333$; $\phi_1 = 53.1^\circ$, $\cos\phi_1 = 0.6$. Hence, line current of this load is

$$I_1 = 60{,}000/\sqrt{3}\times 1000\times 0.6 = 57.8\text{ A}$$

For load 2, $\tan\phi_2 = 120/160 = 0.75$; $\phi_2 = 26.9^\circ$, $\cos\phi_2 = 0.8$. The line current drawn by this load is

$$I_2 = 160{,}000/\sqrt{3}\times 1000\times 0.8 = 115.5\text{ A}$$

If we take the phase voltage as the reference voltage *i.e.* $V_{ph} = (1000/\sqrt{3})\angle 0^\circ = 578\angle 0^\circ$; then I_1 leads this voltage by 53.1° whereas I_2 lags it by 36.9°. Hence, $I_1 = 57.8\angle 53.1^\circ$ and $I_2 = 115.5\angle 36.9^\circ$

$\therefore\ I_{L1} = I_1 + I_2 = 57.8\angle 53.1^\circ + 115.5$

$\angle -36.9^\circ = 171.7\angle 42.3^\circ$ A

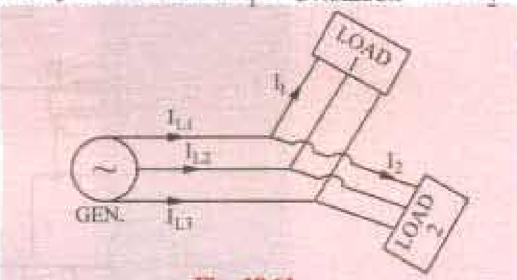


Fig. 19.61

Example 19.46. *A single-phase motor drawing 10A at 0.707 lagging power factor is connected across lines R and Y of a 3-phase supply line connected to a 3-phase motor drawing 15A at a lagging power factor of 0.8 as shown in Fig. 19.62(a). Assuming RYB sequence, calculate the three line currents.*

Solution. In the phasor diagram of Fig. 19.61 (*b*) are shown the three phase voltages and the one line voltage V_{RY} which is ahead of its phase voltage V_R. The current I_1 drawn by single-phase motor lags V_{RY} by $\cos^{-1}$ 0.707 or 45°. It lags behind the reference voltage V_R by 15° as shown. Hence, $I_1 = 10\angle -15^\circ = 9.6 - j2.6$ A. The 3-phase motor currents lag behind their respective phase voltages by $\cos^{-1}$ 0.8 or 36.9°. Hence, $I_{R1} = 15\angle -36.9^\circ = 12 - j9$.

$$I_{Y1} = 15\angle(-120^\circ - 36.9^\circ) = 15\angle -156.9^\circ = -13.8 - j5.9$$

$$I_B = 15\angle(120^\circ - 36.9^\circ) = 15\angle 83.1^\circ$$

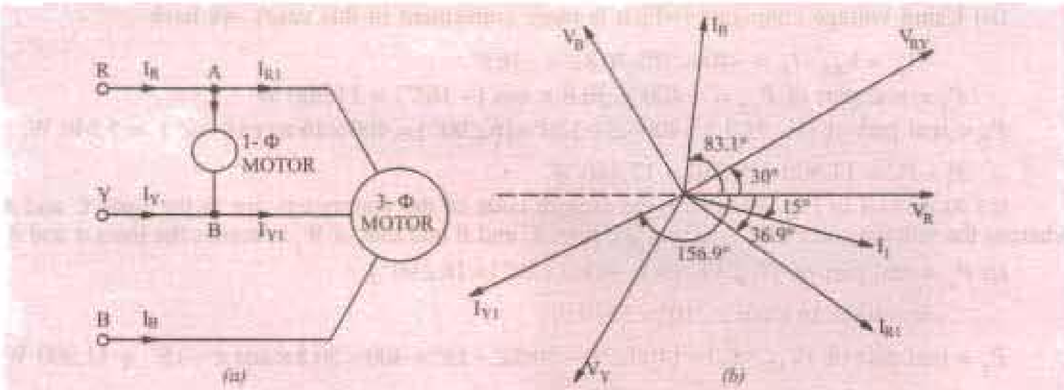


Fig. 19.62

Applying Kirchoff's laws to point A of Fig. 19.62 (a), we get

$$I_R = I_1 + I_{R1} = 9.6 - j2.6 + 12 - j9 = 21.6 - j11.6 = 24.5\angle -28.2°$$

Similarly, applying KCL to point B, we get

$$I_Y + I_1 = I_{Y1} \text{ or } I_Y = I_{Y1} - I_1 = -13.8 - j5.9 - 9.6 + j2.6 = -23.4 - j3.3 = 23.6\angle -172°.$$

Example 19.47. *A 3-ϕ, 434-V, 50-Hz, supply is connected to a 3-ϕ, Y-connected induction motor and synchronous motor. Impedance of each phase of induction motor is (1.25 + j2.17) Ω. The 3-ϕ synchronous motor is over-excited and it draws a current of 120 A at 0.87 leading p.f. Two wattmeters are connected in usual manner to measure power drawn by the two motors. Calculate (i) reading on each wattmeter (ii) combined power factor.*

(Elect. Technology, Hyderabad Univ. 1992)

Solution. It will be assumed that the synchronous motor is Y-connected. Since it is over-exicted it has a leading p.f. The wattmeter connections and phasor diagrams are as shown in Fig. 19.63.

$$\mathbf{Z}_1 = 1.25 + j2.17 = 2.5\angle 60°$$

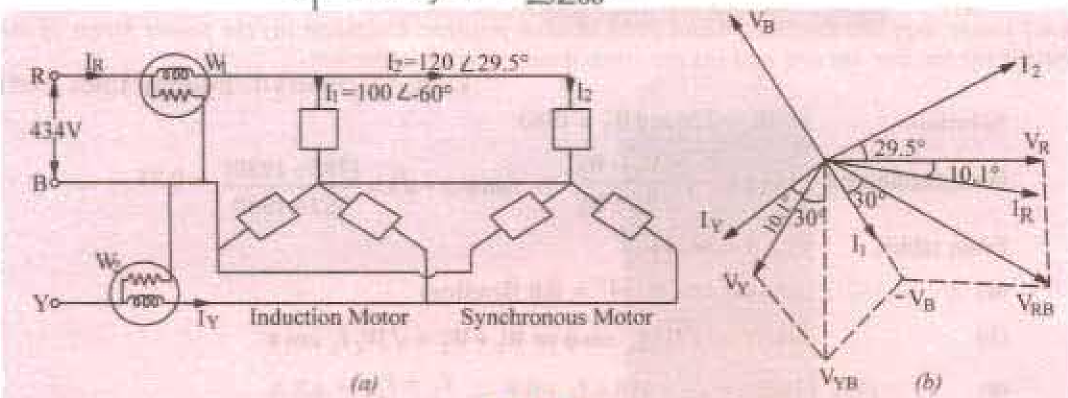


Fig. 19.63

Phase voltage in each case = $434\sqrt{3}$ = 250 V

$I_1 = 250/2.5 = 100$ A lagging the reference vector V_R by 60°. Current $I_2 = 120$ A and leads V_R by an angle = $\cos^{-1}(0.87) = 29.5°$

$$\therefore \quad \mathbf{I}_1 = 100\angle -60° = 50 - j86.6; \; \mathbf{I}_2 = 120\angle 29.5° = 104.6 + j59°$$

$$\mathbf{I}_R = \mathbf{I}_1 + \mathbf{I}_2 = 154.6 - j27.6 = 156.8\angle -10.1°$$

(*a*) As shown in Fig. 19.63 (*b*), I_R lags V_R by 10.1°. Similarly, I_Y lags V_Y by 10.1°.

As seen from Fig. 19.63 (*a*), current through W_1 is I_R and voltage across it is $V_{RB} = V_R - V_B$. As seen, V_{RB} = 434 V lagging by 30°. Phase difference between V_{RB} and I_R is = 30 − 10.1 = 19.9°.

∴ reading of W_1 = 434 × 156.8 × cos 19.9° = **63,970 W**

Current I_Y is also (like I_R) the vector sum of the line currents drawn by the two motors. It is equal to 156.8 A and lags behind its respective phase voltage V_Y by 10.1°. Current through W_2 is I_Y and voltage across it is $\mathbf{V}_{YB} = \mathbf{V}_Y - \mathbf{V}_B$. As seen, V_{YB} = 434 V. Phase difference between V_{YB} and I_Y = 30° + 10.1° = 40.1° (lag).

∴ reading of W_2 = 434 × 156.8 × cos 40.1° = **52,050 W**

(*b*) Combined p.f. = cos 10.1° = **0.9845 (lag)**

Example 19.48. *Power in a balanced 3-phase system is measured by the two-wattmeter method and it is found that the ratio of the two readings is 2 to 1. What is the power factor of the system?* **(Elect. Science-1, Allahabad Univ. 1991)**

Solution. We are given that $W_1 : W_2$ ... 2 : 1. Hence, $W_2 / W_2 = r = 1/2 = 0.5$. As seen from Art. 19.21.

$$\cos\phi = \frac{1+r}{2\sqrt{1-r+r^2}} = \frac{1+0.5}{2\sqrt{1-0.5+0.5^2}} = \mathbf{0.866\ lag}$$

Example 19.49. *A synchronous motor absorbing 50 kW is connected in parallel with a factory load of 200 kW having a lagging power factor of 0.8. If the combination has a power factor of 0.9 lagging, find the kVAR supplied by the motor and its power factor.*

(Elect. Machines, A.M.I.E. Sec B, 1989)

Solution. Load kVA = 200/0.8 = 250

Load kVAR = 250 × 0.6 = 150 (lag) [cos ϕ = 0.8 sin ϕ = 0.6]

Total combined load = 50 + 200 = 250 kW

kVA of combined load = 250/0.9 = 277.8

Combined kVAR = 277.8 × 0.4356 = 121 (inductive) (combined cos ϕ = 0.9, sin ϕ = 0.4356)

Hence, leading kVAR supplied by synch. motor = 150 − 121 = 29 (capacitive)

kVA of motor alone = $\sqrt{(kW^2 + kVAR^2)} = \sqrt{50^2 + 29^2} = 57.8$

p.f. of motor = kW/kVA = 50/57.8 = 0.865 (leading)

Example 19.50. *A star-connected balanced load is supplied from a 3-phase balanced supply with a line voltage of 416 V at a frequency of 50 Hz. Each phase of load consists of a resistance and a capacitor joined in series and the readings on two wattmeters connected to measure the total power supplied are 782 W and 1980 W, both positive. Calculate (a) the power factor of the circuit (b) the line current and (c) the capacitance of each capacitor.*

(Elect. Machinery-I, Bombay Univ. 1985)

Solution. $W_1 = 728 \text{ and } W_2 = 1980$

For a leading p.f. $\tan\phi = -\sqrt{3}\dfrac{W_1 - W_2}{W_1 + W_2}$ $\therefore$ $\tan\phi = -\sqrt{3} \times \dfrac{(782 - 1980)}{782 + 1980} = 0.75$

From tables, $\phi = 36°54'$

(*a*) $\therefore$ $\cos\phi = \cos 36°54' =$ **0.8 (leading)**

(*b*) power = $\sqrt{3}V_L I_L \cos\phi$ or $W_1 + W_2 = \sqrt{3}V_L I_L \cos\phi$

or $(782 + 1980) = \sqrt{3} \times 416 \times I_L \times 0.8$ $\therefore$ $I_L = I_{ph} =$ **4.8 A**

(*c*) Now $V_{ph} = 416/\sqrt{3}$ V $\therefore$ $Z_{ph} = 416/\sqrt{3} \times 4.8 = 50\,\Omega$

$\therefore$ In Fig. 19.64, $Z_{ph} = 50\angle -36°54' = 50(0.8 - j0.6) = 40 - j30$

Capacitive reactance X_C = 30; or $\dfrac{1}{2\pi \times 50 \times C} = 30$ $\therefore$ $C =$ **106 μF**.

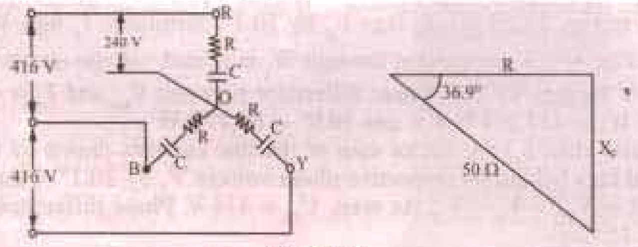


Fig. 19.64

Example 19.51. *The two wattmeters A and B, give readings as 5000 W and 1000 W respectively during the power measurement of 3-φ, 3-wire, balanced load system. (a) Calculate the power and power factor if (i) both meters read direct and (ii) one of them reads in reverse. (b) If the voltage of the circuit is 400 V, what is the value of capacitance which must be introduced in each phase to cause the whole of the power to appear on A. The frequency of supply is 50 Hz.*

(Elect. Engg-I, Nagpur Univ. 1992)

Solution. (*a*) (*i*) Both Meters Read Direct

$W_1 = 5000\text{ W}; W_2 = 1000\text{ W};$ $\therefore$ $W_1 + W_2 = 6000\text{ W}; W_1 - W_2 = 4000\text{ W}$

$$\tan\phi = \sqrt{3}(W_1 - W_2)/(W_1 + W_2) = \sqrt{3}\times 4000/6000 = 1.1547$$

$\therefore$ $\phi = \tan^{-1}(1.1547) = 49.1°$; p.f. = cos 49.1° = 0.655 (lag)

Total power = 5000 + 1000 = 6000 W

(*ii*) One Meter Reads in Reverse

In this case, $\tan\phi = \sqrt{3}(W_1 + W_2)/(W_1 - W_2) = \sqrt{3}\times 6000/4000 = 2.598$

$\therefore$ $\phi = \tan^{-1}(2.598) = 68.95°$; p.f. = cos 68.95° = 0.36 (lag)

Total power = $W_1 + W_2$ = 5000 − 1000 = 4000 W ...Art.

(*b*) The whole of power would be measured by wattmeter W_1 if the load power factor is 0.5 (lagging) or less. It means that in the present case p.f. of the load will have to be reduced from 0.655 to 0.5 In other words, capacitive reactance will have to be introduced in each phase of the load in order to partially neutralize the inductive-reactance.

Now, $\sqrt{3}V_L I_L \cos\phi = 6000$ or $\sqrt{3}\times 400\, I_L \times 0.655 = 6000$

$\therefore$ $I_L = 13.2\text{A}$; $\therefore$ $I_{ph} = 13.2/\sqrt{3} = 7.63$ A

$$Z_{ph} = V_{ph}/I_{ph} = 400/7.63 = 52.4\,\Omega$$

$$X_L = Z_{ph}\sin\phi = 52.4\times\sin 49.1° = 39.6\,\Omega$$

When p.f. = 0.5

$$\sqrt{3}\times 400\times I_L\times 0.5 = 6000; I_L = 17.32A; I_{ph} = 17.32/\sqrt{3} = 10A; Z_{ph} = 400/10 = 40\Omega$$

$$\cos\phi = 0.5; \phi = 60; \sin 60° = 0.886; X = Z_{ph}\sin\phi = 40\times 0.886 = 35.4\,\Omega$$

$\therefore$ $X = X_L - X_C = 35.4$ or $39.6 - X_C = 35.4$; $\therefore$ $X_C = 4.2\,\Omega$.

If C is the required capacitance, then $4.2 = 1/2\pi\times 50\times C$; $\therefore$ $C = 758\mu\text{F}$.

Tutorial Problems No. 19.2

1. Two wattmeters connected to measure the input to a balanced three-phase circuit indicate 2500 W and 500 W respectively. Find the power factor of the circuit (*a*) when both readings are positive and (*b*) when the latter reading is obtained after reversing the connections to the current coil of one instrument.
[(*a*) 0.655 (*b*) 0.3591] (*City & Guilds, London*)

2. A 400-V, 3-phase induction motor load takes 900 kVA at a power factor of 0.707. Calculate the kVA rating of the capacitor bank to raise the resultant power factor of the installation of 0.866 lagging.
Find also the resultant power factor when the capacitors are in circuit and the motor load has fallen to 300 kVA at 0.5 power factor. **[296 kVA, 0.998 leading]** (*City & Guilds, London*)

3. Two wattmeters measure the total power in three-phase circuits and are correctly connected. One reads 4,800 W while other reads backwards. On reversing the latter, it reads 400 W. What is the total power absorbed by the circuit and the power factor? **[4400 W; 0.49]** (*Sheffield Univ. U.K.*)

4. The power taken by a 3-phase, 400-V motor is measured by the two wattmeter method and the readings of the two wattmeters are 460 and 780 watts respectively. Estimate the power factor of the motor and the line current. **[0.913, 1.96 A]** (*City & Guilds, London*)

5. Two wattmeters, W_1 and W_2 connected to read the input to a three-phase induction motor running unloaded, indicate 3 kW and 1 kW respectively. On increasing the load, the reading on W_1 increases while that on W_2 decreases and eventually reverses.
Explain the above phenomenon and find the unloaded power and power factor of the motor.
[2 kW, 0.287 lag] (*London Univ.*)

6. The power flowing in a 3-ϕ, 3-wire, balanced-load system is measured by the two wattmeter method. The reading on wattmeter *A* is 5,000 W and on wattmeter *B* is − 1,000 W
(*a*) What is the power factor of the system?
(*b*) If the voltage of the circuit is 440, what is the value of capacitance which must be introduced into each phase to cause the whole of the power measured to appear on wattmeter A?
[0.359; 5.43 Ω] (*Meters and Meas. Insts. A.M.I.E.E. London*)

7. Two wattmeters are connected to measure the input to a 400 V; 3-phase, connected motor outputting 24.4 kW at a power factor of 0.4 (lag) and 80% efficiency. Calculate the
(*i*) resistance and reactance of motor per phase
(*ii*) reading of each wattmeters.
[(*i*) 2.55 Ω; 5.85 Ω; (*ii*) 34,915 W; − 4850 W] (*Elect. Machines, A.M.I.E. Sec. B, 1993*)

8. The readings of the two instruments connected to a balanced three-phase load are 128 W and 56 W. When a resistor of about 25 Ω is added to each phase, the reading of the second instrument is reduced to zero. State, giving reasons, the power in the circuit before the resistors were added. **[72 W]** *(London Univ.)*

9. A balanced star-connected load, each phase having a resistance of 10 Ω and inductive reactance of 30 Ω is connected to 400-*V*, 50-Hz supply. The phase rotationis read, yellow and blue. Wattmeters connected to read total power have their current coils in the red and blue lines respectively. Calculate the reading on each wattmeter and draw a vector diagram in explanation. **[2190 W, – 583 W]** *(London Univ.)*

10. A 7.46 kW induction motor runs from a 3-phase, 400-V supply. On no-load, the motor takes a line current of 4. A at a power factor of 0.208 lagging. On full load, it operates at a power factor of 0.88 lagging and an efficiency of 89 per cent. Determine the readings on each of the two wattmeters connected to read the total power on (*a*) no load and (*b*) full load. **[1070 W, – 494 W; 5500 W; 2890 W]**

11. A balanced inductive load, connected in star across 415-V, 50-Hz, three-phase mains, takes a line current of 25A. The phase sequence is *RYB*. A single-phase wattmeter has its current coil connected in the *R* line and its voltage coil across the line *YB*. With these connections, the reading is 8 kW. Draw the vector diagram and find (*i*) the kW (*ii*) the kVAR (*iii*) the kVA and (*iv*) the power factor of the load.
[(*i*) 11.45 kW (*ii*) 13.87 kVAR (*iii*) 18 kVA (iv) 0.637] *(City & Guilds, London)*

19.27. Double Subscript Notation

In symmetrically-arranged networks, it is comparatively easier and actually more advantangeous, to use single-subscript notation. But for unbalanced 3-phase circuits, it is essential to use double subscript notation, in order to avoid unnecessary confusion which is likely to result in serious errors.

Suppose, we are given two coils are whose induced e.m.fs. are 60° out of phase with each other [Fig. 19.65 (*a*)]. Next, suppose that it is required to connect these coils in additive series *i.e.*

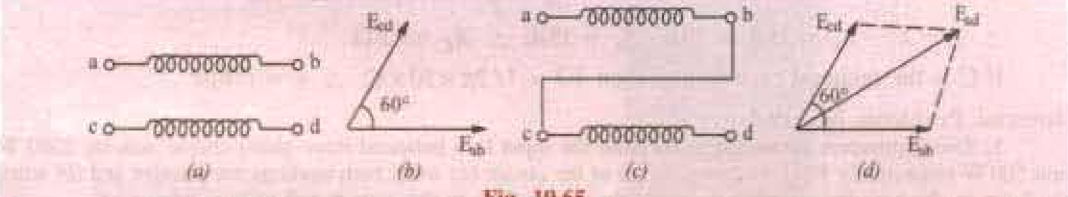


Fig. 19.65

in such a way that their e.m.fs. add at an angle of 60°. From the information given, it is impossible to know whether to connect terminal '*a*' to terminal '*c*' or to terminal '*d*'. But if additionally it were given that e.m.f. from terminal '*c*' to terminal '*d*' is 60° out of phase with that from terminal '*a*' to terminal '*b*', then the way to connect the coils is definitely fixed, as shown in Fig. 19.59 (*b*) and 19.60 (*a*). The double-subscript notation is obviously very convenient in such cases. The order in which these subscripts are written indicates the direction along which the voltage acts (or current flows). For example the e.m.f. '*a*' to '*b*' [Fig. 19.59 (*a*)], may be written as E_{ab} and that from '*c*' to '*d*' as E_{cd}..... The e.m.f. between '*a*' and '*d*' is E_{ad} where $\mathbf{E}_{ad} = \mathbf{E}_{ab} + \mathbf{E}_{bd}$ and is shown in Fig. 19.59 (*b*).

Example 19.52. *If in Fig. 19.60 (a), terminal 'b' is connected to 'd', find E_{ac} if E = 100 V.*

Solution. Vector diagram is shown in Fig. 19.60 (*b*)

Obviously, $\mathbf{E}_{ac} = \mathbf{E}_{ab} + \mathbf{E}_{dc} = \mathbf{E}_{ab}\ (-\mathbf{E}_{cd})$

Hence, $\mathbf{E}_{cd}$ is reversed and *added* to $\mathbf{E}_{ab}$ to get $\mathbf{E}_{ac}$ as shown in Fig. 19.60 (*b*). The magnitude of resultant vector is

$$E_{ac} = 2\times100\cos120°/2 = 100\,\text{V};\ \mathbf{E}_{ac} = 100\angle-60°$$

Example 19.53. *In Fig. 19.66 (a) with terminal 'b' connected to 'd', find E_{ca}.*

Solution. $\mathbf{E}_{ca} = \mathbf{E}_{cd} + \mathbf{E}_{ba} = \mathbf{E}_{cd} + (-\mathbf{E}_{ab})$

As shown in Fig. 19.67, vector $\mathbf{E}_{ab}$ is reversed and then combined with $\mathbf{E}_{cd}$ to get $\mathbf{E}_{ca}$.

Magnitude of $\mathbf{E}_{ca}$ is given by 2 × 100 × cos 60° = 100 V but it leads $\mathbf{E}_{ab}$ by 120°.

$$\therefore \qquad \mathbf{E}_{ca} = 100\angle120°$$

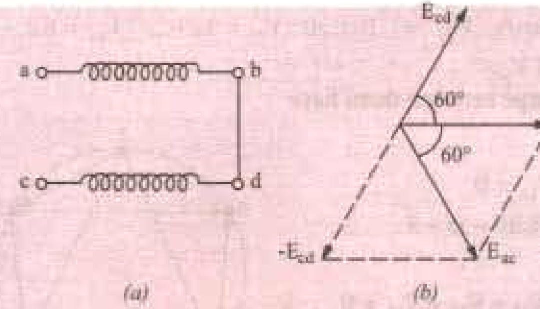


Fig. 19.66

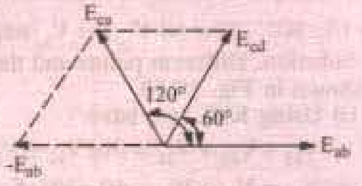


Fig. 19.67

In Fig. 19.68 (*b*) is shown the vector diagram of the e.m.fs induced in the three phases 1, 2, 3 (or *R*, *Y*, *B*) of a 3-phase alternator [Fig.19.68 (*a*)]. According to double subscript notation, each phase e.m.f. may be written as $\mathbf{E}_{01}$, $\mathbf{E}_{02}$ and $\mathbf{E}_{03}$, the order of the subscripts indicating the direction in which the e.m.fs. act. It is seen that while passing from phase 1 to phase 2 through the external circuit, we are in opposition to $\mathbf{E}_{02}$.

$$\mathbf{E}_{12} = \mathbf{E}_{20} + \mathbf{E}_{01} = (-\mathbf{E}_{02}) + \mathbf{E}_{01} = \mathbf{E}_{01} - \mathbf{E}_{02}$$

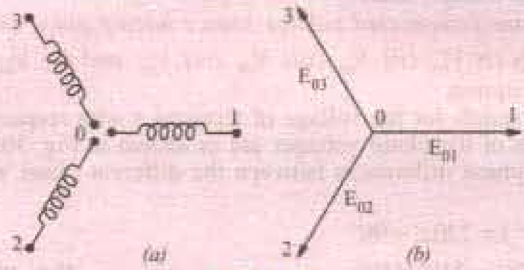


Fig. 19.68

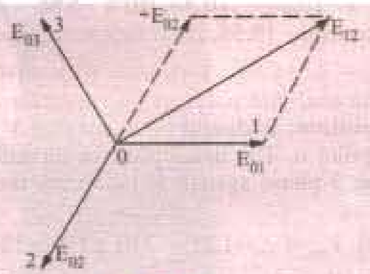


Fig. 19.69

It means that for obtaining $\mathbf{E}_{12}$, $\mathbf{E}_{20}$ has to be reversed to obtain $-\mathbf{E}_{02}$ which is then combined with $\mathbf{E}_{01}$ to get $\mathbf{E}_{12}$ (Fig. 19.69). Similarly,

$$\mathbf{E}_{23} = \mathbf{E}_{30} + \mathbf{E}_{02} = (-\mathbf{E}_{03}) + \mathbf{E}_{02} = \mathbf{E}_{02} - \mathbf{E}_{03}$$

$$\mathbf{E}_{31} = \mathbf{E}_{10} + \mathbf{E}_{03} = (-\mathbf{E}_{01}) + \mathbf{E}_{03} = \mathbf{E}_{03} - \mathbf{E}_{01}$$

By now it should be clear that double-subscript notation is based on lettering every junction and terminal point of diagrams of connections and on the use of two subscripts with all vectors representing voltage or current. The subscripts on the vector diagram, taken from the diagram of connections, indicate that the positive direction of the current or voltage is from the first subscript to the second. For example, according to this notation $\mathbf{I}_{ab}$ represents a current whose + ve direction is from *a* to *b* in the branch *ab* of the circuit in the diagram of connections. In the like manner, $\mathbf{E}_{ab}$ represents the e.m.f. which produces this current. Further, $\mathbf{I}_{ba}$ will represent a current flowing from *b* to *a*, hence its vector will be drawn equal to but in a direction opposite to that of $\mathbf{I}_{ab}$ *i.e.* $\mathbf{I}_{ab}$ and $\mathbf{I}_{ba}$ differ in phase by 180° although they do not differ in magnitude.

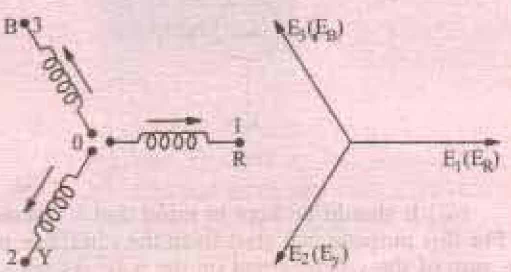


Fig. 19.69 (*a*)

In single subscript notation (*i.e.* the one in which single subscript is used) the + ve directions are fixed by putting arrows on the circuit diagrams as shown in Fig. 19.69 (*a*). According to this notation

$$\mathbf{E}_{12} = -\mathbf{E}_2 + \mathbf{E}_1 = \mathbf{E}_1 - \mathbf{E}_2;\ \mathbf{E}_{23} = -\mathbf{E}_3 + \mathbf{E}_2 = \mathbf{E}_2 - \mathbf{E}_3 \text{ and } \mathbf{E}_{31} = -\mathbf{E}_1 + \mathbf{E}_3 = \mathbf{E}_3 - \mathbf{E}_1$$

or $$\mathbf{E}_{RY} = \mathbf{E}_R - \mathbf{E}_Y;\ \mathbf{E}_{YB} = \mathbf{E}_Y - \mathbf{E}_B;\ \mathbf{E}_{BR} = \mathbf{E}_B - \mathbf{E}_R$$

Example 19.54. *Given the phasors* $V_{12} = 10\angle 30°; V_{23} = 5\angle 0°;\ V_{14} = 6\angle -60°;$ $V_{45} = 10\angle 90°$. *Find (i)* V_{13} *(ii)* V_{34} *and (iii)* V_{25}.

Solution. Different points and the voltage between them have been shown in Fig. 19.70.

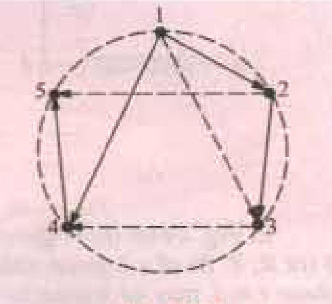


Fig. 19.70

(*i*) Using KVL, we have

$$V_{12} + V_{23} + V_{31} = 0 \text{ or } V_{12} + V_{23} - V_{13} = 0$$

or $V_{13} = V_{12} + V_{23} = 10\angle 30° + 5\angle 0° = 8.86 + j5 + 5$

$$= 13.86 + j5 = 14.7\angle 70.2°$$

(*ii*) Similarly, $V_{13} + V_{34} + V_{41} = 0$ or $V_{13} + V_{34} - V_{14} = 0$

or $V_{34} = V_{14} - V_{13} = 6\angle -60° - 14.7\angle 70.2°$

$$= 3 - j5.3 - 13.86 - j5 = -10.86 - j10.3 = 15\angle 226.5°$$

(*iii*) Similarly, $V_{23} + V_{34} + V_{45} + V_{52} = 0$

or $V_{23} + V_{34} + V_{45} - V_{52} = 0$

or $V_{25} = V_{23} + V_{34} + V_{45} = 5\angle 0° + 15\angle 226.5° + 10\angle 90°$

$$= 5 - 10.86 - j10.3 + j10 = -5.86 - j0.3 = 5.86 - \angle 2.9°.$$

Example 19.55. *In a balanced 3-phase Y-connected voltage source having phase sequence abc,* $V_{an} = 230\angle 30°$. *Calculate analytically (i)* V_{bn} *(ii)* V_{cn} *(iii)* V_{ab} *(iv)* V_{bc} *and (v)* V_{ca}. *Show the phase and line voltages on a phasor diagram.*

Solution. It should be noted that V_{an} stands for the voltage of terminal a with respect to the neutral point *n*. The usual positive direction of the phase voltages are as shown in Fig. 19.71 (*a*). Since the 3-phase system is balanced, the phase differences between the different phase voltages are 120°.

(*i*) $V_{bn} = \angle -120° = 230\ \angle(30° - 120°) = 230\angle -90°$

(*ii*) $V_{cn} = V_{an}\angle 120° = 230\ \angle(30° + 120°) = 230\angle 150°$ ... Fig. 19.71 (*b*)

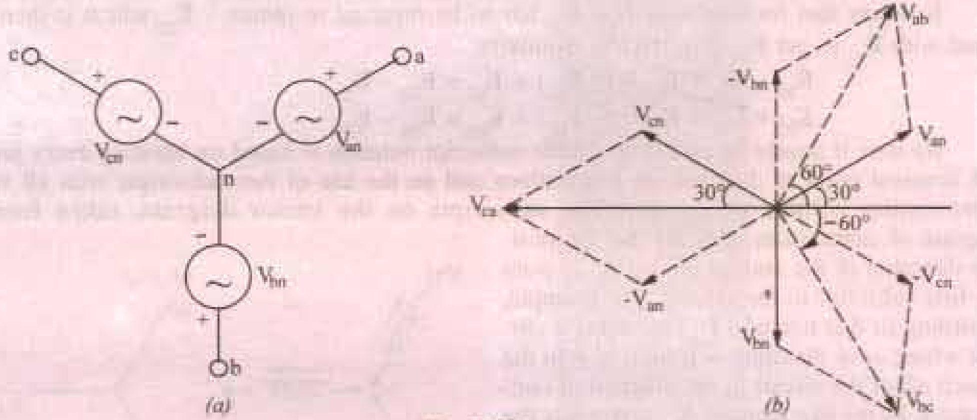


Fig. 19.71

(*iii*) It should be kept in mind that V_{ab} stands for the voltage of point a with respect to point *b*. For this purpose, we start from the reference point *b* in Fig.19.71 (*a*) and go to point *a* and find the sum of the voltages met on the way. As per sign convention given in Art, 19.27 as we go from *b* to *n*, there is a fall in voltage of by an amount equal to V_{bn}. Next as we go from *n* to *a*, there is increase of voltage given by V_{an}.

$$\therefore\ V_{ab} = -V_{bn} + V_{an} = V_{an} - V_{bn} = 230\angle 30° - 230\angle -90°$$

$$= 230(\cos 30° + j\sin 30°) - 230(0 - j\sin 90°)$$

$$= 230\left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right) + j230 = 230\left(\frac{\sqrt{3}}{2} + j\frac{3}{2}\right) = 230\sqrt{3}\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = 400\angle 60°$$

(*iv*) $V_{bc} = V_{bn} - V_{cn} = 230\angle-90° - 230\angle150° = -j230 - 230$

$$\left(-\frac{\sqrt{3}}{2} + j\frac{1}{2}\right) = 230\sqrt{3}\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) = 400\angle-60°$$

(*v*) $$\mathbf{V}_{ca} = \mathbf{V}_{cn} - \mathbf{V}_{an} = 230\angle150° - 230\angle30° = \left(-\frac{\sqrt{3}}{2} + j\frac{1}{2}\right) - 230\left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right) = -400 = 400\angle180°$$

These line voltages along with the phase voltages have been shown in the phasor diagram of Fig. 19.71 (*b*).

Example 19.56. *Three non-inductive resistances, each of 100 Ω are connected in star to a 3-phase, 440-V supply. Three equal choking coils are also connected in delta to the same supply; the resistance of one coil being equal to 100 Ω. Calculate (a) the line current and (b) the power factor of the system.* **(Elect. Technology-II, Sumbal Univ. 1987)**

Solution. The diagram of connections and the vector diagram of the *Y*-and Δ-connected impedances are shown in Fig. 19.72.

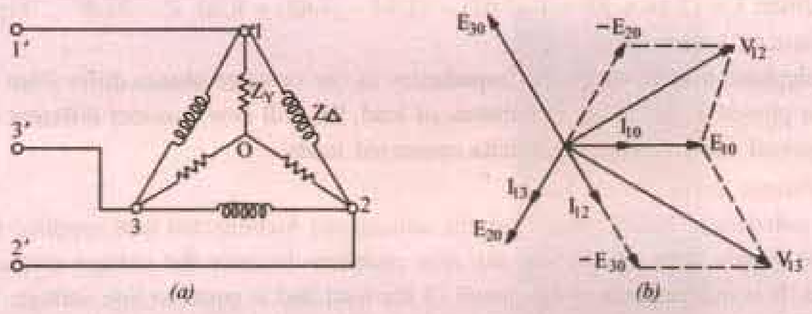


Fig. 19.72

The voltage $\mathbf{E}_{10}$ between line 1 and neutral is taken along the *X*-axis. Since the load is balanced, it will suffice to determine the current in one line only. Applying Kirchhoff's Law to junction 1, we have $\qquad \mathbf{I}'_{11} = \mathbf{I}_{10} + \mathbf{I}_{12} + \mathbf{I}_{13}$

Let us first get the vector expressions for $\mathbf{E}_{10}$, $\mathbf{E}_{20}$ and $\mathbf{E}_{30}$

$$\mathbf{E}_{10} = \frac{440}{\sqrt{3}}(1 + j0) = 254 + j0,,\ \mathbf{E}_{20} = 254\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) = -127 - j220$$

$$\mathbf{E}_{30} = 254\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = -127 + j220$$

Let us now derive vector expressions for $\mathbf{V}_{12}$ and $\mathbf{V}_{31}$.

$\mathbf{V}_{10} = \mathbf{E}_{10} + \mathbf{E}_{02} = \mathbf{E}_{10} - \mathbf{E}_{20} = (254 + j0) - (-127 - j220) = 381 + j220$

$\mathbf{V}_{13} = \mathbf{E}_{10} + \mathbf{E}_{03} = \mathbf{E}_{10} - \mathbf{E}_{30} = (254 - j0) - (-127 + j220) = 381 - j220$

$$\mathbf{I}_{10} = \frac{\mathbf{E}_{10}}{\mathbf{Z}_y} = \frac{254 + j0}{100} = 2.54 + j0.\ \mathbf{I}_{12} = \frac{\mathbf{V}_{13}}{\mathbf{Z}_\Delta} = \frac{381 + j220}{j100} = 2.2 - j3.81 = 4.4\angle-60°$$

$$= \mathbf{I}_{13} = \frac{V_{13}}{Z_\Delta} = \frac{381 - j220}{j100} = -2.2 - j3.81 = 4.4\angle-120°$$

(*a*) $\mathbf{I}'_{11} = (2.54 + j0) + (2.2 - j3.81) + (-2.2 - j3.81) = (2.54 - j7.62) = 8.03\ \angle -71.6°$

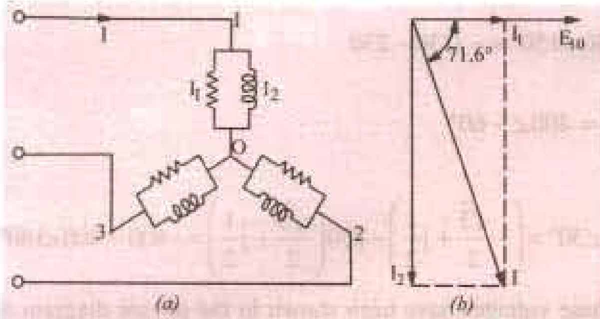


Fig. 19.73

(*b*) p.f. = cos 71.6° = **0.316 (lag)**

Alternative Method

This question may be easily solved by Δ/*Y* conversion. The star equivalent of the delta reactance is 100/3 Ω per phase.

As shown is Fig. 19.73, there are now two parallel circuits across each phase, one consisting of a resistance of 100 Ω and the other of a reactance of 100/3 Ω.

Taking E_{10} as the reference vector, we have

$$\mathbf{E}_{10} = (254 + j0)$$

$$I_1 = \frac{254 + j0}{100} = 2.54 + j0;\ I_2 = \frac{254 + j0}{j100/3} = -j7.62$$

Line current $\mathbf{I} = (2.54 + j0) + (-j7.62) = (2.54 - j7.62) = 8.03\ \angle -71.6°$ ... Fig. 19.73 (*b*)

19.28. Unbalanced Loads

Any polyphase load in which the impedances in one or more phases differ from the impedances of other phases is said to be an unbalanced load. We will now consider different methods to handle unbalanced star-connected and delta-connected loads.

19.29. Unbalanced Δ-connected Load

Unlike unbalanced *Y*-connected load, the unbalanced Δ-connected load supplied from a balanced 3-phase supply does not present any new problems because the voltage across each load phase is fixed. It is independent of the nature of the load and is equal to line voltage. In fact, the problem resolves itself into three independent single-phase circuits supplied with voltages which are 120° apart in phase.

The different phase currents can be calculated in the usual manner and the three line currents are obtained by taking the vector difference of phase currents in pairs.

If the load consists of three different pure resistances, then trigonometrical method can be used with advantage, otherwise symbolic method may be used.

Example 19.57. *A 3-phase, 3-wire, 240 volt, CBA system supplies a delta-connected load in which $Z_{AB} = Z_{AB} = 25 \angle 90°$, $Z_{BC} = 15 \angle Z_{CA} = 20 \angle 0°$ ohms. Find the line currents and total power.* **(Advanced Elect. Machines A.M.I.E. Sec. B, Summer 1991)**

Solution. As explained in Art. 19.2, a 3-phase system has only two possible sequences : *ABC* and *CBA*. In the *ABC* sequence, the voltage of phase *B* lags behind voltage of phase *A* by 120° and that of phase *C* lags behind phase *A* voltage by 240°. In the *CBA* phase with can be written as A → *C* → *B*, voltage of *C* lags behind voltage *A* by 120° and that of *B* lags behind voltage *A* by 240°. Hence, the phase voltage which can be written as

$$E_{AB} = E \angle 0°\ ;\ E_{BC} = E \angle -120°$$

and $\quad E_{CA} = E \angle -240°$ or $E_{CA} = \angle 120°$

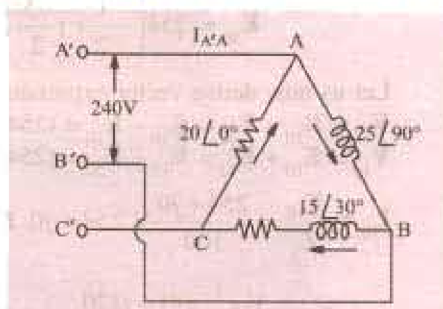


Fig. 19.74

$$\therefore \quad I_{AB} = \frac{E_{AB}}{Z_{AB}} = \frac{240\angle 0°}{25\angle 90°} = 9.6\angle -90° = -j9.6 \text{ A}$$

$$I_{BC} = \frac{E_{BC}}{Z_{BC}} = \frac{240\angle 120°}{15\angle 30°} = 16\angle 90° = j16 \text{ A}$$

$$I_{CA} = \frac{E_{CA}}{Z_{CA}} = \frac{240\angle -120°}{20\angle 0°} = 12\angle -120° = 12(0.5 - j0.866) = (-6 - j10.4) \text{ A}$$

The circuit is shown in Fig. 19.74.

Line current $I_{A'A} = I_{AB} + I_{AC} = I_{AB} - I_{CA} = -j9.6 - (-6 - j10.4) = 6 + j0.08$

Line current $I_{B'B} = I_{BC} - I_{AB} = j16 - (-j9.6) = j25.6$ A

$I_{C'C} = I_{CA} - I_{BC} = (-6 - j10.4) - j\,16 = (-5 - j26.4)$ A

Now, $\quad R_{AB} = 0;\ R_{BC} = 15 \cos 30 = 13\,\Omega;\ R_{CA} = 20\,\Omega$

Power

$W_{AB} = 0;\ W_{BC} = I_{BC}^2\,R_{BC} = 16^2 \times 13 = 3328$ W; $W_{CA} = I_{CA}^2 \times R_{CA} = 27^2 \times 20 = 14{,}580$ W.

Total Power = 3328 + 14580 = 17,908 W.

Example 19.58. *In the network of Fig. 19.75, $E_{na} = 230\angle 0°$ and the phase sequence is abc. Find the line currents I_a, I_b and I_c as also the phase currents I_{AB}, I_{BC} and I_{CA}.*

E_{na}, E_{nb}, E_{nc} is a balanced three-phase voltage system with phase sequence abc.

(Network Theory, Nagpur Univ. 1993)

Solution. Since the phase sequence is *abc*, the generator phase voltages are:

$$E_{na} = 320\angle 0°;\ E_{nb} = 230\angle -120°;\ E_{nc} = 230\angle 120°$$

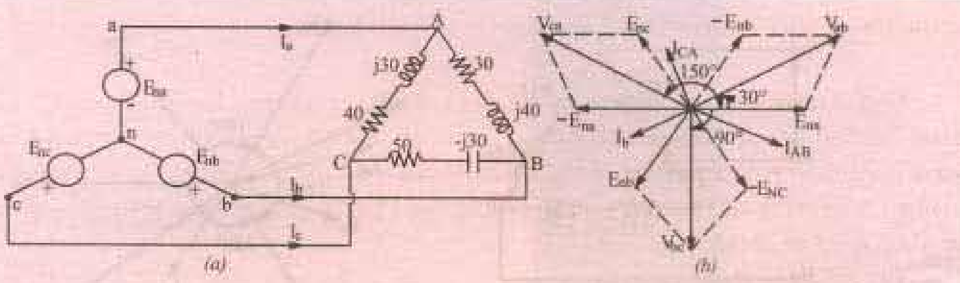


Fig. 19.75

As Seen from the phasor diagram of Fig. 19.75 (*b*), the line voltages are as under :-

$V_{ab} = E_{na} - E_{nb}$; $V_{bc} = E_{nb} = E_{nb} - E_{nc}$; $V_{ca} = E_{nc} - E_{na}$

$\therefore\ V_{ab} = \sqrt{3} \times 230\sqrt{30°} = 400\angle 30°$ *i.e* it is ahead of the reference generator phase voltage E_{na} by 30°.

$V_{bc} = \sqrt{3} \times 230\angle 90° = 400\angle -90°$. This voltage is 90° behind E_{na} but 120° behind V_{ab}.

$V_{ca} = \sqrt{3} \times 230\angle 150° = 400\angle 150°$ or $\angle -210°$. This voltage leads reference voltage E_{na} by 150° but leads V_{ab} by 120°.

These voltages are applied across the unbalanced Δ-connected lead as shown in Fig. 19.75 (*a*).

$Z_{AB} = 30 + j40 = 50\angle 53.1°;\ Z_{BC} = 50 - j30 = 58.3\angle -31°$,

$Z_{CA} = 40 + j30 = 50\angle 36.9°$

$$I_{AB} = \frac{V_{ab}}{Z_{AB}} = \frac{400\angle 30°}{50\angle 53.1°} = 8\angle -23.1° = 7.36 - j3.14$$

$$I_{BC} = \frac{V_{bc}}{Z_{BC}} = \frac{400\angle -90°}{58.3\angle -31°} = 6.86\angle -59° = 3.53 - j5.88$$

$$I_{CA} = \frac{V_{ca}}{Z_{CA}} = \frac{400\angle 150^\circ}{50\angle 36.9^\circ} = 8\angle 113.1^\circ = 3.14 + j7.36$$

$$I_a = I_{AB} - I_{CA} = 7.36 - j3.14 + 3.14 - j7.36 = 10.5 - j10.5 = 14.85\angle -45^\circ$$

$$I_b = I_{BC} - I_{AB} = 3.53 - j5.88 - 7.36 + j3.14 = -3.83 - j2.74 = 4.71\angle -215.6^\circ$$

$$I_C = I_{CA} - I_{BC} = -3.14 + j7.36 - 3.53 + j5.88 = -6.67 + j13.24 = 14.8\angle 116.7^\circ$$

Example 19.59. *For the unbalanced* Δ-connected *load of Fig. 19.76 (a), find, the phase currents, line currents and the total power consumed by the load when phase sequence is (a) abc and (b) acb.*

Solution. (*a*) **Phase sequence abc** (Fig. 19.76).

Let $\quad \mathbf{V}_{ab} = 100\angle 0^\circ = 100 + j0$

$$\mathbf{V}_{bc} = 100\angle -120^\circ = 100\left(-\frac{1}{2}\,\mathrm{j}\frac{\sqrt{3}}{2}\right) = -50 - \mathrm{j}86.6$$

$$\mathbf{V}_{ca} = 100\angle 102^\circ = 100\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = -50 + j86.6$$

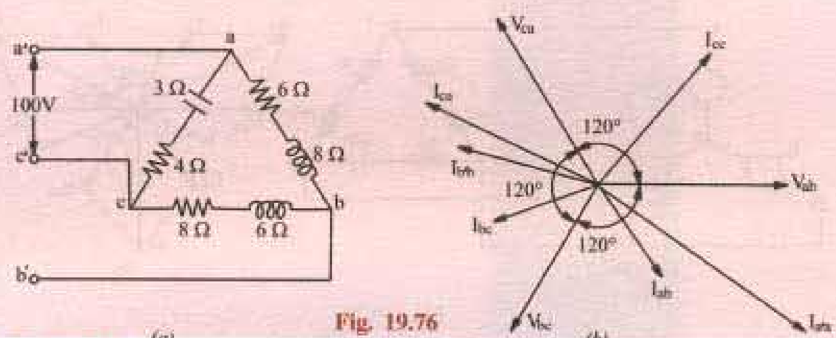


Fig. 19.76

(*i*) **Phase currents**

$$\text{Phase current, } \mathbf{I} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_{ab}} = \frac{100 + j0}{6 + j8} = 6 - j8 = 10\angle -53^\circ 8'$$

$$\text{Similarly, } \mathbf{I}_{bc} = \frac{\mathbf{V}_{bc}}{\mathbf{Z}_{bc}} = \frac{-30 - j86.6}{8 + j6} = -9.2 - j3.93 = 10\angle -156^\circ 52'$$

$$\mathbf{I}_{ca} = \frac{\mathbf{V}_{ca}}{\mathbf{Z}_{ca}} = \frac{-50 + j86.6}{4 - j3} = -18.39 + j7.86 = 20\angle 156^\circ 52'$$

(*ii*) **Line Currents**

$$\text{Line Current } \mathbf{I}_{a'a} = \mathbf{I}_{ab} + \mathbf{I}_{ac} = \mathbf{I}_{ab} - \mathbf{I}_{ca} = (6 - j8) - (-18.39 + j7.86)$$
$$= 24.39 - j15.86 = 29.1\angle -33^\circ 2'$$

$$\text{Similarly, } \mathbf{I}_{b'b} = \mathbf{I}_{bc} + \mathbf{I}_{ba} = \mathbf{I}_{bc} - \mathbf{I}_{ab} = (-9.2 - j3.93) - (6 - j8) = -15.2 + j4.07 = 15.73\angle 165^\circ 30'$$

$$\mathbf{I}_{c'c} = \mathbf{I}_{ca} + \mathbf{I}_{cb} = \mathbf{I}_{ca} - \mathbf{I}_{bc} = (-18.39 + j7.86) - (-9.2 + j3.93)$$
$$= 9.19 + j11.79 = 14.94\angle 52^\circ 3'$$

Check $\qquad \Sigma\mathbf{I} = 0 + j0$

(*iii*) **Power**

$$W_{ab} = I_{ab}^2 R_{ab} = 10^2 \times 6 = 600\text{ W}$$

$$W_{bc} = I_{bc}^2 R_{bc} = 10^2 \times 8 = 800\text{ W}$$

$$W_{ca} = I_{ca}^2 R_{ca} = 20^2 \times 4 = 1600\text{ W}$$

Total = 3000 W

(*b*) **Phase sequence acb** (Fig. 19.77)

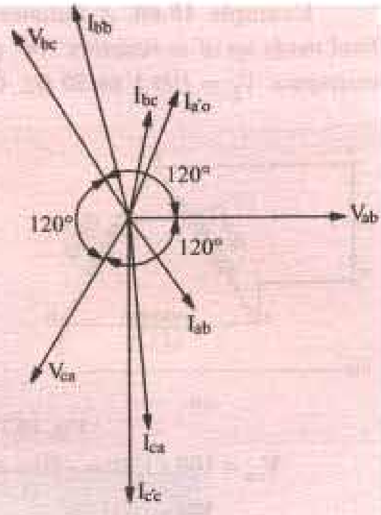


Fig. 19.77

Here, $\mathbf{V}_{ab} = 100\angle 0° = 100 + j9$

$\mathbf{V}_{bc} = 100\angle 120° = -50 + 86.6$

$\mathbf{V}_{ca} = 100\angle -120° = -50 - j86.6$

(*i*) **Phase Currents**

$$\mathbf{I}_{ab} = \frac{100}{6 + j8} = 6 - j8 = 10\angle -53°8'$$

$$\mathbf{I}_{bc} = \frac{-50 + j86.6}{8 + j6} = (1.2 + j9.93) = 10\angle 83°8'$$

$$I_{ca} = \frac{-50 + j86.6}{7 + j3} = (2.4 - j19.86) = 20 - 83°8'$$

(*ii*) **Line Currents**

$$\mathbf{I}_{a'a} = \mathbf{I}_{ab} + \mathbf{I}_{ac} = \mathbf{I}_{ab} - \mathbf{I}_{ca}$$

$$= (6 - j8) - (2.4 - j19.86) = (3.6 + j11.86) = 12.39\angle 73°6'$$

$$\mathbf{I}'_{bb} = (1.2 + j9.93) - (6 - j8) = (-4.8 + j17.93) = 18.56\angle 105°$$

$$\mathbf{I}'_{cc} = (2.4 - j19.86) - (1.2 - j9.93) = (1.2 - j29.79) = 29.9\angle -87°42'$$

It is seen that $\Sigma\mathbf{I} = 0 + j0$

(*iii*) **Power**

$$W_{ab} = 10^2 \times 6 = 600\text{W}$$

$$W_{bc} = 10^2 \times 8 = 800\text{ W}$$

$$W_{ca} = 20^2 \times 4 = 1600\text{W}$$

Total = 3000 W — as before

It will be seen that the effect of phase reversal on an unbalanced Δ-connected load is as under:

(*i*) phase currents change in angle only, their magnitudes remaining the same

(*ii*) consequently, phase powers remain unchanged

(*iii*) line currents change both in magnitude and angle.

The adjoining tabulation emphasizes the effect of phase sequence on the line currents drawn by an unbalanced 3-phase load.

Line	*Ampere Sequence* *a b c*	*Sequence* *c b a*
a	$29.1\angle -33°2'$	$12.39\angle 73.1°$
b	$15.73\angle 165°$	$18.56\angle 105°$
c	$14.94\angle 52°3'$	$29.9\angle -87.7°$

Example 19.60. *A balanced 3-phase supplies an unbalanced 3-phase delta-connected load made up of to resistors 100 Ω and a reactor having an inductance of 0.3 H with negligible resistance. V_L = 100 V at 50 Hz. Calculate (a) the total power in the system.*

(Elect. Engineering-I, Madras Univ. 1988)

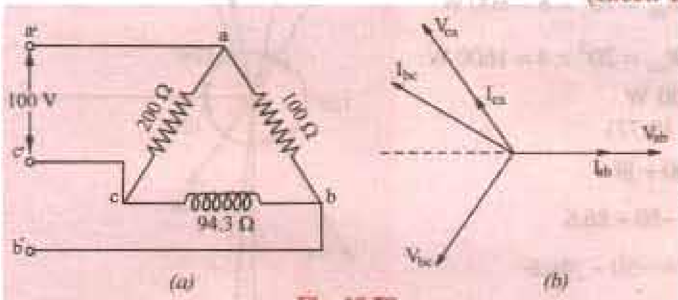


Fig. 19.78

Solution. The Δ-connected load and its phasor diagram are shown in Fig. 19.78 (*a*).

$$X_L = \omega L = 314.2 \times 0.2 = 94.3\Omega$$

Let $\mathbf{V}_{ab} = 100\angle 0° = 100 + j0$

$$\mathbf{V}_{bc} = 100\angle -120° = -50 - j86.6$$

$$\mathbf{V}_{ca} = 100\angle 120° = -50 + j86.6$$

$$\mathbf{I}_{ab} = \frac{\mathbf{V}_{ab}}{\mathbf{Z}_{ab}} = \frac{100\angle 0°}{100\angle 0°} = 13\angle 0° = 1 + j0$$

$$\mathbf{I}_{bc} = \frac{100\angle 0°}{100\angle 0°} = 1.06\angle -210° = -0.92 + j0.53$$

$$\mathbf{I}_{ca} = \frac{100\angle 120°}{200\angle 0°} = 0.5\angle 120° = -0.25 + j0.43$$

Watts in branch $ab = V_{ab}{}^2 / R_{ab} = 100^2/100$ = 100 W; VARs = 0

Watts in branch $bc = 0$; VRAs = 100 × 1.06 = 106 (lag)

Watts in branch $ca = V_{ca}{}^2 / R_{ca} = 100^2/200$ = 50 W; VRAs = 0

(a) Total power = 100 + 50 = 150 W; VARs = **106 (lag)**

19.30. Four-wire Star-connected Unbalanced Load

It is the simplest case of an unbalanced load and may be treated as three separate single-phase systems with a common return wire. It will be assumed that impedance of the line wires and source phase windings is zero. Should such an assumption be unacceptable, these impedances can be added to the load impedances. Under these conditions, source and load line terminals are at the same potential.

Consider the following two cases:

(*i*) Neutral wire of zero impedance

Because of the presence of neutral wire (assumed to behaving zero impedance), the star points of the generator and load are tied together and are at the same potential. Hence, the voltages across the three load impedances are *equalized* and each is equal to the voltage of the corresponding phase of the generator. In other words, due to the provision of the neutral, each phase voltage is a *forced* voltage so that the three phase voltages are balanced when line voltages are balanced even though phase impedances are unbalanced. However, it is worth noting that a break or open ($\mathbf{Z}_N = \infty$) in the nuetral wire of a 3-phase, 2-wire system with *unbalanced* load always causes large (in most cases inadmissible) changes in currents and phase voltages. It is because of this reason that no fuses and circuit breakers are ever used in the neutral wire of such a 3-phase system.

The solution for currents follows a pattern similar to that for the unbalanced delta.

Obviously, the vector sum of the currents in the three lines is not zero but is equal to neutral current.

(ii) Neutral wire with impedance Z_N

Such a case can be easily solved with the help of Node-pair Voltage method as detailed below. Consider the general case of a *Y*- to -*Y* system with a neutral wire of impedance Z_n as shown in Fig. 19.79 (*a*). As before, the impedance of line wires and source phase windings would be assumed to be zero so that the line and load terminals, *R,Y, B* and R', Y', B' are the same respective potentials.

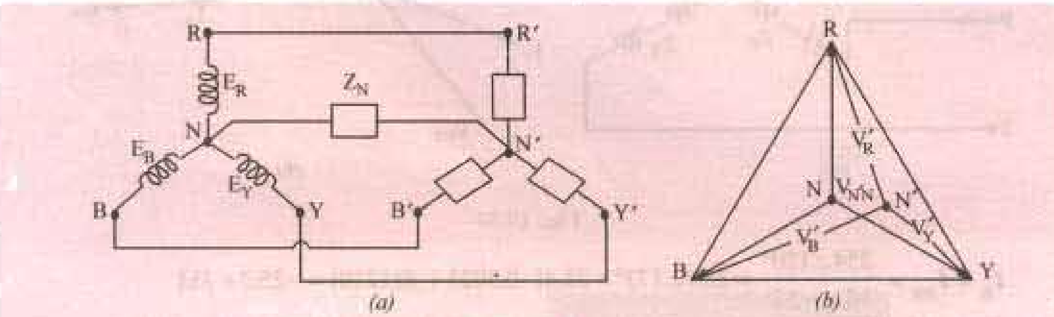


Fig. 19.79

According to Node-pair Voltage method, the above star-to-star system can be looked upon as multi-mesh network with a single pair of nodes *i.e.* neutral points *N* and *N*'. The node potential *i.e.* the potential difference between the supply and local neutrals is given by

$$V'_{NN} = \frac{E_R Y_R + E_Y Y_Y + E_B Y_B}{Y_R + Y_Y + Y_B + Y_N}$$

where $\mathbf{Y}_R$, $\mathbf{Y}_Y$ and $\mathbf{Y}_B$ represent the *load phase* admittances. Obviously, the load neutral N' does not coincide with source neutral *N*. Hence, load phase voltages are no longer equal to one another even when phase voltages are as seen from Fig. 19.79 (*b*).

The load phase voltage are given by

$$V_R' = E_R - V'_{NN}, V_Y' = E_Y - V_{NN} \text{ and } V_{B'} = E_B - V'_{NN}$$

The phase currents are

$$\mathbf{I}_R = \mathbf{V}_R'\mathbf{Y}_R,\ \mathbf{I}_Y' = \mathbf{V}_Y'\mathbf{Y}_Y \text{ and } \mathbf{I}_B = \mathbf{V}_B'\mathbf{Y}_B$$

The current in the neutral wire is $I_N = V_N\, Y_N$

Note. In the above calculations. $I_R = I_R = I_{RR}$

Similarly, $I_Y = I_Y' = I'_{YY}$ and $I_B' = I_{BB}$.

Example 19.61. *A 3-phase, 4-wire system having a 254-V line-to-neutral has the following loads connected between the respective lines and neutral; $Z_R = 10\angle 0°$ ohm; $Z_Y = 10\angle 37°$ ohm and $Z_B = 10\angle -53°$ ohm. Calculate the current in the neutral wire and the power taken by each load when phase sequence is (i) RYB and (ii) RBY.*

Solution. (*i*) **Phase sequence RYB** (Fig. 19.80)

$$V_{RN} = 254\angle 0°;\ V_{YN} = 254\angle -120°;\ V_{BN} = 254\angle 120°$$

$$\mathbf{I}_R = \mathbf{I}_{RN} = \frac{\mathbf{V}_{RN}}{\mathbf{R}_R} = \frac{254\angle 0°}{10\angle 0°} = 25.4\angle 0\,*$$

$$\mathbf{I}_r = \mathbf{I}_{YN} = \frac{254\angle -120°}{10\angle 37°} = 25.4\angle -157° = 25.4(-0.9205 - j0.3907) = -23.38 - j9.95$$

*This method is similar to Millman's Theorem of Art. 19.32.

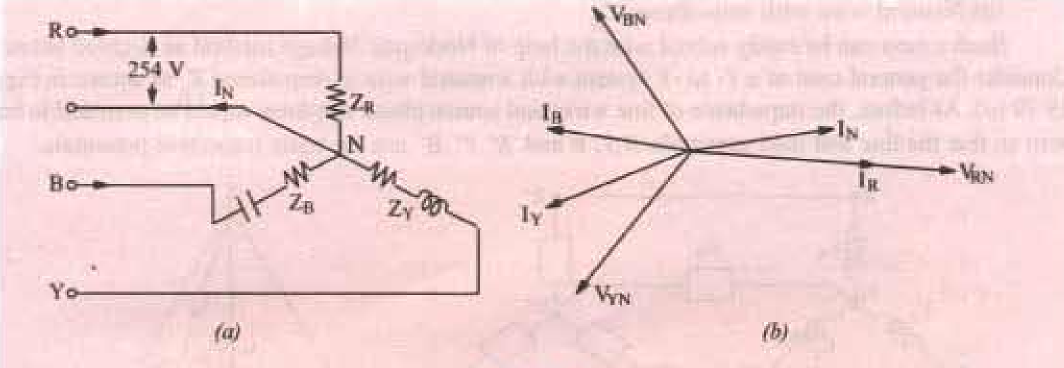


Fig. 19.80

$$I_B = I_{BN} = \frac{254\angle 120°}{10\angle -53°} = 25.4\angle 173° = 25.4(-0.9925 + j0.1219) = -25.2 + j3.1$$

$$I_N = -(I_R + I_Y + I_B) = -[25.4 + (-23.38 - j9.55) + (-25.21 + j3.1)] = 23.49 + j6.85$$
$$= 24.46\angle 16°15'$$

Now $\quad R_R = 10\,\Omega;\ R_Y = 10\cos 37° = 8\,\Omega;\ R_B = 10\cos 53° = 6\,\Omega$

$$W_R = 25.4^2 \times 10 = \mathbf{6.452\,W};\ W_Y = 25.4^2 \times 8 = \mathbf{5{,}162\,W}$$

$$W_B = 25.4^2 \times = \ 3{,}871\ \text{W}$$

(ii) **Phase sequence RBY** [Fig. 19.81]

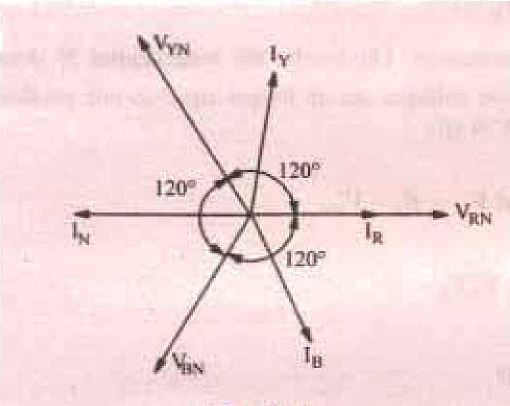


Fig. 19.81

$$V_{RN} = 254\angle 0°;\ V_{YN} = 254\angle 120°$$

$$V_{BN} = 254\angle -120°$$

$$I_R = \frac{254\angle 0°}{10\angle 0°} = 25.4\angle 0°$$

$$I_Y = \frac{254\angle 120°}{10\angle 37°} = 25.4\angle 83°$$
$$= (3.1 + j25.2)$$

$$I_B = \frac{254\angle -120°}{10\angle -53°} = 25.4\angle -67° = (9.95 - j23.4)$$

$$\mathbf{I}_N = -(\mathbf{I}_R + \mathbf{I}_Y + \mathbf{I}_B) = -(38.45 + j1.8)$$
$$= -38.45 - j1.8 = 38.5\angle -177.3°\ .$$

Obviously, power would remain the same because magnitude of branch currents is unaltered. From the above, we conclude that phase reversal in the case of a 4-wire unbalanced load supplied from a balanced voltage system leads to the following changes:

(i) it changes the angles of phase currents but not their magnitudes.

(ii) however, power remains unchanged.

(iii) it changes the magnitude as well as angle of the neutral current $\mathbf{I}_N$.

Example 19.62. *A $3-\phi$, 4-wire, 380-V supply is connected to an unbalanced load having phase impedances of: ZR = (8 + j6) Ω, Z_Y = (8 – j6) Ω and Z_B = 5 Ω. Impedance of the neutral wire is Z_N = (0.5 + j1) Ω.*

Ignoring the impedances of line wires and internal impedances of the e.m.f. sources, find the phase currents and voltages of the load.

Solution. This question will be solved by using Node-pair Voltage method discussed in Art. 19.30. The admittances of the various branches connected between nodes *N* and *N'* in Fig. 19.82 (a).

$$\mathbf{Y}_R = 1/\mathbf{Z}_R = 1/(8 + j6) = (0.08 - j0.06)$$
$$\mathbf{Y}_Y = 1/\mathbf{Z}_Y = 1/(8 - j6) = (0.08 + j0.06)$$
$$\mathbf{Y}_B = 1/2/\mathbf{Z}_B = 1/(5 + j0) = 0.2$$
$$\mathbf{Y}_N = 1/\mathbf{Z}_N = 1/(0.5 + j1) = (0.4 - j0.8)$$

Let $\mathbf{E}_R = (380/\sqrt{3})\angle 0° = 220\angle 0° = 220 + j0$

$$\mathbf{E}_Y = 220\angle -120° = 220(-0.5 - j0.866) = -110 - j190$$
$$\mathbf{E}_B = 220\angle 120° = 220(-0.5 + j0.866) = -110 + j190$$

The node voltage between N' and N is given by

$$V'_{NN} = \frac{E_R Y_R + E_Y Y_Y + E_B Y_B}{Y_R + Y_Y + Y_B + Y_N}$$
$$= \frac{200(0.08 - j0.06) + (-110 - j190)(0.08 + j0.06) + (-110 + j190) \times 0.2}{(0.08 - j0.06) + (0.08 + j0.06) + 0.2 + (0.4 - j0.8)}$$
$$= \frac{-1.8 + j3}{0.76 - j0.8} = -3.41 + j0.76$$

The three load phase voltages are as under:

$$\mathbf{V}'_R = \mathbf{E}_R - \mathbf{V}'_{NN} = 220 + 3.41 - j0.76 = 223.41 - j0.76$$
$$\mathbf{V}'_Y = \mathbf{E}_Y - \mathbf{V}'_{NN} = (-110 - j90) + 3.41 - j0.76 = -106.59 - j190.76$$
$$V'_B = E_B - V'_{NN} = (-110 - j90) + 3.41 - j0.76 = -106.59 + j190.76$$

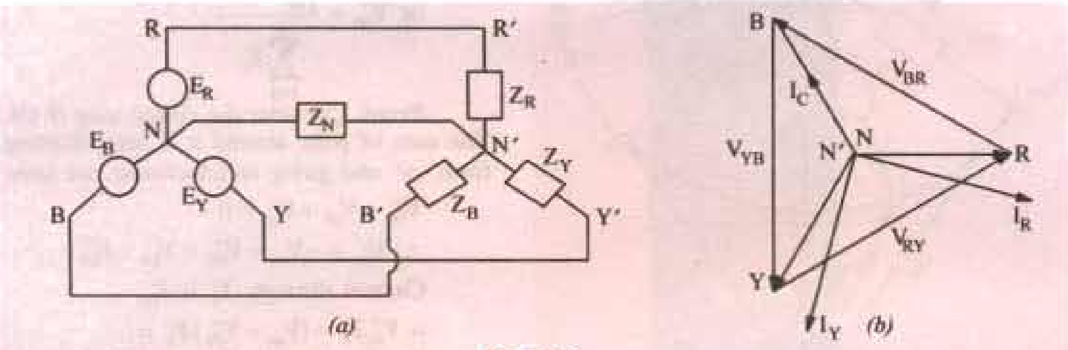


Fig 19.82

$$I'_R = V'_R Y_R = (223.41 - j0.76)(0.08 - j0.06) = 17.83 - j13.1 = 22.1\angle -36.3°\text{ A}$$
$$I'_Y = V'_Y Y_Y = (-106.59 + j190.76)(0.08 + j0.06) = 2.92 - j21.66 = 21.86\angle 82.4°\text{ A}$$
$$I'_B = V'_B Y_B = (-106.59 + j190.76) \times 0.2 = -21.33 + j37.85 = 43.45\angle 119.4°\text{ A}$$
$$I_N = V_N Y_N = (-3.41 + j0.76)(0.4 - j0.8) = -0.76 - j3.03 = 3.12\angle 104.1°\text{ A}$$

These voltage and currents are shown in the phasor diagram of Fig. 19.82 (b) where displacement of the neutral point has not been shown due to the low value of V'_{NN}.

Note. It can be shown that $I_N = I'_R + I'_Y + I'_B$

19.31. Unbalanced Y-connected Load Without Neutral

When a star-connected load is unbalanced and it has no neutral wire. Then its star point is isolated from the star-point of the generator. The potential of the load star-point is different from that of the generator star-point. The potential of the former is subject to variations according to the imbalance of the load and under certain conditions of loading, the potentials of the two star- point may differ considerably. Such an isolated load star-point or neutral point is called 'floating' neutral point because its potential is always changing and is not fixed.

All *Y*-conected unbalanced loads supplied from polyphase systems have floating neutral points without a neutral wire. Any unbalancing of the load causes variations not only of the potential of the star-point but also of the voltages across the different branches of the load. *Hence, in that case, phase voltage of the load is not* $1/\sqrt{3}$ *of the line voltage.*

There are many methods to tackel such unbalanced Y-connected loads having isolated neutral points.

1. By converting the *Y*-connected load to an equivalent unbalanced Δ-connected load by using Y-Δ conversion theorem. The equivalent Δ-connection can be solved in Fig. 19.80. The line currents so calculated are equal in magnitude and phase to those taken by the original unbalanced *Y*-connected load.
2. By applying Kirchhoff's Laws.
3. By applying Millman's Theorem.
4. By using Maxwell's Mesh or Loop Current Method.

19.32 Millman's Theorem

Fig. 19.83 shows a number of linear bilateral admittances, Y_1, Y_2, ... connected to a common point or node O'. The voltages of the free ends of these admittances with respect to another common point O are $V_{10'}$ V_{20} ... $V_{no'}$. Then, according to this theorem, the voltage of O' with respect to O is given by $\mathbf{V}'_{00} = \dfrac{\mathbf{V}_{10}\mathbf{Y}_1 + \mathbf{V}_{20}\mathbf{Y}_2 + \mathbf{V}_{30}\mathbf{Y}_3 + ..\mathbf{V}_{no}\mathbf{Y}_n}{\mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + ...\mathbf{Y}_n}$.

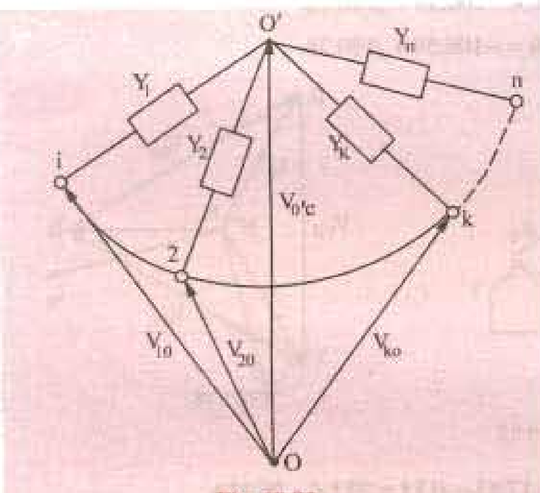


Fig. 19.83

$$\text{or } V'_{00} = \frac{\sum_{k=1}^{n} V_{k0} Y_k}{\sum_{k=1}^{n} Y_k}$$

Proof. Consider the closed loop 0' *Ok*. The sum of p.ds. around it is zero. Starting from O' and going anticlockwise, we have

$$V'_{00} + V_{ok} + V'_{ko} = 0$$

$$\therefore\ V'_{ko} = -V_{ok} - V'_{00} = V_{ko} - V'_{00}$$

Current through Y_k is I'_{ko}

$$= V'_{ko} Y_k = (V_{ko} - V'_{00}) Y_k =$$

By Kirchhoff's Current Law, sum of currents meeting at point O' is zero.

$$\therefore\ I'_{10} + I'_{20} + ... I'_{ko} .... + I'_{no} = 0$$

$$(\mathbf{V}'_{10} - \mathbf{V}'_{00})\mathbf{Y}_1 + (\mathbf{V}_{20} - \mathbf{V}'_{00})\mathbf{Y}_2 + ...(\mathbf{V}_{ko} - \mathbf{V}'_{00})\mathbf{Y}_k + ... = 0$$

$$\text{or}\quad \mathbf{V}'_{10}\mathbf{Y}_1 + \mathbf{V}_{20}\mathbf{Y}_2 + ...\mathbf{V}_{ko}\mathbf{Y}_k + ... = \mathbf{V}'_{00}(\mathbf{Y}_1 + \mathbf{Y}_2 + ... + \mathbf{Y}_k + ...)$$

$$V'_{00} = \frac{V_{10}Y_1 + V_{20}Y_2 + ...}{Y_1 + Y_2 + ...}$$

19.33. Application of Kirchhoff's Laws

Consider the unbalanced *Y*-connected load of Fig. 19.84. Since the common point of the three load impedances is not at the potential of the neutral, it is marked 0' instead of *N**. Let us assume the phase sequence sequence V_{ab}, V_{bc}, V_{ca} *i.e.* V_{ab} leads V_{bc} and V_{bc} leads V_{ca}. Let the three branch impedances be Z_{ca}, Z_{ab} and Z_{ac}, however, since double subscript notation is not necessary for a *Y*-connected impedances in order to indicate to which phase it belongs, single-subscript notation may be used with advantage. Therefore Z_{oa}, Z_{ab}, Z_{oc} can be written as Z_a, Z_b, C_z respectively. It may be

*For the sake of avoiding printing difficulties, we will take the load star point as 0 instead of 0' for this article.

pointed out that double-subscript notation is essential for mesh-connected impedances in order to make them definite.

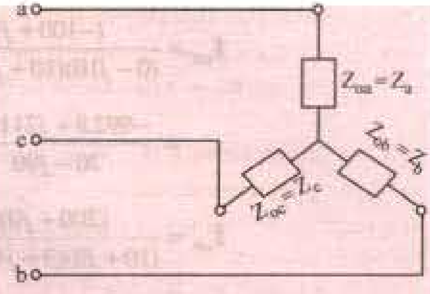


Fig. 19.84

From Kirchoff's laws, we obtain

$$V_{ab} = I_{ao}Z_a + I_{ab}Z_b \quad \text{... (1)}$$

$$V_{bc} = I_{bo}Z_b + I_{oc}Z_c \quad \text{... (2)}$$

$$V_{ca} = I_{co}Z_c + I_{oa}Z_a \quad \text{... (3)}$$

and $I_{ao} + I_{bo} + I_{co} = 0$ – point law ... (4)

Equation (1), (3) and (4) can be used for finding I_{bo}.

Adding (1) and (3), we get

$$V_{ab} + V_{ca} = I_{ao}Z_a + I_{ao}Z_b + I_{co}Z_c + I_{oc}Z_a$$

$$= I_{ob}Z_b + I_{co}Z_c + I_{oa}Z_a - I_{oa}Z_a = I_{ob}Z_b + I_{co}Z_c \quad \text{... (5)}$$

Substituting $\mathbf{I}_{oa}$ from equation (4) in equation (3), we get

$$V_{ca} = I_{co}Z_c + (I_{bo} + I_{co})Z_a = I_{co}(Z_c + Z_a) + I_{bo}Z_a \quad \text{... (6)}$$

Putting the value of I_{co} from equation (5) in equation (6), we have

$$V_{ca} = (Z_c + Z_a)\frac{(V_{ab} + V_{ca}) - I_{ab}Z_b}{Z_c} + I_{bo}Z_a$$

$$V_{ca}Z_c = -I_{ob}Z_bZ_c - I_{ob}Z_bZ_a + I_{bo}Z_aZ_c + V_{ab}Z_a + V_{ab}Z_c + V_{ca}Z_c + V_{ca}Z_a$$

$$\therefore \quad I_{ob} = \frac{(V_{ab} + V_{ca})Z_a + V_{ab}Z_a}{Z_aZ_b + Z_bZ_c + Z_cZ_a}$$

Since $V_{ab} + V_{bc} + V_{ca} = 0$ $\quad \therefore \quad I_{ob} = \dfrac{V_{ab}Z_c - V_{bc}Z_a}{Z_aZ_b + Z_bZ_c + Z_cZ_a}$ ... (7)

From the symmetry of the above equation, the expressions for the other branch currents are,

$$I_{oc} = \frac{V_{bc}Z_a - V_{ca}Z_b}{Z_aZ_b + Z_bZ_c + Z_cZ_a} \quad \text{... (8)} \qquad I_{oa} = \frac{V_{ca}Z_b - V_{ab}Z_c}{Z_aZ_b + Z_bZ_c + Z_cZ_a} \quad \text{... (9)}$$

Note. Obviously, the three line currents can be written as

$$I_{ao} = -I_{oa} = \frac{V_{ab}Z_c - V_{ca}Z_b}{\sum Z_aZ_b} \quad \text{... (10)} \qquad I_{bo} = -I_{ob}\frac{V_{bc}Z_a - V_{ab}Z_c}{\sum Z_aZ_b} \quad \text{... (11)}$$

$$I_{co} = -I_{oc}\frac{V_{ca}Z_b - V_{bc}Z_a}{\sum Z_aZ_b} \quad \text{... (12)}$$

Example 19.63. *If in the unbalanced Y-connected load of Fig. 19.78, $Z_a = (10 + j0)$, $Z_b = (3 + j4)$ and $Z_c = (0 - j10)$ and the load is put across a 3-phase, 200-V circuit with balanced voltages, find the three line currents and voltages across each branch impedance. Assume phase sequence of V_{ab}, V_{bc}, V_{ca}.*

Solution. Take V_{ab} along the axis of reference. The vector expressions for the three voltages are

$$V_{ab} = 200 + j0$$

$$V_{bc} = 200\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) = -100 - j173.2; \quad V_{ca} = 200\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = -100 + j173.2$$

From equation (9) given above

$$\mathbf{I}_{oa} = \frac{(-100 + j173.2)(3 + j4) - (200 + j0)(0 - j10)}{(0 - j10)(10 + j0) + (10 + j0)(3 + j4) + (3 + j4)(0 - j10)}$$

$$= \frac{-992.8 + j2119.6}{70 - j90} = -20.02 + j4.54$$

$$\mathbf{I}_{ob} = \frac{(200 + j0)(0 - j10) - (-100 - j173.2)(10 + j0)}{(10 + j0)(3 + j4) + (3 + j4)(0 - j10) + (0 - j10)(10 + j0)}$$

$$= \frac{1000 - j268}{70 - j90} = 7.24 + j5.48$$

Now, $\mathbf{I}_{oc}$ may also be calculated in the same way or it can be found easily from equation (4) of Art. 19.33.

$$\mathbf{I}_{oc} = \mathbf{I}_{ao} + \mathbf{I}_{bo} = -\mathbf{I}_{oa} - \mathbf{I}_{ob} = 20.02 - j4.54 - 7.24 - j5.48 = 12.78 - j10.02$$

Now $\quad \mathbf{V}_{oa} = \mathbf{I}_{oa}\mathbf{Z}_a = (-20.02 + j4.54)(10 + j0) = 200.2 + j45.4$

$\mathbf{V}_{ob} = \mathbf{I}_{ob}\mathbf{Z}_b = (7.24 + j5.48)(3 + j4) = -0.2 + j45.4$

$\mathbf{V}_{oc} = \mathbf{I}_{oc}\mathbf{Z}_c = (12.78 - j10.02)(0 - j10) = -100.2 - j127.8$

As a check, we may combine $\mathbf{V}_{oa}$, $\mathbf{V}_{ob}$ and $\mathbf{V}_{oc}$ to get the line voltages which should be equal to the applied line voltages. In passing from *a* to *b* through the circuit internally, we find that we are in opposition to $\mathbf{V}_{oa}$ but in the same direction as the positive direction of $\mathbf{V}_{ob}$.

$$\mathbf{V}_{ab} = \mathbf{V}_{ao} + \mathbf{V}_{ob} = -\mathbf{V}_{oa} + \mathbf{V}_{ob} = -(-200.2 + j45.4) + (-0.2 + j45.4) = 200 + j0$$

$$\mathbf{V}_{bc} = \mathbf{V}_{bo} + \mathbf{V}_{oc} = -\mathbf{V}_{ob} + \mathbf{V}_{oc} = -(-0.2 + j45.4) + (-100.2 - j127.8) = -100 - j173.2$$

$$\mathbf{V}_{ca} = \mathbf{V}_{co} + \mathbf{V}_{oa} = -\mathbf{V}_{oc} + \mathbf{V}_{oa} = -(-100.2 + j127.8) + (-200.2 + j45.4) = -100 + j173.2$$

19.34. Delta/Star and Star/Delta Conversions

Let us consider the unbalanced Δ-connected load of Fig. 19.85 (*a*) and *Y*-connected load of Fig. 19.85 (*b*). If the two systems are to be equivalent, then the impedances between corresponding pairs of terminals must be the same.

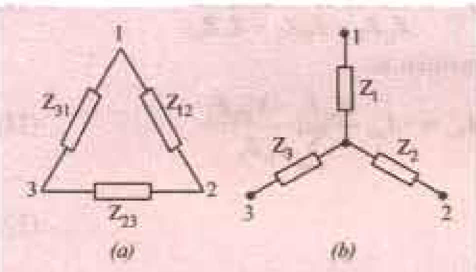


Fig. 19.85

(*i*) Delta/Star Conversion

For *Y*-load, total impedance between terminals 1 and 2 is $= \mathbf{Z}_1 + \mathbf{Z}_2$ (it should be noted that double subscript notation of $\mathbf{Z}_{01}$ and $\mathbf{Z}_{02}$ has been purposely avoided).

Considering terminals 1 and 2 of Δ-load, we find that there are two parallel paths having impedances of $\mathbf{Z}_{12}$ and $(\mathbf{Z}_{31} + \mathbf{Z}_{23})$. Hence, the equivalent impedance between terminals 1 and 2 is given by

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_{12}} + \frac{1}{\mathbf{Z}_{23} + \mathbf{Z}_{31}} \quad \text{or} \quad \mathbf{Z} = \frac{\mathbf{Z}_{12}(\mathbf{Z}_{23} + \mathbf{Z}_{31})}{\mathbf{Z}_{12} + \mathbf{Z}_{23} + \mathbf{Z}_{31}}$$

Therefore, for equivalence between the two systems $\mathbf{Z}_1 + \mathbf{Z}_2 = \dfrac{\mathbf{Z}_{12}(\mathbf{Z}_{23} + \mathbf{Z}_{31})}{\mathbf{Z}_{12} + \mathbf{Z}_{23} + \mathbf{Z}_{31}}$ ... (1)

Similarly $\mathbf{Z}_2 + \mathbf{Z}_3 = \dfrac{\mathbf{Z}_{23}(\mathbf{Z}_{31} + \mathbf{Z}_{12})}{\mathbf{Z}_{12} + \mathbf{Z}_{23} + \mathbf{Z}_{31}}$ ... (2) $\quad \mathbf{Z}_3 + \mathbf{Z}_1 = \dfrac{\mathbf{Z}_{31}(\mathbf{Z}_{12} + \mathbf{Z}_{23})}{\mathbf{Z}_{12} + \mathbf{Z}_{23} + \mathbf{Z}_{31}}$ ... (3)

Adding equation (3) to (1) and substracting equation (2), we get

$$2\mathbf{Z}_1 = \frac{\mathbf{Z}_{12}(\mathbf{Z}_{23}+\mathbf{Z}_{31})+\mathbf{Z}_{31}(\mathbf{Z}_{12}+\mathbf{Z}_{23})-\mathbf{Z}_{23}(\mathbf{Z}_{31}+\mathbf{Z}_{12})}{\mathbf{Z}_{12}+\mathbf{Z}_{23}+\mathbf{Z}_{31}} = \frac{2\mathbf{Z}_{12}\mathbf{Z}_{31}}{\mathbf{Z}_{12}+\mathbf{Z}_{23}+\mathbf{Z}_{31}}$$

$$\therefore \quad \mathbf{Z}_1 = \frac{\mathbf{Z}_{12}\mathbf{Z}_{31}}{\mathbf{Z}_{12}+\mathbf{Z}_{23}+\mathbf{Z}_{31}} \quad \text{... (4)}$$

The other two results may be written down by changing the subscripts cyclically

$$\therefore \quad \mathbf{Z}_2 = \frac{\mathbf{Z}_{23}\mathbf{Z}_{12}}{\mathbf{Z}_{12}+\mathbf{Z}_{23}+\mathbf{Z}_{31}}; \quad \text{... (5)} \quad \mathbf{Z}_3 = \frac{\mathbf{Z}_{31}\mathbf{Z}_{23}}{\mathbf{Z}_{12}+\mathbf{Z}_{23}+\mathbf{Z}_{31}} \quad \text{... (6)}$$

The above expression can be easily obtained by remembering that (Art. 2.19)

$$\text{Start } \mathbf{Z} = \frac{\text{Product of } \Delta\mathbf{Z}\text{'s connected to the same terminals}}{\text{Sum of } \Delta\mathbf{Z}\text{'s}}$$

In should be noted that all **Z**' are to be expressed in their complex form.

(iii) Star/Delta Conversion

The equations for this conversion can be obtained by rearranging equations (4), (5) and (6). Rewriting these equations, we get

$$\mathbf{Z}_1(\mathbf{Z}_{12}+\mathbf{Z}_{23}+\mathbf{Z}_{31}) = \mathbf{Z}_{12}\mathbf{Z}_{31} \quad \text{... (7)}$$
$$\mathbf{Z}_2(\mathbf{Z}_{12}+\mathbf{Z}_{23}+\mathbf{Z}_{31}) = \mathbf{Z}_{23}\mathbf{Z}_{12} \quad \text{... (8)}$$
$$\mathbf{Z}_3(\mathbf{Z}_{12}+\mathbf{Z}_{23}+\mathbf{Z}_{31}) = \mathbf{Z}_{31}\mathbf{Z}_{23} \quad \text{... (9)}$$

Dividing equation (7) by (9), we get $\frac{\mathbf{Z}_1}{\mathbf{Z}_3} = \frac{\mathbf{Z}_{12}}{\mathbf{Z}_{23}} \quad \therefore \mathbf{Z}_{23} = \mathbf{Z}_{12}\frac{\mathbf{Z}_3}{\mathbf{Z}_1}$

Dividing equation (8) by (9), we get $\frac{\mathbf{Z}_2}{\mathbf{Z}_3} = \frac{\mathbf{Z}_{12}}{\mathbf{Z}_{31}} \quad \therefore \mathbf{Z}_{31} = \mathbf{Z}_{12}\frac{\mathbf{Z}_3}{\mathbf{Z}_2}$

Substituting these values in equation (7), we have $Z_1\left(Z_{12}+Z_{12}\frac{Z_3}{Z_1}+Z_{31}\right) = Z_{12}.Z_{12}\frac{Z_3}{Z_2}$

$$\text{or} \quad \mathbf{Z}_1\mathbf{Z}_{12}\left(1+\frac{\mathbf{Z}_3}{\mathbf{Z}_1}+\frac{\mathbf{Z}_{31}}{\mathbf{Z}_{12}}\right) = \mathbf{Z}_{12}\mathbf{Z}_{12}\frac{\mathbf{Z}_3}{\mathbf{Z}_2} \quad \therefore \mathbf{Z}_1\mathbf{Z}_2+\mathbf{Z}_2\mathbf{Z}_3+\mathbf{Z}_3\mathbf{Z}_1 = \mathbf{Z}_{12}\times\mathbf{Z}_3$$

$$\mathbf{Z}_{12} = \frac{\mathbf{Z}_1\mathbf{Z}_2+\mathbf{Z}_2\mathbf{Z}_3+\mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_3} \text{ or } \mathbf{Z}_{12} = \mathbf{Z}_1+\mathbf{Z}_2+\frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_3}$$

Similarly, $\mathbf{Z}_{23} = \mathbf{Z}_2+\mathbf{Z}_3+\frac{\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_1} = \frac{\mathbf{Z}_1\mathbf{Z}_2+\mathbf{Z}_2\mathbf{Z}_3+\mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_1}$

$$\mathbf{Z}_{31} = \mathbf{Z}_3+\mathbf{Z}_1+\frac{\mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_2} = \frac{\mathbf{Z}_1\mathbf{Z}_2+\mathbf{Z}_2\mathbf{Z}_3+\mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_2}$$

As in the previous case, it is to be noted that all impedances must be expressed in their complex form.

Another point for noting is that *the line currents of this equivalent delta are the currents in the phases of the Y-connected load.*

Example 19.64. *An unbalanced star-connected load has branch impedances of $Z_1 = 10\angle 30°\ \Omega$, $Z_2 = 10\angle -45°\ \Omega$, $Z_3 = 20\angle 60°\ \Omega$ and is connected across a balanced 3-phase, 3-wire supply of 200 V. Find the line currents and the voltage across each impedance using Y/Δ conversion method.*

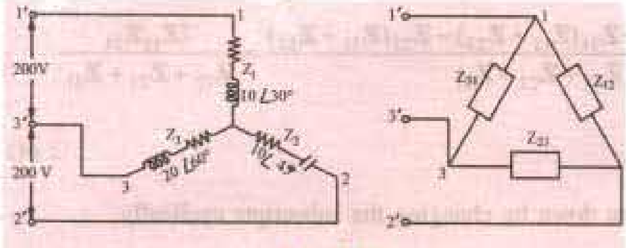


Fig. 19.86

Solution. The unbalanced *Y*-connected load and its equivalent Δ-connected load are shown in Fig. 19.86.

Now $\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1 = (10 \angle 30°)(10 \angle -45°) + (10 \angle -45°)(20 \angle 60°) + (20 \angle 60°)(10 \angle 30°) = 100 \angle -15° + 200 \angle 15° + 200 \angle 90°$

Converting these into their cartesian form, we get

$$= 100\,[\cos(-15°) - j \sin 15°] + 200\,(\cos 15° + j \sin 15°) + 200\,(\cos 90° + j \sin 90°)$$
$$= 96.6 - j25.9 + 193.2 + j51.8 + 0 + j200 = 289.8 + j225.9 = 368 \angle 38°$$

$$\mathbf{Z}_{12} = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_3} = \frac{368\angle 38°}{20\angle 60°} = 18.4\angle -22° = 17.0 - j6.9$$

$$\mathbf{Z}_{23} = \frac{368\angle 38°}{10\angle -30°} = 36.8\angle 8° = 36.4 + j5.1$$

$$\mathbf{Z}_{31} = \frac{368\angle 38°}{10\angle -45°} = 36.8\angle 83° = 4.49 + j36.5$$

Assuming clockwise phase sequence of voltages $\mathbf{V}_{12}$, $\mathbf{Z}_{23}$ and $\mathbf{V}_{31}$, we have

$$\mathbf{V}_{12} = 200\angle 0°,\ \mathbf{V}_{23} = 200\angle -120°,\ \mathbf{V}_{31} = 200\angle 120°$$

$$\mathbf{I}_{12} = \frac{\mathbf{V}_{12}}{\mathbf{Z}_{12}} = \frac{200\angle 0°}{18.4\angle -22°} = 10.86\angle 22° = 10.07 + j4.06$$

$$\mathbf{I}_{23} = \frac{\mathbf{V}_{23}}{\mathbf{V}_{23}} = \frac{200\angle -120°}{36.8\angle 8°} = 5.44\angle -128° = -3.35 - j4.29$$

$$\mathbf{I}_{31} = \frac{\mathbf{V}_{31}}{\mathbf{Z}_{31}} = \frac{200\angle 120°}{36.8\angle 83°} = 5.44\angle 37° = 4.34 + j3.2$$

Line current $= \mathbf{I}_{11}' = \mathbf{I}_{12} + \mathbf{I}_{13} = \mathbf{I}_{12} - \mathbf{I}_{31}$

$$= (10.07 + j4.06) - (4.34 + j3.2) = 5.73 + j0.86 = 5.76\angle 8°32'$$

$$\mathbf{I}_{22}' = \mathbf{I}_{23} - \mathbf{I}_{12} = (-3.35 - j4.29) - (10.07 + j4.06) = -13.42 - j8.35 = 15.79\angle -148°6'$$

$$\mathbf{I}_{33}' = \mathbf{I}_{31} - \mathbf{I}_{23} = (4.34 - j3.2) - (-3.35 - j4.29) = 7.69 + j7.49 = 10.73\angle°44°16'$$

These are currents in the *phases of the Y-connected unbalanced load*. Let us find voltage drop across each star-connected branch impedance.

Voltage drop across $\mathbf{Z}_1 = \mathbf{V}_{10} = \mathbf{I}_{11}'\mathbf{Z}_1 = 5.76\angle 8°32'.10\angle 30° = 57.6\angle 38°32'$

Voltage drop across $\mathbf{Z}_2 = \mathbf{V}_{20} = \mathbf{I}_{22}'\,\mathbf{Z}_2 = 15.79\angle -148°6.10\angle -45° = 157.9\angle -193°6'$

Voltage drop across $\mathbf{Z}_3 = \mathbf{V}_{30} = \mathbf{I}_{33}'\,\mathbf{Z}_3 = 10.73\angle 44°16'.20°\angle 60° = 214.6 * 104°16'$

Example 19.65. *A 300-V (line) 3-phase supply feeds! star-connected load consisting of non-inductive resistors of 15, 6 and 10 Ω connected to the R, Y and B lines respectively. The phase sequence is RYB. Calculate the voltage across each resistor.*

Solution. The Y-connected unbalanced load and its equivalent Δ-connected load are shown in Fig. 19.87. Using Y/Δ conversion method we have

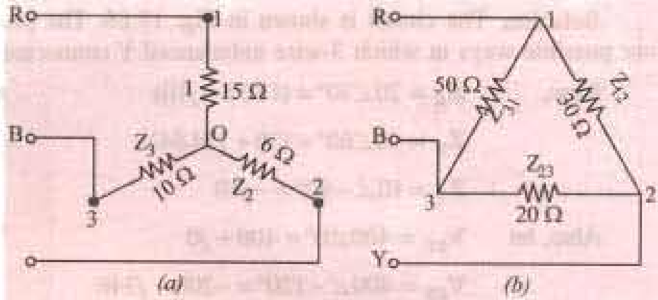


Fig. 19.87

$$\mathbf{Z}_{12} = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_3}$$

$$= \frac{90+60+150}{10} = 30\,\Omega$$

$$\mathbf{Z}_{23} = 300/15 = 20\,\Omega$$

$$\mathbf{Z}_{31} = 300/6 = 50\,\Omega$$

Phase current $I_{RY} = V_{RY}/Z_{12} = 300/30 = 10\text{ A}$

Similarly $\quad I_{YB} = V_{YB}/Z_{23} = 300/20 = 15\text{ A}$

$$I_{BR} = V_{BR}/Z_{31} = 300/50 = 6\text{ A}$$

Each current is in phase with its own voltage because the load is purely resistive.

The line currents for the delta connection are obtained by compounding these phase currents in pairs, either trigonometrically or by phasor algebra. Using phasor algebra and choosing $\mathbf{V}_{RY}$ as the reference axis, we get

$$\mathbf{I}_{RY} = 10 + j0;\ \mathbf{I}_{YB} = 15\left(-\frac{1}{2} - j\sqrt{3}/2\right) = -7.5 - j13.0;\ \mathbf{I}_{BR} = 6\left(-\frac{1}{2} + j\sqrt{3}/2\right) = -3.0 + j5.2$$

Line currents for delta-connection [Fig. 19.66 (*b*)] are

$\mathbf{I}_R = \mathbf{I}_{RY} + \mathbf{I}_{RB} = \mathbf{I}_{RY} - \mathbf{I}_{BR} = (10 + j0) - (-3 + j5.2) = 13 - j5.2$ or 14 A in magnitude

$\mathbf{I}_Y = \mathbf{I}_{YR} + \mathbf{I}_{YB} = \mathbf{I}_{YB} - \mathbf{I}_{RY} = (-7.5 - j13.0) - (10 + j0) = -17.5 - j13$ or 21.8 A in magnitude

$\mathbf{I}_B = \mathbf{I}_{BR} + \mathbf{I}_{BY} = \mathbf{I}_{BR} - \mathbf{I}_{YB} = (-3.0 + j5.2) - (-7.5 - j13.0) = 4.5 + j18.2 = 18.7$ A magnitude

These line currents for Δ-connection are the phase currents for Y-connection. Voltage drop across each limb of Y-connected load is

$\mathbf{V}_{RN} = \mathbf{I}_R\mathbf{Z}_1 = (13 - j5.2)(15 + j0) = 195 - 78$ volt or 210 V

$\mathbf{V}_{YN} = \mathbf{I}_Y\mathbf{Z}_2 = (-17.5 - j13.0)(6 + j0) = -105 - j78$ volt or 131 V

$\mathbf{V}_{BN} = \mathbf{I}_B\mathbf{Z}_3 = (4.5 + j18.2)(10 + j0) = 45 + j182$ volt or 187 V

As a check, it may be verified that the difference of phase voltages taken in pairs should give the three line voltages. Going through the circuit internally, we have

$\mathbf{V}_{RY} = \mathbf{V}_{RN} + \mathbf{V}_{NY} = \mathbf{V}_{RN} - \mathbf{V}_{YN} = (195 - j78) - (105 - j78) = 300\ \angle 0°$

$\mathbf{V}_{YB} = \mathbf{V}_{YN} - \mathbf{V}_{BN} = (-105 - j78) - (45 + j182) = -150 - j260 = 300\ \angle -120°$

$\mathbf{V}_{BR} = \mathbf{V}_{BN} - \mathbf{V}_{RN} = (45 + j182) - (195 - j78) = -150 + j260 = 300\ \angle 120°$

This question could have been solved by direct geometrical methods as shown in Ex. 19.52.

Example 19.66. *A Y-connected load is supplied from a 400-V, 3-phase, 3-wire symmetrical system RYB. The branch circuit impedances are*

$$Z_R = 10\sqrt{3} + j10;\ Z_Y = 20 + j20\sqrt{3};\ Z_B = 0 - j10$$

Determine the current in each branch. Phase sequence is RYB.

(Network Analysis, Nagpur Univ. 1993)

Solution. The circuit is shown in Fig. 19.88. The problem will be solved by using all the four possible ways in which 3-wire unbalanced Y connected load can be handled.

Now, $\mathbf{Z}_R = 20\angle 30° = (17.32 + j10)$

$\mathbf{Z}_Y = 40\angle 60° = (20 + j34.64)$

$\mathbf{Z}_B = 10\angle -90° = -j10$

Also, let $\mathbf{V}_{RY} = 400\angle 0° = 400 + j0$

$\mathbf{V}_{RB} = 400\angle -120° = -200 - j346$

$\mathbf{V}_R = 400\angle 120° = -200 + j346$

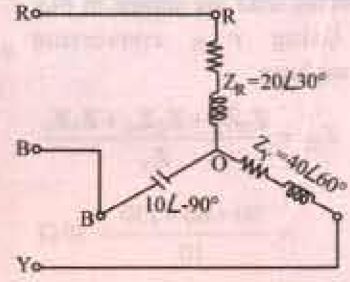


Fig. 19.88

(a) By applying Kirchhoff's Laws

With reference to Art. 19.33, it is seen that

$$I_{RO} = I_R = \frac{V_{RY}\mathbf{Z}_B - V_{BR}\mathbf{Z}_Y}{\mathbf{Z}_R\mathbf{Z}_Y + \mathbf{Z}_Y\mathbf{Z}_B + \mathbf{Z}_B\mathbf{Z}_R}; I_{YO} = \mathrm{I}_Y = \frac{V_{YB}\mathbf{Z}_R - V_{RY}\mathbf{Z}_B}{\mathbf{Z}_R\mathbf{Z}_Y + \mathbf{Z}_Y\mathbf{Z}_B + \mathbf{Z}_B\mathbf{Z}_R};$$

$$I_{BO} = I_B = \frac{V_{BR}\mathbf{Z}_Y - V_{YB}\mathbf{Z}_R}{\mathbf{Z}_R\mathbf{Z}_Y + \mathbf{Z}_Y\mathbf{Z}_B + \mathbf{Z}_B\mathbf{Z}_R}$$

Now, $\mathbf{Z}_R\mathbf{Z}_Y + \mathbf{Z}_Y\mathbf{Z}_B + \mathbf{Z}_B\mathbf{Z}_R$

$$= 20\angle 30°.40\angle 60° + 40\angle 60°.10\angle -90° + 10\angle -90°.20\angle 30°$$

$$= 800\angle 90° + 400\angle -30° + 200\angle -60° = 446 + j426 = 617\angle 43.7°$$

$$\mathbf{V}_{RY}\mathbf{Z}_B - \mathbf{V}_{BR}\mathbf{Z}_Y = 400 \times 10\angle -90° - 400\angle 120°.40\angle 60°$$

$$= 16{,}000 - j4000 = 16{,}490\angle -14°3'$$

$$\therefore \qquad \mathbf{I}_R = \frac{16{,}490\angle -14°3'}{617\angle 43.7°} = 26.73\angle -57°45'$$

$$\mathbf{V}_{YB}\mathbf{Z}_R - \mathbf{V}_{RY}\mathbf{Z}_B = 400\angle -120°.20\angle 30° - 400.10\angle -90° = -j4000 = 4000\angle -90°$$

$$\mathbf{I}_Y = \frac{4000\angle -90°}{617\angle 43.7°} = 6.48\angle -133.7°$$

$$\mathbf{V}_{BR}\mathbf{Z}_Y - \mathbf{V}_{YB}\mathbf{Z}_R = 400\angle 120°.40\angle 60° - 400\angle -120° 20.\angle 30°$$

$$= -16{,}000 + j8{,}000 = 17{,}890\angle 153°26'$$

$$\therefore \qquad \mathbf{I}_B = \frac{17{,}890\angle 153°26'}{617\angle 43.7°} = 29\angle 109°45'$$

(b) By Star/Delta Conversion (Fig. 19.89)

The given star may be converted into the equivalent delta with the help of equations given in Art. 19.34.

$$\mathbf{Z}_{RY} = \frac{\mathbf{Z}_R\mathbf{Z}_Y + \mathbf{Z}_Y\mathbf{Z}_B + \mathbf{Z}_B\mathbf{Z}_R}{\mathbf{Z}_B} = \frac{617\angle 43.7°}{10\angle -90°} = 61.73\angle 133.7°$$

$$\mathbf{Z}_{YB} = \frac{\Sigma \mathbf{Z}_R\mathbf{Z}_Y}{\mathbf{Z}_R} = \frac{617\angle 43.7°}{20\angle 30°} = 30.87\angle 13.7°$$

$$\mathbf{Z}_{BR} = \frac{\Sigma \mathbf{Z}_R\mathbf{Z}_Y}{\mathbf{Z}_Y} = \frac{617\angle 43.7°}{40\angle 60°} = 15.43\angle -16.3°$$

$$\mathbf{I}_{RY} = \frac{\mathbf{V}_{RY}}{\mathbf{Z}_{RY}} = \frac{400}{61.73\angle 133.7°} = 6.48\angle -133.7° = (-4.47 - j4.68)$$

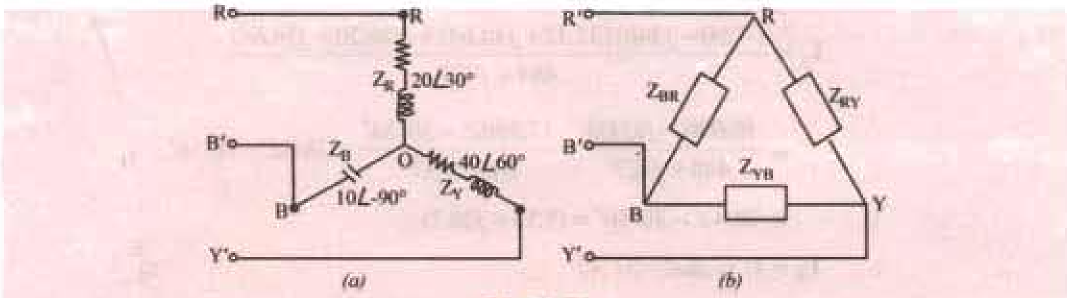


Fig. 19.89

$$\mathbf{I}_{YB} = \frac{\mathbf{V}_{YB}}{\mathbf{Z}_{YB}} = \frac{400\angle -120^\circ}{30.87\angle 13.7^\circ} = 12.95\angle -133.7^\circ = (-8.95 - j9.35)$$

$$\mathbf{I}_{BR} = \frac{\mathbf{V}_{BR}}{\mathbf{Z}_{BR}} = \frac{400\angle 120^\circ}{15.43\angle -16.3^\circ} = 25.9\angle 136.3^\circ = (-18.7 + j17.9)$$

$$\mathbf{I}'_{RR} = \mathbf{I}_{RY} - \mathbf{I}_{BR} = 14.23 - j22.58 = 26.7\angle -57^\circ 48'$$

$$\mathbf{I}'_{YY} = \mathbf{I}_{YB} - \mathbf{I}_{RY} = -4.48 - j4.67 = 6.47\angle -134^\circ 6'$$

$$\mathbf{I}'_{YB} = \mathbf{I}_{BR} - \mathbf{I}_{YB} = -9.85 + j27.25 = 29\angle 109^\circ 48'$$

$$\Sigma\mathbf{I} = (0 + j0) \qquad \text{–as a check}$$

As explained in Art. 19.34, these line currents of the equivalent delta represent the phase currents of the star-connected load of Fig. 19.89 (*a*).

Note. Minor differences are due to accumulated errors.

(*c*) By Using Maxwell's Loop Current Method

Let the loop or mesh currents be as shown in Fig. 19.90. It may be noted that

$$\mathbf{I}_R = \mathbf{I}_1; \mathbf{I}_Y = \mathbf{I}_2 - \mathbf{I}_1 \text{ and } \mathbf{I}_B = -\mathbf{I}_2$$

Considering the drops across *R* and *Y*-arms, we get

$$\mathbf{I}_1\mathbf{Z}_R + \mathbf{Z}_Y(\mathbf{I}_1 - \mathbf{I}_2) = \mathbf{V}_{RY}$$

or $$\mathbf{I}_1(\mathbf{Z}_R + \mathbf{Z}_Y) - \mathbf{I}_2\mathbf{Z}_Y = \mathbf{V}_{RY} \qquad \text{... (i)}$$

Similarly, considering the legs *Y* and *B*, we have

$$\mathbf{Z}_Y(\mathbf{I}_2 - \mathbf{I}_1) + \mathbf{Z}_B\mathbf{I}_2 = V_{YB}$$

or $$-\mathbf{I}_1\mathbf{Z}_Y + \mathbf{I}_2(\mathbf{Z}_B + \mathbf{Z}_Y) = \mathbf{V}_{YB} \qquad \text{... (ii)}$$

Solving for $\mathbf{I}_1$ and $\mathbf{I}_2$, we get

$$\mathbf{I}_1 = \frac{\mathbf{V}_{RY}(\mathbf{Z}_Y + \mathbf{Z}_B) + \mathbf{Z}_{YB}\mathbf{V}_Y}{(\mathbf{Z}_R + \mathbf{Z}_Y)(\mathbf{Z}_Y + \mathbf{Z}_B) - \mathbf{Z}_Y^2};$$

$$\mathbf{I}_2 = \frac{\mathbf{V}_{YB}(\mathbf{Z}_R + \mathbf{Z}_Y) + \mathbf{V}_{RY}\mathbf{Z}_Y}{(\mathbf{Z}_R + \mathbf{Z}_Y)(\mathbf{Z}_Y + \mathbf{Z}_B) - \mathbf{Z}_Y^2}$$

$$\mathbf{I}_1 = \frac{400(20 + j24.64) + 400\angle -120^\circ.40\angle 60^\circ}{(37.32 + j44.64)(20 + j24.64) - 1600\angle 120^\circ}$$

$$= \frac{16{,}000 - j4{,}000}{448 + j427} = \frac{16{,}490\angle -14^\circ 3'}{617\angle 43.7^\circ}$$

$$= 26\angle -57^\circ 45' = (13.9 - j22)$$

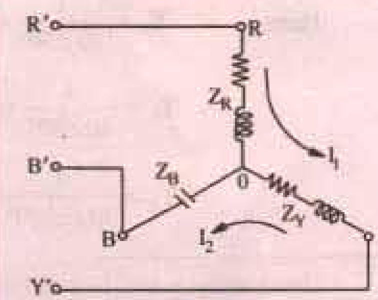


Fig. 19.90

$$\mathbf{I}_2 = \frac{(-200 - j346)(37.32 + j44.64) + 400(20 + j34.64)}{484 + j427}$$

$$= \frac{16{,}000 - j8{,}000}{448 + j427} = \frac{17{,}890\angle -26°34'}{617\angle 43.7°} = 28.4\angle -70°16'$$

$$= 28.4\angle -70°16' = (9.55 - j26.7)$$

$\therefore$ $$\mathbf{I}_R = \mathbf{I}_1 = 26\angle -57°45'$$

$$\mathbf{I}_Y = \mathbf{I}_2 - \mathbf{I}_1 = (9.55 - j26.7) - (13.9 - j22) = -4.35 - j4.7 = 6.5\angle -134°$$

$$\mathbf{I}_B = -\mathbf{I}_2 = -28.4\angle -70°16' = 28.4\angle 109°44'$$

(d) By Using Millman's Theorem

According to this theorem, the voltage of the load star point O' with respect to the star point or neutral O of the generator or supply (normally zero potential) is given by

$$\mathbf{V}_{O'O} = \frac{\mathbf{V}_{RO}\mathbf{Y}_R + \mathbf{V}_{YO}\mathbf{Y}_Y + \mathbf{V}_{BO}\mathbf{Y}_B}{\mathbf{Y}_R + \mathbf{Y}_Y + \mathbf{Y}_B}$$

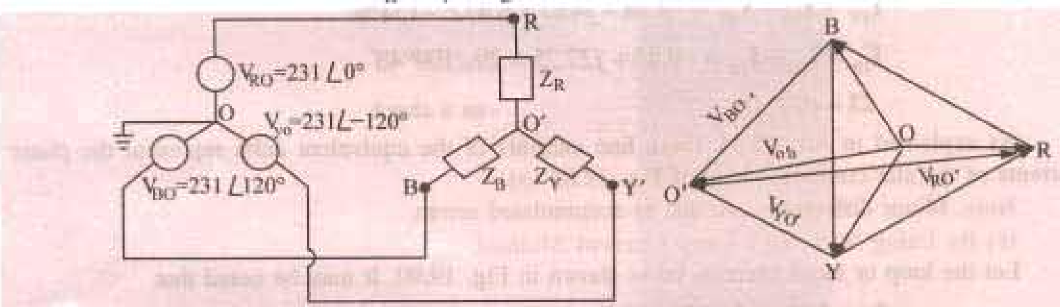


Fig. 19.91

where $\mathbf{V}_{RO}$, $\mathbf{V}_{YO}$ and $\mathbf{V}_{BO}$ are the phase voltages of the generator or 3-phase supply.

As seen from Fig. 19.91, voltage across each phase of the load is

$$V_{RO'} = V_{RO} - V_{O'O} \quad V_{YO'} = V_{YO} - V_{O'O} \quad V_{BO'} = V_{BO} - V_{O'O}$$

Obviously, $\mathbf{I}_{RO'} = (\mathbf{V}_{RO} - \mathbf{V}_{O'O})\mathbf{Y}_R; \mathbf{I}_{YO'} = (\mathbf{V}_{YO} - \mathbf{V}_{O'O})\mathbf{Y}_Y$ and

$$\mathbf{I}_{BO'} = (\mathbf{V}_{BO} - \mathbf{V}_{O'O})\mathbf{Y}_B$$

Here $$\mathbf{Y}_R = \frac{1}{20\angle 30°} = 0.05\angle -30° = (0.0433 - j0.025)$$

$$\mathbf{Y}_Y = \frac{1}{40\angle 60°} = 0.025\angle -60° = (0.0125 - j0.0217)$$

$$\mathbf{Y}_B = \frac{1}{10\angle -90°} = 0.1\angle 90° = 0 + j0.1$$

*Incidentally, it may be noted that the p.d. between load neutral and supply neutral is given by

$$V_{OO'} = \frac{V_{RO'} + V_{YO'} + V_{BO'}}{3}$$

$$\therefore \quad \mathbf{Y}_R + \mathbf{Y}_Y + \mathbf{Y}_B = 0.0558 + j0.0533 = 0.077\angle 43.7°$$

Let $$\mathbf{V}_{RO} = \frac{400}{\sqrt{3}}\angle 0° = (231 + j0)$$

$$\mathbf{V}_{BO} = 231\angle -120° = -115.5 - j200$$

$$\mathbf{V}_{BO} = 231\angle 120° = -115.5 + j200$$

$$\mathbf{V}'_{OO} = \frac{231.0.05\angle -30° + 231\angle -120°.0.025\angle -60° + 231\angle 120°.01\angle 90°}{0.077\angle 43.7°}$$

$$= \frac{-15.8 - j17.32}{0.077\angle 43.7°} = \frac{23.5\angle -132.4\angle}{0.077\angle 43.7} = 305\angle -176.1° = (304.5 - j20.8)$$

$$\mathbf{V}_{R\acute{O}} = \mathbf{V}_{RO} - \mathbf{V}_{O\acute{O}} = 231 - (-304.5 - j20.8) = 535.5 + j20.8 = 536\angle 2.2°$$

$$\mathbf{V}_{Y\acute{O}} = (-115.5 - j200) - (-304.5 - j20.8) = 189 - j179 = 260\angle -43°27'$$

$$\mathbf{V}_{B\acute{O}} = (-115.5 + j200) - (-304.5 - j20.8) = 189 + j221 = 291\angle 49°27'$$

$$\therefore \quad \mathbf{I}'_{RO} = 536\angle 2.2° \times 0.05\angle -30° = 26.5\angle -27.8°$$

$$\mathbf{I}'_{;O} = 260\angle -43°27' \times 0.025 < -60° = 6.5\angle -103°27'$$

$$\mathbf{I}'_{BO} = 291\angle 49°27' \times 0.1\angle 90° = 29.1\angle 139°27'$$

Note. As seen from above, $\mathbf{V}'_{RO} = \mathbf{V}'_{RO} - \mathbf{V}'_{OO}$

Substituting the value of $\mathbf{V}'_{OO}$, we have

$$\mathbf{V}'_{RO} = \mathbf{V}_{RO} - \left(\frac{\mathbf{V}_{RO}\mathbf{Y}_R + \mathbf{V}_{YO}\mathbf{Y}_Y + \mathbf{V}_{BO}\mathbf{Y}_B}{\mathbf{V}_R + \mathbf{V}_Y + \mathbf{V}_B}\right)$$

$$= \frac{(\mathbf{V}_{RO} - \mathbf{V}_{YO})\mathbf{Y}_Y + (\mathbf{Y}_{RO} - \mathbf{V}_{BO})\mathbf{Y}_B}{\mathbf{Y}_R + \mathbf{Y}_Y + \mathbf{Y}_B}$$

$$= \frac{\mathbf{V}_{RY}\mathbf{Y}_Y + \mathbf{V}_{RB}\mathbf{Y}_B}{\mathbf{Y}_R + \mathbf{Y}_Y + \mathbf{Y}_B}$$

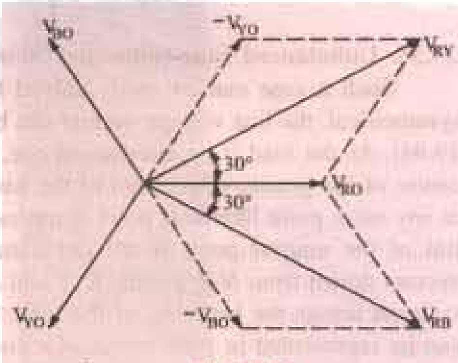


Fig. 19.92

Since $\mathbf{V}_{RO}$ is taken as the reference vector, then as seen from Fig. 19.92.

$$\mathbf{V}_{RY} = 400\angle 30° \text{ and } \mathbf{V}_{RB} = 400\angle -30°$$

$$\therefore \mathbf{V}'_{RO} = \frac{400\angle 30° \times 0.025\angle -60° + 400\angle -30° \times 0.1\angle 90°}{0.077\angle 43.7°}$$

$$= \frac{28.6 + j29.64}{0.077\angle 43.7°} = \frac{41\angle 46°}{0.077\angle 43.7°} = 532.5\angle 2.3°$$

$$= \mathbf{I}'_{RO} = \mathbf{V}'_{RO}\mathbf{Y}_R = 532.5\angle 2.3° \times 0.05 \times \angle -30° = 26.6\angle -27.7°$$

Similarly, $\mathbf{V}'_{YO}$ and $\mathbf{V}'_{BO}$ may be found and $\mathbf{I}_Y$ and $\mathbf{I}_B$ calculated thereform.

Example 19.67. *Three impedances, Z_R, Z_Y and Z_B are connected in star across a 440-V, 3-phase supply. If the voltage of star-point relative to the supply neutral is 200 $\angle$ 150° volt and Y and B line currents are 10 $\angle$ – 90° A and 20 $\angle$ 90° A respectively, all with respect to the voltage between the supply neutral and the R line, calculate the values of Z_R, Z_Y and Z_B.*

(Elect Circuit; Nagpur Univ. 1991)

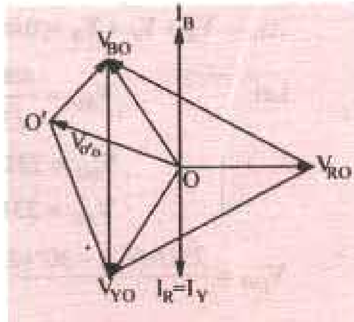


Fig. 19.93

Solution. Let O and O' be the supply and load neutrals respectively. Also, let,

$$\mathbf{V}_{RO} = \frac{440}{\sqrt{3}}\angle 0° = 254\angle 0° = 254 + j0$$

$$\mathbf{V}_{YO} = 254\angle -120° = -127 - j220$$

$$\mathbf{V}_{BO} = 254\angle 120° = -127 + j220$$

$$\mathbf{I}_Y = 10\angle -90° = -j10;\ \mathbf{I}_B = 20\angle 90° = j20$$

$$\mathbf{I}_R = -(\mathbf{I}_Y + \mathbf{I}_B) = -j10$$

Also, $\mathbf{V}'_{OO} = 200\angle 150° = -173 + j100$

$$\mathbf{V}_{RO} - \mathbf{V}'_{OO} = 254 - (-173 + j100) = 427 - j100 = 438.5\angle -13.2°$$

$$\mathbf{V}_{YO} - \mathbf{V}'_{OO} = (-127 - j220) - (-173 + j100) = 46 - j320 = 323\angle -81.6°$$

$$\mathbf{V}_{BO} - \mathbf{V}'_{OO} = (-127 + j220) - (-173 + j100) = 46 + j120 = 128.6\angle 69°$$

As seen from Art. 19.32.

$$\mathbf{Z}_R = \frac{\mathbf{V}_{RO} - \mathbf{V}'_{00}}{\mathbf{I}_R} = \frac{438.5\angle -13.2°}{10\angle -90°} = 43.85\angle 76.8°$$

$$\mathbf{Z}_Y = \frac{\mathbf{V}_{YO} - \mathbf{V}'_{OO}}{\mathbf{I}_Y} = \frac{323}{10\angle -90°} = 32.3\angle 8.4°$$

$$\mathbf{Z}_B = \frac{\mathbf{V}_{BO} - \mathbf{V}'_{OO}}{\mathbf{I}_B} = \frac{128.6\angle 69°}{20\angle 90°} = 6.43\angle -21°$$

19.35. Unbalanced Star-connected Non-inductive Load

Such a case can be easily solved by direct geometrical method. If the supply system is symmetrical, the line voltage vectors can be drawn in the form of an equilateral triangle RYB (Fig. 19.94). As the load is an unbalanced one, its neutral point will not, obviously, coincide with the centre of the gravity or centroid of the triangle. Let it lie at any other point like N. If point N represents the potential of the neutral point if the unbalanced load, then vectors drawn from N to points, R, Y and B represent the voltages across the branches of the load. These voltages can be represented in their rectangular co-ordinates with respect to the rectangular axis drawn through N. it is seen that taking co-ordinates of N as (0, 0), the co-ordinates of point R are $[(V/2 - x), -y]$

of point Y are $[-(V/2 + x), -y]$

and of point B are $[-x, (\sqrt{3}V/2 - y)]$

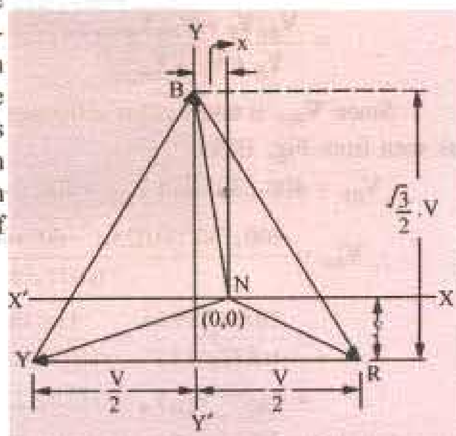


Fig. 19.94

$$V_{RN} = \left(\frac{V}{2} - x\right) - jy;\ V_{YN} = -\left(\frac{V}{2} + x\right) - iy$$

$$V_{BN} = -x + j\left(\frac{\sqrt{3}V}{2} - y\right)$$

Let R_1, R_2 and R_3 be the respective branch impedances, Y_1, Y_2 and Y_3 the respective admittances and $\mathbf{I}_R$, $\mathbf{I}_Y$ and $\mathbf{I}_B$ the respective currents in them.

Then $\qquad \mathbf{I}_R = \mathbf{V}_{RN} / \mathbf{R}_1 = \mathbf{V}_{RN}\mathbf{Y}_1$

Similarly, $\qquad \mathbf{I}_Y = \mathbf{V}_{YN}\mathbf{Y}_2$ and $\mathbf{I}_B = \mathbf{V}_{BN}\mathbf{Y}_3$. Since $\mathbf{I}_R + \mathbf{I}_Y + \mathbf{I}_B = 0$

$\therefore \qquad \mathbf{V}_{RN}\mathbf{Y}_1 + \mathbf{V}_{YN}\mathbf{Y}_2 + \mathbf{V}_{BN}\mathbf{Y}_3 = 0$

$$\text{or } Y_1\left[\left(\frac{V}{2} - x\right) - jy\right] + Y_2\left[-\left(\frac{V}{2} + x\right) - jy\right] + Y_3\left[-x + j\left(\frac{\sqrt{3}V}{2} - y\right)\right] = 0$$

$$\text{or } -x(Y_1 + Y_2 + Y_3) + \frac{V}{2}(Y_1 - Y_2) + j\left[Y_3\frac{\sqrt{3}V}{2} - y(Y_1 + Y_2 + Y_3)\right] = 0$$

$$\therefore \qquad -x(Y_1 + Y_2 + Y_3) + \frac{V}{2}(Y_1 - Y_2) = 0 \qquad \therefore \; x = \frac{V(Y_1 - Y_2)}{2Y_1 + Y_2 + Y_3}$$

$$\text{Also} \qquad Y_3\frac{\sqrt{3}V}{2} - y(Y_1 + Y_2 + Y_3) = 0 \qquad \therefore \; y = \frac{\sqrt{3}V}{2}\frac{Y_3}{(Y_1 + Y_2 + Y_3)}$$

Knowing the values of x, the values of $\mathbf{V}_{RN}$, $\mathbf{V}_{YN}$ and $\mathbf{V}_{BN}$ and hence, of $\mathbf{I}_R$, $\mathbf{I}_Y$ and $\mathbf{I}_B$ can be found as illustrated by Ex. 19.68.

Example 19.68. *Three non-inductive resistances of 5, 10 and 15 Ω are connected in star and supplied from a 230-V symmetrical 3-phase system. Calculate the line currents (magnitudes).*

(Principles of Elect. Engg. Jadavpur Univ. 1987)

Solution.

(*a*) Star/Delta Conversion Method

The Y-connected unbalanced load and its equivalent Δ-connected load are shown in Fig. 19.95 (*a*) and (*b*) respectively. Using Y/Δ conversion, we have

$$\mathbf{Z}_{12} = \frac{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1}{\mathbf{Z}_3} = \frac{50 + 150 + 75}{15} = \frac{55}{3}\,\Omega$$

$$\mathbf{Z}_{23} = 275/5 = 55\,\Omega \text{ and } \mathbf{Z}_{31} = 275/10 = 27.5\,\Omega$$

Phase current $\mathbf{I}_{RY} = \mathbf{V}_{RY}/\mathbf{Z}_{12} = 230/(55/3) = 12.56$ A

Similarly, $\mathbf{I}_{YB} = \mathbf{V}_{YB}/\mathbf{Z}_{23} = 230/55 = 4.18$ A; $\mathbf{I}_{BR} = \mathbf{V}_{BR}/\mathbf{Z}_{31} = 230/27.5 = 8.36$ A

The line currents for Δ-connection are obtained by compounding the above phase currents trigonometrically or vectorially. Choosing vector addition and taking $\mathbf{V}_{RY}$ as the reference vector, we get;

$\mathbf{I}_{RY} = (12.56 + j0)$

$$\mathbf{I}_{YB} = 4.18\left(-\frac{1}{2}j\frac{\sqrt{3}}{2}\right) = -2.09 - j3.62$$

$$\mathbf{I}_{BR} = 8.36\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = -4.18 + j7.24$$

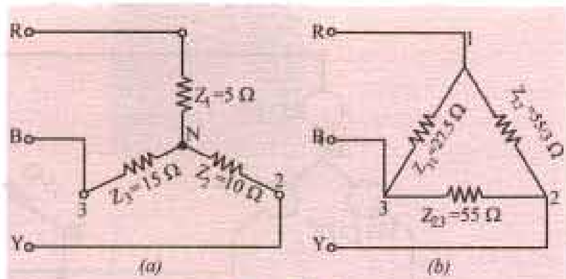


Fig. 19.95

Hence, line currents for Δ-connection of Fig. 19.95 (*b*) are

$$\mathbf{I}_R = \mathbf{I}_{RY} + \mathbf{I}_{RB} = \mathbf{I}_{RY} - \mathbf{I}_{BR}$$
$$= (12.56 + j0) - (-4.18 + j7.24) = 16.74 - j7.24 \text{ or } 18.25 \text{ A – in magnitude}$$

$$\mathbf{I}_Y = \mathbf{I}_{YR} + \mathbf{I}_{YB} = \mathbf{I}_{YB} - \mathbf{I}_{RY}$$
$$= (-2.09 - j3.62) - (12.56 + j0) = -14.65 - j3.62 \text{ or } 15.08 \text{ A – in magnitude}$$

$$\mathbf{I}_B = \mathbf{I}_{BR} + \mathbf{I}_{BY} = \mathbf{I}_{BR} - \mathbf{I}_{YB}$$
$$= (-4.18 + j7.24) - (-2.09 - j3.62) = -2.09 + j10.86 \text{ or } 11.06 \text{ A –in magnitude}$$

(b) Geometrical Method

Here, $R_1 = 5\Omega$, $R_2 = 10\Omega$ and $R_3 = 15\Omega$ $Y_1 = 1/5S$; $Y_2 = 1/10S$; $Y_3 = 1/15S$

As found above in Art. 19.35 $x = \frac{V}{2}(Y_1 - Y_2)/(Y_1 + Y_2 + Y_3)$

$$= \frac{230}{2}\left(\frac{1}{5} - \frac{1}{10}\right) \Big/ \left(\frac{1}{5} + \frac{1}{10} + \frac{1}{15}\right) = 31.4$$

$$y = \left(\frac{\sqrt{3}V}{2}.Y_3\right) \Big/ (Y_1 + Y_2 + Y_3) = (\sqrt{3} \times 115 \times 1/15)/(11/30) = 36.2$$

$$V_{RN} = \left(\frac{V}{2} - x\right) - jy = (115 - 31.4) - j36.2 = 83.6 - j36.2$$

$$V_{YN} = -\left(\frac{V}{2} + x\right) - jy = -146.4 - j36.2$$

$$V_{BN} = -x + j\left(\frac{\sqrt{3}V}{2} - y\right) = -31.4 + j163$$

$$\mathbf{I}_R = \mathbf{V}_{RN}\mathbf{Y}_1 = (83.6 - j36.2) \times 1/5 = 16.72 - j7.24$$

$$\mathbf{I}_Y = \mathbf{V}_{YN}\mathbf{Y}_2 = (-146.4 - j36.2) \times 1/10 = -14.64 - j3.62$$

$$\mathbf{I}_B = \mathbf{V}_{BN}\mathbf{Y}_3 = (-31.4 + j163) \times 1/15 = -2.1 \times j10.9$$

These are the same currents as found before.

(c) Solution by Millman's Theorem

$\mathbf{Y}_R = 1/5\angle 0°$; $\mathbf{Y}_Y = 1/10\angle 0°$; $\mathbf{Y}_B = 1/15\angle 0°$ and $\mathbf{Y}_R + \mathbf{Y}_Y + \mathbf{Y}_B = 11/30\angle 0°$ Siemens

Let the supply voltages be represented (Fig. 19.96) by

$\mathbf{V}_{RO} = 230/\sqrt{3}\angle 0° = 133\angle 0°$; $\mathbf{V}_{YO} = 133\angle -120°$; $\mathbf{V}_{BO} = 133\angle 120°$

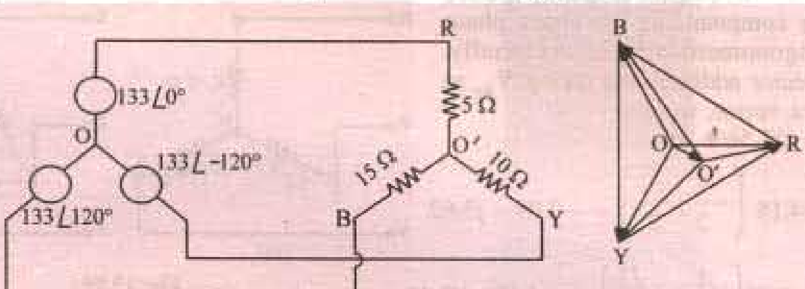


Fig. 19.96

The p.d. between load and supply neutral is

$$\mathbf{V}_{OO'} = \frac{133/5 + (133/10)\angle -120° + (133/15)\angle 120°}{30/11\angle 0°}$$

$$= 42.3 - j10.4 = 43.6\angle -13.8°$$

$$\mathbf{V}_{RO}{}' = 133 - (42.3 - j10.4) = 90.7 + j10.4$$

$$\mathbf{V}_{YO}{}' = 133\angle -120^\circ - (42.3j - 10.4)$$
$$= (-66.5 - j115) - (42.3 - j10.4) = -108.8 - j104.6$$

$$\mathbf{V}_{BO}{}' = 133\angle 120^\circ - V_{OO}{}' = (-66.5 + j115) - (42.3 - j10.4) = -108.8 + j125.4$$

$$\mathbf{I}_R = \mathbf{V}_{RO}{}'\mathbf{Y}_R = 1/5.(90.7 + j10.4) = 18.1 + j2.1 \text{ or } 18.22 \text{ A in magnitude}$$

$$I_Y = -10.88 - j10.5 \text{ or } 15.1 \text{ A in magnitude}$$

$$I_B = -7.5 + j8.4 \text{ or } 11.7 \text{ A in magnitude}$$

Example 19.69. *The unbalanced circuit of Fig. 19.97 (a) is connected across a symmetrical 3-phase supply of 400-V. Calculate the currents and phase voltages. Phase sequence is RYB.*

Solution. The line voltages are represented by the sides of an equilateral triangle *ABC* in Fig. 19.97 (*b*). Since phase impedances are unequal, phase voltages are unequal and are represented by lengths, *NA*, *NB* and *NC* where *N* is the neutral point which is shifted from its usual position. *CM* and *ND* are drawn perpendicular to horizontal side *AB*. Let co-ordinates of point *N* be (0, 0). Obviously, $AM = BM = 200$ V, $CM = \sqrt{3} \times 200$ V, $CM = \sqrt{3} \times 200 = 346$ V. Let $DM = x$ volts and $ND = y$ volts.

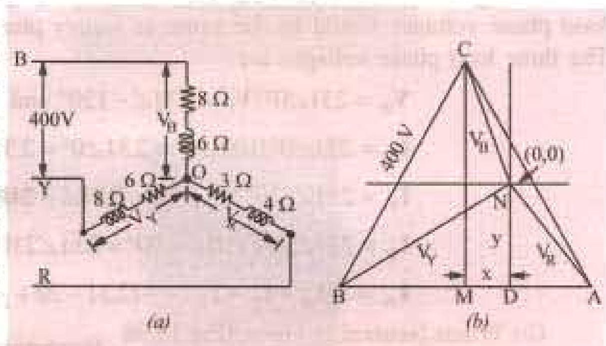


Fig. 19.97

Then, with reference to point *N*, the vector expressions for phase voltages are

$$\mathbf{V}_R = (200 - x) - jy,\ \mathbf{V}_Y = -(200 + x) - jy,\ \mathbf{V}_B = -x + j(346 - y)$$

$$\mathbf{I}_R = \frac{\mathbf{V}_R}{\mathbf{Z}_R} = \frac{(200 - x) - jy}{3 + j4} \times \frac{3 - j4}{3 - j4} = (24 - 0.12x - 0.16y) + j(-32 + 0.16x - 0.12y)$$

$$\mathbf{I}_Y = \frac{\mathbf{V}_Y}{\mathbf{Z}_Y} = \frac{-(200 + x) - jy}{6 + j8} \times \frac{6 - j8}{6 - j8}$$
$$= (-12 - 0.06x - 0.08y) + j(16 + 0.08x - 0.06y)$$

$$\mathbf{I}_B = \frac{\mathbf{V}_B}{\mathbf{Z}_B} = \frac{-x + j(346 - y)}{8 + j6} \times \frac{8 - j6}{8 - j6}$$
$$= (20.76 - 0.08x - 0.06y) + j(27.68 + 0.06x - 0.08y)$$

Now, $\mathbf{I}_R + \mathbf{I}_Y + \mathbf{I}_B = 0$

$\therefore \quad (32.76 - 0.26x - 0.3y) + j(11.68 + 0.3x - 0.26y) = 0$

Obviously, the real component as well as the *j*-component must be zero.

$\therefore \quad 32.76 - 0.26 \times -0.3y = 0$ and $11.68 + 0.3x - 0.26y = 0$

Solving these equations for *x* and *y*, we have $x = 31.9$ V and $y = 81.6$ V

$$\mathbf{V}_R = (200 - 31.9) - j81.6 = 168 - j81.6 = 186.7\angle -25.9^\circ$$

$$\mathbf{V}_Y = -(200 - 31.9) - j81.6 = -231.9 - j81.6 = 245.8\angle 199.4^\circ$$

$$\mathbf{V}_B = -31.9 + j(346 - 81.6) = -31.9 + j264.4 = 266.3\angle 83.1^\circ$$

Substituting these values of *x* and *y* in the expressions for currents, we get

$\mathbf{I}_R = (24 - 0.12\times 31.9 - 0.16\times 81.6) + j(-32 + 0.16\times 31.9 - 0.12\times 81.6)$

$= 7.12 - j36.7$

Similarly $\mathbf{I}_Y = -20.44 + j13.65; I_B = 13.3 + j23.06$

$\Sigma I = (0. + j0)$ – as a check

Example 19.70. *A $3-\phi$, 4-wire, 400-V symmetrical system supplies a Y-connected load having following branch impedances:*

$Z_R = 100\Omega, Z_Y = j10\Omega$ *and* $Z_B = -j10\Omega$

Compute the values of load phase voltages and currents and neutral current. Phase sequence is RYB.

How will these values change in the event of an open in the neutral wire?

Solution. (*a*) **When Neutral Wire is Intact.** [Fig 19.98 (*a*)]. As discussed in Art. 19.30, the *load* phase voltages would be the same as *supply* phase voltages despite imbalance in the load. The three load phase voltages are:

$$\mathbf{V}_R = 231\angle 0^\circ, \mathbf{V}_Y = 231\angle -120^\circ \text{ and } \mathbf{V}_B = 231\angle 120^\circ$$

$$\mathbf{I}_R = 231\angle 0^\circ / 100\angle 0^\circ = 2.31\angle 0^\circ = 2.31 + j0$$

$$\mathbf{I}_Y = 231\angle 120^\circ / 10\angle 90^\circ = 2.31\angle -210^\circ = -20 + j11.5$$

$$\mathbf{I}_B = 231\angle 120^\circ / 10\angle -90^\circ = 23.1\angle 210^\circ = -20 - j11.5$$

$$\mathbf{I}_N = -(\mathbf{I}_R + \mathbf{I}_Y + \mathbf{I}_B) = -(2.31 - 20 + j11.5 - 20 - j11.5) = 37.7\text{ A}$$

(*b*) **When Neutral is Open** [Fig.19.98 (*b*)]

In this case, the load phase voltages will be no longer equal. The node pair voltage method will be used to solve the question. Let the supply phase voltages be given by

$\mathbf{E}_R = 231\angle 0^\circ, \mathbf{E}_Y = 231\angle -120^\circ$

$= -115.5 - j200$

$\mathbf{E}_B = 231\angle 120^\circ = -115\cdot 5 + j200$

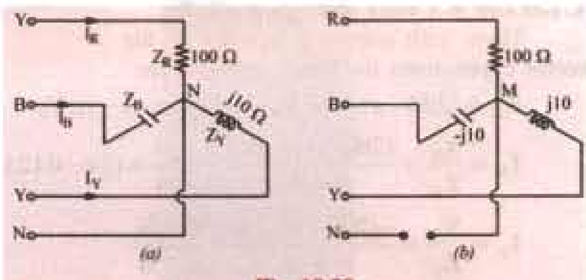


Fig. 19.98

$\mathbf{Y}_R = 1/100 = 0.01; \mathbf{Y}_Y = 1/j10 = -j0.1$ and $\mathbf{Y}_B = 1/-j10 = j0.1$

$$\mathbf{V}'_{NN} = \frac{231\times 0.01 + (-j0.1)(-115.5 - j200) + j0.1(-115.5 + j200)}{0.01 + (-j0.1) + j0.1} = -3769 + j0$$

The load phase voltages are given by

$V'_R = E_R - V'_{NN} = (231 + j0) - (-3769 + j0) = 4000\text{ V}$

$\mathbf{V}'_Y = \mathbf{E}_Y - \mathbf{V}'_{NN} = -115.5 - j200 - (-3769 + j0) = (3653.5 - j200)$

$\mathbf{V}_B' = \mathbf{E}_B - \mathbf{V}'_{NN} = -115.5 + j200 - (-3769 + j0) = (3653.5 + j0)$

$\mathbf{I}_R = \mathbf{V}_R'\mathbf{Y}_R = 4000\times 0.01 = -40\text{ A}$

$\mathbf{I}_Y = (-j0.1)(3653.5 - j200) = (20 - j3653.5)$

$\mathbf{I}_B = (j0.1)(3653.5 + j200) = -20 + j3653.5$

Obviously, the neutral current will just not exist.

Note. As hinted in Art. 19.30 (*i*), the load phase voltages and currents become abnormally high.

Example 19.71. *For the circuit shown in Fig. 19.99 find the readings on the two wattmeters W_a and W_c.*

Solution. The three line currents for this problem have already been determined in Example 19.43.

$$\mathbf{I}_{ao} = 20.02 - j4.54$$
$$\mathbf{I}_{bo} = -7.24 - j5.48$$
$$\mathbf{I}_{co} = -12.78 + j10.12$$

The line voltages are given by

$$\mathbf{V}_{ab} = 200 + j0$$
$$\mathbf{V}_{bc} = -100 - j\,173.2$$
$$\mathbf{V}_{ca} = -100 + j\,173.2$$

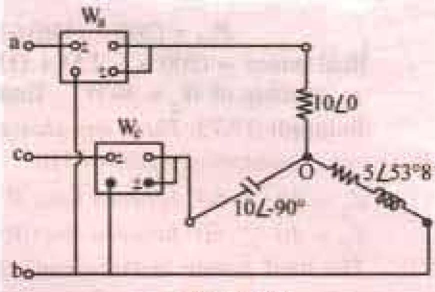


Fig. 19.99

Wattmeter W_a carries a current of using $\mathbf{I}_{ao} = 20.02 - j\,4.54$ and has voltage $\mathbf{V}_{ab}$ impressed across its pressure coil. Power can be found by using current conjugate.

$$\mathbf{P}_{VA} = (200 + j\,0)\,(20.02 + j\,4.54) = (200)\,(20.02) + j\,(200)(4.54)$$

Actual power = 200 × 20.02 = 4004 W $\therefore$ W_a = **4004 W**

The other wattmeter W_c carries current of $\mathbf{I}_{co} = -12.78 + j\,10.02$ and has a voltage $\mathbf{V}_{cb} = -\mathbf{V}_{bc} = 100 + j\,173.2$ impressed across it. By the same method, wattmeter reading is

$$W_c = (100 \times -12.78) + (173.2 \times 10.02) = -1278 + 1735.5 = \mathbf{457.5\ W}$$

Example 19.72. *Three resistors 10, 20 and 20 Ω are connected in star to the terminals A, B and C of a 3-φ, 3 wire supply through two single-phase wattmeters for measurement of total power with current coils in lines A and C and pressure coils between A and B and C and B. Calculate (i) the line currents (ii) the readings of each wattmeter. The line voltage is 400-V.* **(Electrical Engineering-I, Bombay Univ. 1987)**

Solution. Let $\mathbf{V}_{AB} = 400\angle 0°$; $\mathbf{V}_{BC} = 400\angle -120°$ and $\mathbf{V}_{CA} = 400\angle 120°$

As shown in Fig. 19.100, current through wattmeter W_1 is $\mathbf{I}_{AO}$ or $\mathbf{I}_A$ and that through W_2 is $\mathbf{I}_{CO}$ or $\mathbf{I}_C$ and the voltages are $\mathbf{V}_{AB}$ and $\mathbf{V}_{CB}$ respectively. Obviously,

$$\mathbf{Z}_A = 10\angle 0°; \mathbf{Z}_B = 20\angle 0°, \mathbf{Z}_C = 20\angle 0°$$

The currents $\mathbf{I}_A$ and $\mathbf{I}_C$ may be found be applying either Kirchhoff's laws (Art. 19.33) or Maxwell's Mesh Method. Both methods will be used for illustration.

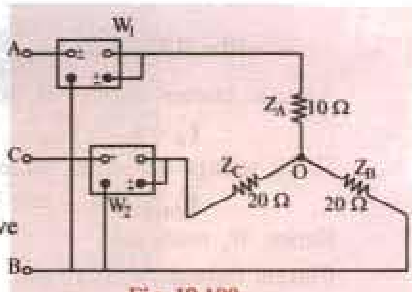


Fig. 19.100

(*a*) From Eq. (10), (11) and (12) of Art. 19.33, we have

$$\mathbf{I}_A = \frac{400 \times 20 - 20(-200 + j346)}{(10 \times 20) + (20 \times 20) + (20 \times 10)}$$
$$= \frac{12{,}000 - j6{,}920}{800} = 15 - j8.65\,\text{A}$$

$$\mathbf{I}_C = \frac{20(-200 + j346) - 10(-200 - j346)}{800} = \frac{-2000 + j10{,}380}{800} = -2.5 + j13$$

(*b*) From Eq. (*i*) and (*ii*) of solved Example 17.48 (*c*) we get

$$\mathbf{I}_A = \mathbf{I}_1 = \frac{400 \times 40 + 20(-200 - j346)}{30 \times 40 - 20^2} = 15 - j8.65\,\text{A}$$

$$\mathbf{I}_C = -\mathbf{I}_2 = \frac{30 \times (-200 - j346) + 400 \times 20}{800} = -25 + j13$$

As seen, wattmeter W_1 carries current $\mathbf{I}_A$ and has a voltage $\mathbf{V}_{AB}$ impressed across its pressure coil. Power may be found by using voltage conjugate.

$$\mathbf{P}_{VA} = (400 - j0)(15 - j8.65) = 6000 - j\,3{,}460$$

$\therefore$ reading of W_1 = 6000 W = **6 kW**

Similarly, W_2 carries $\mathbf{I}_C$ and has voltage $\mathbf{V}_{CB}$ impressed across its pressure coil.

Now, $\mathbf{V}_{CB} = -\mathbf{V}_{BC} = (200 + j346)$. Using voltage conjugate, we get

$$P_{VA} = (200 - j346)(-2.5 + j13)$$

Real power = (200 × − 2.5) + (13 × 346) = 4000 W

∴ reading of W_2 = **4kW** Total power = 10 kW

Example 19.73. *Three impedances Z_A, Z_B and Z_C are connected in delta to a 200-V, 3-phase three-wire symmetrical system RYB.*

$Z_A = 10\ \angle\ 60°$ between lines R and Y ; $Z_B = 10\ \angle\ 0°$ between lines Y and B

$Z_C = 10\ \angle\ 60°$ between lines B and R

The total power in the circuit is measured by means of two wattmeters with their current coils in lines R and B and their corresponding pressure coils across R and Y and B and Y respectively. Calculate the reading on each wattmeter and the total power supplied. Phase sequence RYB.

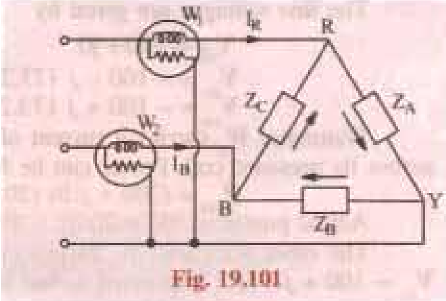


Fig. 19.101

Solution. The wattmeter connections are shown in Fig. 19.101.

$$\mathbf{V}_{RY} = 200\angle 0° = 200 + j0$$

$$\mathbf{V}_{YB} = 200\angle -120° = -100 - j173.2$$

$$\mathbf{V}_{BR} = 200\angle 120° = -100 + j173.2$$

$$\mathbf{I}_{BR} = \frac{200\angle 0°}{10\angle 60°} = 20\angle -60° = 10 - j17.32\,\Omega$$

$$\mathbf{I}_{YB} = \frac{200\angle -120°}{10\angle 0°} = 20\angle -120°$$

$$= -10 - j17.32; \mathbf{I}_{BR} = \frac{200\angle 120°}{10\angle 60°} = 20\angle 60° = 10 + j17.32$$

As seen, current through W_1 is $\mathbf{I}_R$ and voltage across its pressure coil is $\mathbf{V}_{RY}$

$$\mathbf{I}_R = \mathbf{I}_{RY} - \mathbf{I}_{BR} = -j34.64\text{ A}$$

Using voltage conjugate, we have

$$\mathbf{P}_{VA} = (200 - j0)(-j34.64) = 0 - j6{,}928$$

Hence, W_1 reads zero.

Current through W_2 is $\mathbf{I}_B$ and voltage across its pressure coil is $\mathbf{V}_{BY}$

$$\mathbf{I}_B = \mathbf{I}_{BR} - \mathbf{I}_{YB} = 20 + j34.64;\ \mathbf{V}_{BY} = -\mathbf{V}_{YB} = 100 + j173.2$$

Again using voltage conjugate, we get

$$\mathbf{P}_{YA} = (100 - j173.2)(20 + j34.64)$$

$$= 8000 + j0$$

∴ reading of W_2 = **8000 W**

19.36. Phase Sequence Indicators

In unbalanced 3-wire star-connected loads, phase voltages change considerably if the phase sequence of the supply is reversed. One or the other load phase voltage becomes dangerously large which may result in damage to the equipment. Some phase voltage becomes too small which is equally deterimental to some types of electrical equipment. Since phase

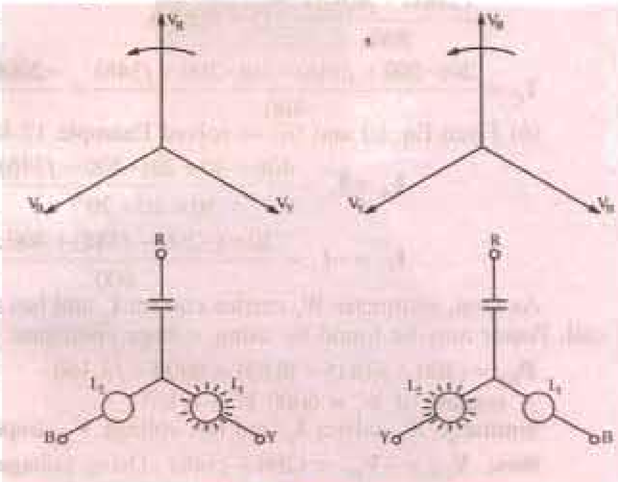


Fig. 19.102

voltage depends on phase sequence, this fact has been made the basis of several types of phasesequence indicators.* A simple phase sequence indicator may be made by connecting two suitable incandescent lamps and a capacitor in a *Y*-connection as shown in Fig. 19.102. It will be found that for phase sequence *RYB*, lamp L_1 will glow because its phase voltage will be large whereas L_2 will not glow because of low voltage across it.

When, phase sequence is *RBY*, opposite conditions develop so that this time L_2 glows but not L_1.

Another method of determining the phase sequence is by means of a small 3-phase motor. Once direction of rotation with a known sequence is found, the motor may be used thereafter for determining an unknown sequence.

Tutorial Problem No. 19.3

1. Three impedances $\mathbf{Z}_1$, $\mathbf{Z}_2$ and $\mathbf{Z}_3$ are mesh-connected to a symmetrical 3-phase, 400-V, 50-Hz supply of phase sequence $R \to Y \to B$.

$\mathbf{Z}_1 = (10 + j0)$ ohm – between *R and Y lines*
$\mathbf{Z}_2 = (5 + j6)$ ohm – between *Y* and *B* lines
$\mathbf{Z}_3 = (5 - j5)$ ohm – between *B* and *R* lines

Calculate the phase and line currents and total power consumed.

[40 A, 40 A, 56.6 A ; 95.7 A, 78.4 A, 35.2 A; 44.8 kW]

2. A symmetrical $3\text{-}\phi$, 380-V supply feeds a mesh-connected load as follows :

Load *A* : 19 kVA at p.f. 0.5 lag ; Load *B* : 20 kVA at p.f. 0.8 lag ; Load *C* : 10 kVA at p.f. 0.9 load

Determine the line currents and their phase angles for *RYB* sequence.

[74.6 $\angle -51°$ A, 98.6 $\angle 172.7°$ A ; 68.3 $\angle 41.8°$ A]

3. Determine the line corrents in an unbalanced *Y* connected load supplied from a symmetrical $3\text{-}\phi$, 440-V, 3-wire system. The branch impedances of the load are : $\mathbf{Z}_1 = 5\ \angle 30°$ ohm, $\mathbf{Z}_2 = 10\ \angle 45°$ ohm and $\mathbf{Z}_3 = 10\ \angle 45°$ ohm and $\mathbf{Z}_3 = 10\ \angle 60°$ ohm. The sequence is *RYB*. **[35.7 A, 32.8 A; 27.7 A]**

4. A $3\text{-}\phi$, *Y*-connected alternator supplies an unbalanced load consisting of three impedances (10 + *j*20), (10 – *j*20) and $10\,\Omega$ respectively, connected in star. There is no neutral connection. Calculate the voltage between the star point of the alternator and that of the load. The phase voltage of the alternator is 230 V. **[–245.2 V]**

5. Non-reactive resistors of 10, 20 and 25 Ω are star-connected to the *R*, *Y* are *B* phases of a 400-V, symmetrical system. Determine the current and power in each resistor and the voltage between star point and neutral. Phase sequence, *RYB*. **[16.5 A, 2.72 kW ; 13.1 A, 3.43 kW; 11.2 A, 3.14 kW ; 68 V]**

6. Determine the line current in an unbalanced, star-connected load supplied from a symmetrical 3-phase, 440-V system. The branch impedance of the load are $\mathbf{Z}_R = 5\ \angle 30°\ \Omega$, $Z_Y = 10\angle 45°\ \Omega$ and $Z_B = 10\angle 60°\ \Omega$. The phase sequence is *RYB*. **[35.7 A, 32.8 A, 27.7 A]**

7. Three non-reactive resistors of 3, 4 and 5-Ω respectively are star-connected to a 3-phase, 400-V symmetrical system, phase sequence *RYB*. Find (*a*) the current in each resistor (*b*) the power dissipated in each resistor (*c*) the phase angles between the currents and the corresponding line voltages (*d*) the star-point potential. Draw to scale the complete vector diagram.

[(*a*) 66.5 A, 59.5 A, 51.8 A (*b*) 13.2, 14.15, 13.4 kW (*c*) 26°24′, 38°10′, 25°20 (*d*) 34 V]

8. An unbalanced *Y*-connected load is supplied from a 400-V, $3\text{-}\phi$, 3-wire symmetrical system. The branch circuit impedances and their connection are (2 + *j*2) Ω, *R* to *N* ; (3 – *j*3) Ω, *Y* to *N* and (4 + *j*1) Ω, *B* to *N* of the load. Calculate (*i*) the value of the voltgage between lines *Y* and *N* and (*ii*) the phase of this voltage relative to the voltage between line *R* and *Y*. Phase sequence *RYB*.

[(*i*) (–216–j 135.2) or 225.5 V (*ii*) 2° or –178°]

9. A star-connection of resistors $R_a = 10\ \Omega$; $R_b = 20\ \Omega$ is made to the terminals *A*, *B* and *C* respectively of a symmetrical 400-V, ϕ supply of phase sequence $A \to B \to C$. Find the branch voltages and currents and star-point voltage to neutral.

**[$\mathbf{V}_A = 148.5 + j28.6$; $\mathbf{I}_A = 14.85 + j2.86$; $\mathbf{V}_B = -198 - j171.4$; $\mathbf{I}_B = -9.9 - j8.57$
$\mathbf{V}_C = -198 + j228.6$; $\mathbf{I}_C = -4.95 + j5.71$. $\mathbf{V}_N = 82.5 - j28.6$ (to be subtracted from supply voltage)]**

* It may, however, be noted that phase sequence of *currents* in an unbalanced load is not necessarily the same as the *voltage* phase sequence. Unless indicated otherwise, voltage phase sequence is implied.

10. Three non-reactive resistance of 5, 10 and 5 ohm are star-connected across the three lines of a 230-V 3-phase,

3-wire supply. Calculate the line currents **[(18.1 + J21.1) A ; (– 10.9–j 10.45) A ; (– 7.3 + j8.4) A]**

11. A 3- ϕ , 400-V symmetrical supply feeds a star-connected load consisting of non-reactive resistors of 3, 4 and 5 Ω connected to the *R*, *Y* and *B* lines respectively. The phase sequence is *RYB*. Calculate (*i*) the load star point potential (*ii*) current in each resistor and power dissipated in each resistor.

[(*i*) 34.5 V (*ii*) 66.4 A, 59.7 A, 51.8 A (*iii*) 13.22 kW, 14.21 kW, 13.42 kW]

12. A 20- Ω resistor is connected between lines *R* and *Y*, a 50- Ω resistor between lines *Y* and *B* and a 10-Ω resistor between lines *B* and *R* of a 415-V, 3-phase supply. Calculate the current in each line and the reading on each of the two wattmeters connected to measure the total power, the respective current coils of which a connected in lines *R* and *Y*. **[(25.9 – j9); (– 24.9 – j7.2); (– 1.04 + j16.2); 8.6 kW; 7.75 kW]**

13. A three-phase supply, giving sinusoidal voltage of 400 V at 50 Hz is connected to three terminals marked *R*, *Y* and *B*. Between *R* and *Y* is connected a resistance of 100 Ω , between *Y* and *B* an inductance of 318 mH and negligible resistance and between *B* and *R* a capacitor of 31.8 μF . Determine (*i*) the current flowing in each line and (*ii*) the total power supplied. Determine (*iii*) the resistance of each phase of a balanced star-connected, non-reactive load, which will take the same total power when connected across the same supply. **[(*i*) 7.73 A, 7.73 A, (*ii*) 1,600 W (*iii*) 100 Ω (*London Univ.*)]**

14. An unbalanced, star-connected load is fed from a symmetrical 3-phase system. The phase voltages across two of the arms of the load are $V_B = 295 \angle 97° 30'$ and $V_R = 206 \angle -25°$. Calculate the voltage between the star-point of the load and the supply neutral. **[52.2 $\angle$–49.54°]**

15. A symmetrical 440-V, 3-phase system supplies a star-connected load with the following branch impedances: $Z_R = 100\ \Omega$, $Z_Y = j\,5\ \Omega$, $Z_B = -j5\ \Omega$. Calculate the voltage drop across each branch and the potential of the neutral point to earth. The phase sequence is *RYB*. Draw the vector diagram.

[$8800 \angle -30°$, $8415\angle -31.5°$, $8420 - \angle -28.5°$, $8545\angle 150°$]

16. Three star-connected impedances, $Z_1 = (20 + j37.7)\ \Omega$ per phase are in parallel with three delta-connected impedances, $Z_2 = (30 - j159.3)\,\Omega$ per phase. The line voltage is 398 V. Find the line current, power factor, power and reactive volt-amperes taken by the combination.

[3.37 $\angle$10.4°; 0.984 lag; 2295 lag; 2295 W; 420 VAR.]

17. A 3-phase, 440-V, delta-connected system has the loads: branch *RY*, 20 KW at power factor. 1.0: branch *YB*, 30 kVA at power factor 0.8 lagging; branch *BR*, 20 kVA at power factor 0.6 leading. Find the line currents and readings on watt-meteres whose current coils are in phases *R* and *B*.

[90.5 $\angle$ 176.5°; 111.4 $\angle$ 14°; 36.7 $\angle$– 119; °39.8 kW; 16.1 kW]

18. A 415 V, 50 Hz, 3-phase supply of phase sequence *RYB* is connected to a delta connected load in which branch *RY* consists of $R_1 = 100\ \Omega$, branch *YB* conists of $R_2 = 20\ \Omega$ in series with $X_2 = 60\ \Omega$ and branch *BR* consists of a capacitor $C = 30\ \mu$F . Take V_{RY} as the reference and calculate the line currents. Draw the complete phasor diagrams. (*Elect. Machines, A.M.I.E. Sec. B. 1989*)

[$I_R = 7.78 \angle 14.54°$, $I_Y = 10.66 \angle 172.92°$, $I_B = 4.46 \angle -47°$]

19. Three resistances of 5, 10 and 15 Ω are connected in delta across a 3-phase supply. Find the values of the three resistors, which if connected in star across the same supply, would take the same line currents.

If this star-connected load is supplied from a 4-wire, 3-phase system with 260 V between lines, calculate the current in the neutral. **[2.5 Ω, 1.67 Ω, 5 Ω; 52 A]** (*London Univ.*)

20. The impedances of the three phase of a star-connected load (no neutral wire) are (5 + *j*20), (12 + *j*0) and (1 – *j*10) in that order. The voltage is 400 volts. Find the line currents.

[(0.5 – j 29.65], (16.24 – j11.5); (– 16.74 + j41.15)] (*Elect. Meas. London Univ.*)]

21. Three voltmeters having resistances of 10, 10, and 5 kΩ respectively, are connected in star to a balanced 3-phase 4-wire supply. The line voltage is 440 V. Determine the readings of the three voltmeters.

[190 V, 290 V, 290 V] (*I.E.C. London*)

22. A 440-V, symmetrical, 3-ϕ supply feeds a *Y*-connected load consisting of three non-inductive resistances of 10, 5, and 12 Ω connected to the *R*, *Y* and *B* phases respectively.

Calculate the line currents and the voltage across each resistor. Phase sequence *RYB*.

[28.9 A, 36.5 A, 25.4 V, 290 V, 182 V, 304 V (*London University*)

23. A symmetrical 3-ϕ, 440-V system supplies a *Y*-connected load, of which the branch resistances are $A = 10\ \Omega$, $B = 13\ \Omega$, $C = 15\ \Omega$. Calculate the voltage to earth of the load star-point, assuming the neutral of the supply to be earthed. Phase sequence *A*, *B*, *C*. **[31 V]** (*I.E.E. London*)

24. Three non-reactive resistors, *A*, *B*, *C* of 5Ω, 10 Ω and 15 Ω respectively are star-connected across the lines of a 230 V, 3-ϕ, 3-wire supply. Calculate the lines currents.

[I_A = 15.16A, I_B = 11.1 A, I_C = 18.25 A] (*City & Guilds, London*)

25. The impedances, in ohms, of a three-phase star-connected load are ; $\mathbf{Z}_1 = 10 + j0$; $\mathbf{Z}_2 = 10 + j5$; $\mathbf{Z}_3 = 5 - j10$. The load is connected to a three phase 50-Hz supply having a balanced line voltage of 440 V. Calculate the reading of single-phase wattmeter which has its current coil in line 1 and its voltage coil between lines 1 and 3. Assume the phase sequence to be 1, 2, 3. Draw a vector diagram showing the relationships of the voltages and currents. **[11.75 kW]** (*London Univ.*)

26. An unbalanced star-connected load is supplied from a symmetrical 400-V, three-phase, three-wire system. The impedances are ; $(1 + j2)\ \Omega$ between line *R* and neutral point of load; $(2 + j3)\ \Omega$ between line *Y* and neutral point of load; $(3 - j3)\ \Omega$, between line *B* and neutral point of load. Calculate the current in line. *B*. Phase sequence *RYB*. **[35.5 A]** (*London Univ.*)

27. An unbalanced star-connected load is supplied from symmetrical three-phase mains. The load impedance are Z_a, Z_b, Z_c and the positive phase sequences is *a*, *b*, *c*. Calculate the value of the positive phase-sequence current when the line voltage of the three-phase balanced supply is 400 V and $\mathbf{Z}_a = (0 + j10)\ \Omega$, $Z_b = (0 - j5)\ \Omega$ and $\mathbf{Z}_c = (10 + j0)\Omega$. **[36.6 A]** (*Elect. Theory and Meas. London Univ.*)

28. The impedances are mesh-connected to a symmetrical three-phase, 40-V, 50, Hz supply of phase sequence *RYB*.

$\mathbf{Z}_1 = (5 + j10)\Omega$ is connected between lines *R* and *Y*

$\mathbf{Z}_2 = (5 + j5)\Omega$ is connected between lines *Y* and *B*

$\mathbf{Z}_3 = (6 + j4)\Omega$ is connected between lines *B* and *R*.

Two wattmeters connected to measure input have their current coils in lines *R* and *Y* respectively. Calculate the line currents and the wattmeter readings.

[95.5 A, 79.4 A, 43.4 A; 39.8 kW, 9.6 kW] (*London Univ.*)

29. In Fig. 19.103 three non-reactive resistors, $R_1 = 10\ \Omega$, $R_2 = 20\ \Omega$ and $R_3 = 30\ \Omega$ are connected in star to a symmetrical 3-phase supply of line voltage 400 V and phase sequence *RYB*. The wattmeter *W* is connected as shown, with the current coil in the *R* line and the voltage circuit connected between lines *R* and *Y*. Calculate the currents in the three lines and the reading of the wattmeter.

[15.9 @, 13.1 A, 9.8A, 5.82 kW] (*I.E.E. London*)

30. A 3ϕ -*Y*-connected load is connected between three line terminals, *R*, *Y*, *B*, the impedances (in ohms) of the load being: $(1 + j2)$ between *R* and the star point, $(2 + j3)$ between *Y* and the star point and $(3 + j4)$ between *B* and the star point. The phase sequence is *RYB*. If the line voltage is 400 V, calculate the voltage between *R* and the star point. **[181 V]** (*London Univ.*)

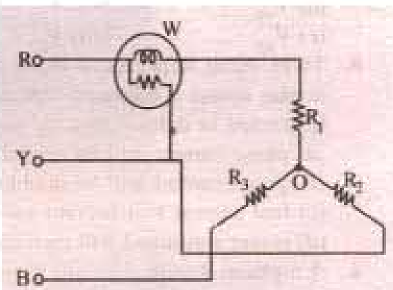


Fig. 19.103

31. A 3-phase 3-wire star-connected load consists of a capacitive reactance of 100 Ω in the red phase and a resistor of 100 Ω in each of the other phases. The line voltage is 440 V and the phase sequence is *RYB*. If the power is measured by the 2-wattmeter method, the current coils being connected in the red and yellow leads respectively, determine the reading of each instrument. **[– 86.5 W, 1.640 W]** (*I.E.E. London*)

32. The impedances, in ohms, of the branches of a three-phase star-connected load are as follows: $\mathbf{Z}_R = (20 + j0)$, $\mathbf{Z}_Y = (10 + j10)$; $\mathbf{Z}_B = (0 - j25)$. If this load, with its neutral point isolated, is connected to a symmetrical three-phase, four-wire system, with 400 V between the line wires, determine the p.d. between the neutral point of the supply system and the neutral point of the load. Phase sequence *RYB*.

[267.5 V] (*London Univ.*)

OBJECTIVE TESTS – 19

1. Electric power is almost exclusively generated, transmitted and distributed, by three phase system because it
   (a) is more efficient
   (b) uses less material for a given capacity
   (c) costs less than single-phase apparatus
   (d) all of the above
2. The voltages induced in the three windings of a 3-phase alternator are ---- degree apart in time phase.
   (a) 120 (b) 60
   (c) 90 (d) 30
3. If positive phase sequence of a 3-phase load is *abc*, the negative sequence would be
   (a) *bac* (b) *cba*
   (c) *acb* (d) all of the above
4. In the balanced 3-phase voltages system generated by a *Y*-connected alternator, V_{YB} lags E_R by ------- electrical degrees.
   (a) 90 (b) 120
   (c) 60 (d) 30
5. The power taken by a 3-ϕ load is given by the expression
   (a) $3V_L I_L \cos\phi$ (b) $\sqrt{3}V_L I_L \cos\phi$
   (c) $3V_L I_L \sin\phi$ (d) $\sqrt{3}V_L I_L \sin\phi$
6. In a balanced 3-phase voltage generator, the different phase voltages reach their maximum values ------ degree apart.
   (a) 120 (b) 60
   (c) 240 (d) 30
7. If the B-phase of a 3-phase, *Y*-connected alternator become reverse connected by mistake, it will not affect
   (a) V_{YB} (b) V_{RY}
   (c) V_{BR} (d) V_{BY}
8. Three equal impedances are first connected in star across a balanced 3-phase supply. If connected in delta across the same supply.
   (a) phase current will be tripled
   (b) phase current will be doubled
   (c) line current will become on-third
   (d) power consumed will increase three-fold.
4. A 3-phase, 4-wire, 230/440-V system is supplying lamp load at 230 V. If a 3-phase motor is now switched on across the same supply, then
   (a) neutral current will increase
   (b) all line currents will decrease
   (c) neutral current will remain unchanged
   (d) power factor will be improved.
10. Power factor improvement
   (a) does not affect the performance characteristics of the original load
   (b) employs series resonance
   (c) increase the active power drawn by the load
   (d) increases the reactive power taken by the load
11. The chief disadvantage of a low power factor is that
   (a) more power is consumed by the load
   (b) current required for a given load power is higher
   (c) active power developed by a generator exceeds its rated output capacity
   (d) heat generated is more than the desired amount
12. In the 2-wattmeter method of measuring 3-phase power, the two wattmeters indicate equal and opposite readings when load power factor angle is ------ degrees lagging.
   (a) 60 (b) 0
   (c) 30 (d) 90
13. When phase sequence at the 3-phase load is reversed
   (a) phase powers are changed
   (b) phase currents are changed
   (c) phase currents change in angle but not in magnitude
   (d) total power consumed is changed.
14. Phase reversal of a 4-wire unbalanced load supplied from a balanced 3-phse supply changes.
   (a) magnitude of phase currents
   (b) magnitudes as well as phase angle of neutral current
   (c) the power consumed
   (d) only the magnitude of neutral current.

ANSWERS

1. d	2. a	3. d	4. a	5. b	6. b	7. b
8. d	9. c	10. a	11. b	12. d	13. c	14. b
15. b	16. a					

20 HARMONICS

20.1. Fundamental Wave and Harmonics

Upto this stage, while dealing with alternating voltages and currents, it has been assumed that they have sinusoidal waveform or shape. Such a waveform is an ideal one and much sought after by the manufactures and designers of alternators. But it is nearly impossible to realize such a waveform in practice. All the alternating waveforms deviate, to a greater or lesser degree, from this ideal sinusoidal shape. Such waveforms are referred to as *non-sinusoidal* or *distorted* or *complex waveforms.*

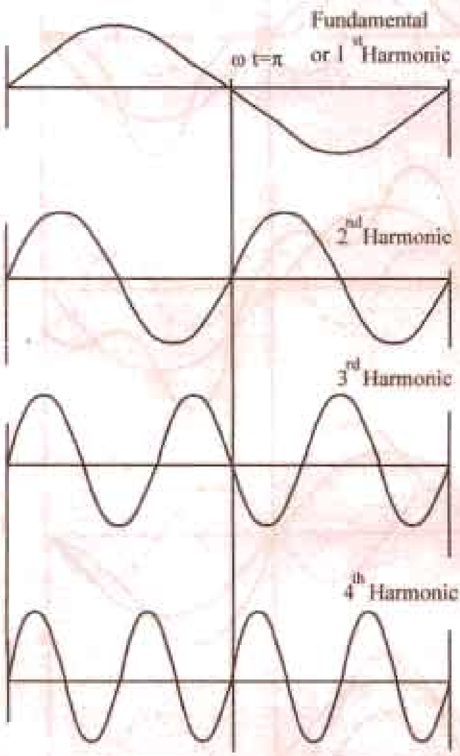


Fig. 20.1

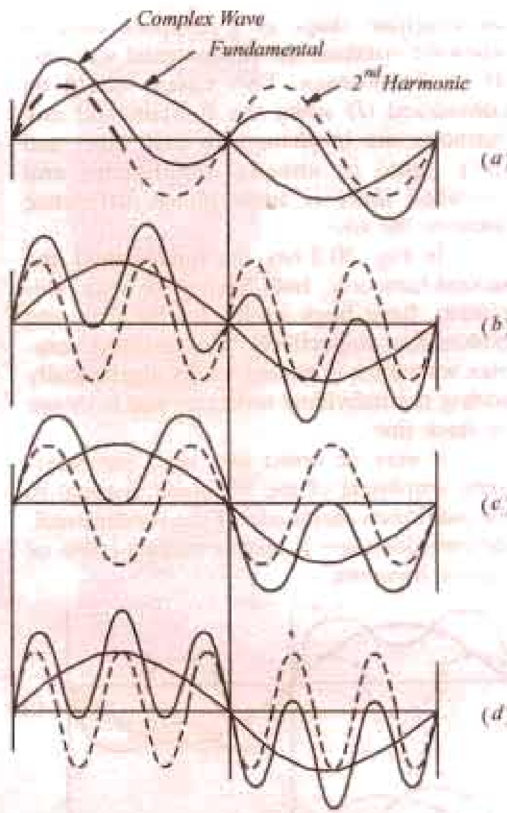


Fig. 20.2

Complex waveforms are produced due to the superposition of sinusoidal waves of different frequencies. Such waves occur in speech, music, TV, rectifier outputs and many other applications of electronics. On analysis, it is found that a complex wave essentially consists of

(*a*) *a fundamental* wave – it has the lowest frequency, say 'f'

(*b*) a number of other sinusoidal waves whose frequencies are an integral multiple of the fundamental or basic frequency like $2f$, $3f$ and $4f$ etc.

The fundamental and its higher multiples form a *harmonic* series.

As shown in Fig. 20.1, fundamental wave itself is called the *first* harmonic. The *second* harmonic has frequency *twice* that of the fundamental, the *third* harmonic has frequency *thrice* that of the fundamental and so on.

Waves having frequencies of 2*f*, 4*f* and 6*f* etc. are called *even harmonics* and those having frequencies of 3*f*, 5*f* and 7*f* etc. are called *odd* harmonics. Expressing the above is angular frequencies, we may say that successive odd harmonics have frequencies of 3ω, 5ω and 7ω etc. and even harmonics have frequencies of 2ω, 4ω and 6ω etc.

As mentioned earlier, harmonics are introduced in the output voltage of an alternator due to many reason such as the irregularities of the flux distribution in it. Considerations of waveform and form factor are very important in the transmission of a. c. power but they are of much greater importance in radio work where the intelligibility of a signal is critically dependent on the faithful transmission of the harmonic structure of sound waves. In fact, it is only the rich harmonic content of the consonants and lesser at still plentiful harmonic content of vowels which helps the ear to distinguish a well regulated speech from a more rhythmical succession of musical sounds.

20.2. Different Complex Waveforms

Let us now find out graphically what the resultant shape of a complex wave is when we combine the fundamental with one of its harmonics. Two cases would be considered (*i*) when the fundamental and harmonic are in phase with each other and have equal or unequal amplitudes and (*ii*) when there is some phase difference between the two.

In Fig. 20.2 (*a*), the fundamental and second harmonic, both having the same amplitude, have been shown by the firm and broken line respectively. The resultant complex waveform is plotted out by algebraically adding the individual ordinates and is shown by thick line.

It may be noted that since the maximum amplitude of the harmonic is equal to the maximum amplitude of the fundamental, the complex wave is said to contain 100% of second harmonic.

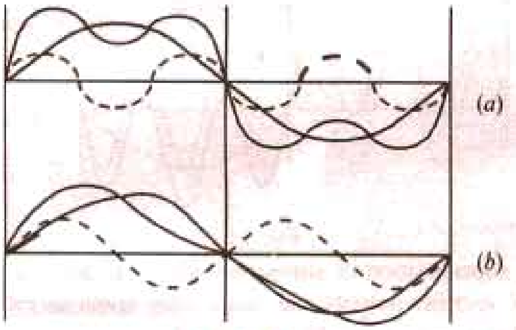


Fig. 20.3

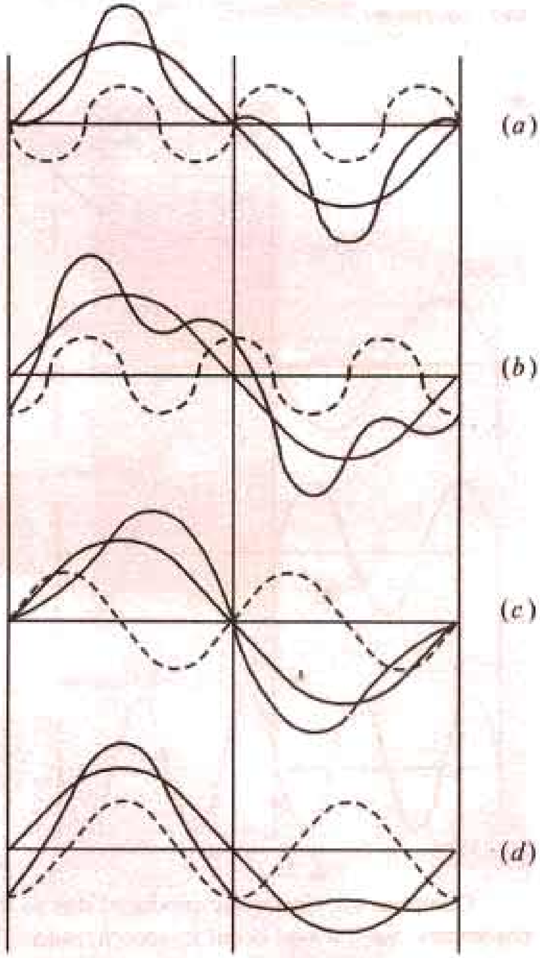


Fig. 20.4

The complex wave of Fig. 20.2 (*b*) is made up of the fundamental and 4th harmonic, that of Fig. 20.2 (*c*) consists of the fundamental and 3rd harmonic whereas that shown in Fig. 20.2 (*d*) is made up of the fundamental and 5th harmonic. Obviously, in all these cases, these is no phase difference between the fundamental and the harmonic.

Fig. 20.2 (*a*) and (*c*) have been reconstructed as Fig. 20.3 (*a*) and (*b*) respectively with the only difference that in this case, the amplitude of harmonic is half that of the fundamental *i.e.* the harmonic content is 50%.

The effect of phase difference between the fundamental and the harmonic on the shape of the resultant complex wave has been illustrated in Fig. 20.4.

Fig. 20.4 (*a*) shows the fundamental and second harmonic with phase difference of $\pi/2$ and Fig. 20.4 (*b*) shows the same with a phase difference of π. In Fig. 20.4 (*c*) and (*d*) are shown the fundamental and third harmonic with a phase difference of $\pi/2$ and π respectively. In all these figures, the amplitude of the harmonic has been taken equal to half that of the fundamental.

A careful examination of the above figures leads us to the following conclusions :

1. With *odd* harmonics, the positive and negative halves of the complex wave are symmetrical whatever the phase difference between the fundamental and the harmonic. In other words, the first and third quarters (*i.e.* ωt from 0 to $\pi/2$ and ωt from $3\pi/2$) and the second and fourth quarters (*i.e.* ωt from $\pi/2$ to π and ωt from $3\pi/2$ to 2π) are respectively similar.

2. (*i*) When *even* harmonics are present and their phase difference with the fundamental is 0 or π, then the first and fourth quarters of the complex wave are of the same phase but inverted and the same holds good for the second and third quarters.

(*ii*) When *even* harmonics are present and their phase difference with the fundamental is $\pi/2$ or $3\pi/2$, then there is no symmetry as shown in Fig. 20 (*a*).

3. It may also be noted that the resultant displacement of the complex wave (whether containing odd or even harmonics) is zero at $\omega t = 0$ only when the phase difference between the fundamental and the harmonics is either 0 to π.

The above conclusions are of great help in analysing a complex waveform into its harmonic constituents because a visual inspection of the complex wave enables us to rule out the presence of certain harmonics. For example, if the positive and negative half-cycles of a complex wave are symmetrical (*i.e.* the wave is symmetrical about $\omega t = 0$), then we need not look for even harmonics. In some cases, we *may* be able to forecast the types of harmonics to be expected from their mode of production. For example, in alternators which are symmetrically designed, we should expect *odd* harmonics only.

20.3. General Equation of a Complex Wave

Consider a complex wave which is built up of the fundamental and a few harmonics, each of which has its own peak value of phase angle. The fundamental may be represented by

$$e_1 = E_{1m}\sin(\omega t + \Psi_1)$$

the second harmonic by $\quad e_2 = E_{2m}\sin(2\omega t + \Psi_2)$

the third harmonic by $\quad e_3 = E_{3m}\sin(3\omega t + \Psi_3)$ and so on.

The equation for the instantaneous value of the complex wave is given by

$$e = e_1 + e_2 + ... e_n = E_{1m}\sin(\omega t + \Psi_1) \times E_{2m}\sin(2\omega t + \Psi_2) + ... + I_{nm}\sin(n\omega t + \Psi_n)$$

when E_{1m}, E_{2m} and E_{nm} etc. denote the maximum values or the amplitudes of the fundamental, second harmonic and *n*th harmonic etc. and Ψ_1, Ψ_2 and Ψ_n represent the phase differences with respect to the complex wave* (*i.e.* angle between the zero value of complex wave and the corresponding zero value of the harmonic).

The number of terms in the seriers depends on the shape of the complex wave. In relatively simple waves, the number of terms in the series would be less, in others, more.

Similarly, the instantaneous value of the complex wave is given by

$$i = I_{1m}\sin(\omega t + \phi_1) + I_{2m}\sin(2\omega t + \phi_2) + ... + I_{nm}\sin(n\omega t + \phi_n)$$

* We could also express these phase anlges with respect to the fundamentals wave instead of the complex wave.

Obviously $(\Psi_1 - \phi_1)$ is the phase difference between the harmonic voltage and current for the fundamental, $(\Psi_2 - \phi_2)$ for the second harmonic and $(\Psi_n - \Phi_2)$ for the *n*th harmonic.

20.4. R.M.S. Value of a Complex Wave

Let the equation of the given complex current wave be

$$i = I_{1m}\sin(\omega t + \phi_1) + I_{2m}\sin(2\omega t + \phi_2)\ldots.I_{nm}\sin(n\omega t + \phi_n)$$

Its r.m.s. value is given by $I = \sqrt{\text{average value of } i^2 \text{ over whole cycle}}$

Now $\quad i^2 = [I_{1m}\sin(\omega t+\phi_1)+I_{2m}\sin(2\omega t+\phi_1)+\ldots.I_{nm}\sin(n\omega t+\phi_n)]^2$

$$= I_{1m}^2\sin^2(\omega t+\phi_1)+I_{2m}^2\sin^2(2\omega t+\phi_2)+\ldots I_{nm}^2\sin^2(n\omega t+\phi_n)]^2$$

$$+2I_{1m}I_{2m}\sin(\omega t+\phi_1)\sin(2\omega t+\phi_2)+2I_{1m}I_{3m}\sin(\omega t+\Phi_1)\sin(\omega t+\Phi_2)+\ldots.$$

The right-hand side of the above equation consists of two types of terms

(*i*) harmonic self-products, the general expression for which is $I_{pm}^2\sin^2(p\omega t+\phi_p)$ for the *p*th harmonic and

(*ii*) the products of different harmonics of the general form $2I_{pm}I_{qm}\sin(p\omega t+\phi_n)\sin(q\omega t+\phi_q)$

The average value of i^2 is the sum of the average values of these individual terms in the above equation. Let us now find the average value of the general term $I_{pm}^2\sin^2(p\omega t+\phi_p)$ over a whole cycle.

$$\text{Average value} = \frac{1}{\pi}\int_0^{2\pi} I_{pm}^2\sin^2(p\omega t+\phi_p)\,d(\omega t) = \frac{I_{pm}^2}{2\pi}\int_0^{2\pi}\sin^2(p\theta+\phi_p)\,d\theta$$

$$= \frac{I_{pm}^2}{2\pi}\int_0^{2\pi}\left(\frac{1-\cos 2(p\theta+\phi_p)}{2}\right)d\theta = \frac{I_{pm}^2}{2\pi}\left|\theta - \sin^2(p\theta+\phi_p)\right|_0^{2\pi}$$

$$= \frac{I_{pm}^2}{4\pi}\times 2\pi = \frac{I_{pm}^2}{2}$$

From this result, we can generalize that

Average value of $I_{1m}^2\sin^2(\omega t+\phi_1) = \dfrac{I_{1m}^2}{2}$

Average value of $I_{2m}^2\sin^2(2\omega t+\phi_2) = \dfrac{I_{2m}^2}{2}$

Average value of $I_{nm}^2\sin^2(n\omega t+\phi_n) = \dfrac{I_{nm}^2}{2}$ and so on.

Now, the average value of the product terms is

$$= \frac{1}{2\pi}\int_0^{2\pi} I_{pm}I_{qm}\sin(p\omega t+\phi_p)\sin(q\omega t+\phi_q)\,d(\omega t)$$

$$= \frac{I_{pm}I_{qm}}{5\pi}\int_0^{2\pi}\sin(p\theta+\phi_p)\sin(q\theta+\phi_q)\,d\theta = 0$$

$\therefore$ Average value of $i^2 = \dfrac{I_{1m}^2}{2}+\dfrac{I_{2m}^2}{2}+\ldots+\dfrac{I_{nm}^2}{2}$

$$\therefore \quad \text{r.m.s. value, } I = \sqrt{\text{average value of } i^2} = \sqrt{\frac{I_{1m}^2}{2}+\frac{I_{2m}^2}{2}+\ldots.\frac{I_{nm}^2}{2}} \qquad \ldots(i)$$

$$= 0.707\sqrt{I_{1m}^2+I_{2m}^2+\ldots+I_{nm}^2}$$

Equation (*i*) above may also be put in the form

$$I = \sqrt{\left(\frac{I_{1m}}{\sqrt{2}}\right)^2 + \left(\frac{I_{2m}}{\sqrt{2}}\right)^2 + \ldots + \left(\frac{I_{nm}}{\sqrt{2}}\right)^2} = \sqrt{I_1^2 + I_2^2 + \ldots I_n^2}$$

where $I_1 = I_{1m}/\sqrt{2}$ – r.m.s. value of fundamental

$I_2 = I_{2m}/\sqrt{2}$ – r.m.s. value of 2nd harmonic

$I_n = I_{nm}/\sqrt{2}$ – r.m.s. value of *n*th harmonic

Similarly, the r.m.s. value of a complex voltage wave is

$$E = 0.707\sqrt{E_{1m}^2 + E_{2m}^2 + \ldots E_{nm}^2} = \sqrt{E_1^2 + E_2^2 + \ldots E_n^2}$$

Hence, the **rule** *is that the r.m.s. value of the complex current (or voltage) wave is given by the square-root of the sum of the squares of the r.m.s. values of its individual components.*

Note. If complex current wave contains a d.c component of constant value I_D then its equation is given by

$$i = I_D + I_{1m}\sin(\omega t + \phi_1) + I_{2m}\sin(2\omega t + \phi_2) + \ldots I_{nm}\sin(n\omega t + \phi_n)$$

r.m.s. value, $I = \sqrt{I_D^2 + (I_{1m}/\sqrt{2})^2 + (I_{2m}/\sqrt{2})^2 \ldots (I_{mn}/\sqrt{2^2})} = I_D^2 + I_1^2 + I_2^2 + \ldots I_n^2$

20.5. Form Factor of a Complex Wave

In general, it may be defined as $k_f = \dfrac{\text{R.M.S. value}}{\text{average value}}$

A general expression for form factor in some simple cases may be found as under :

(*i*) **Sine Series.** Suppose the equation of a complex voltage wave is

$$v = V_{1m}\sin\omega t \pm V_{3m}\sin 3\omega t \pm V_{5m}\sin 5\omega t$$

$$= V_{1m}\sin\theta \pm V_{3m}\sin 3\theta \pm V_{5m}\sin 5\theta \quad \text{where} \quad \omega = 2\pi/T.$$

Obviously, zeros occurs at $t = 0$ or at $\theta = 0°$ and $\theta = 180°$ or $t = T/2$.

Mean value over half-cycle is

$$V_{av} = \frac{1}{\pi}\int_0^{\pi} v\,d\theta$$

$$= \frac{1}{\pi}\left[V_{1m}\int_0^{\pi}\sin\theta.d\theta \pm V_{3m}\int_0^{\pi}\sin 3.d\theta \pm V_{5m}\int_0^{\pi}\sin 5\theta.d\theta\right] = \frac{2}{\pi}\left(\frac{V_{1m}}{1} \pm \frac{V_{3m}}{3} \pm \frac{V_{5m}}{5}\right)$$

As found in Art. 20.4,

$$V = (V_1^2 + V_3^2 + V_5^2)^{1/2} = \frac{1}{\sqrt{2}}(V_{1m}^2 + V_{3m}^2 + V_{5m}^2)^{1/2}$$

$$\therefore \quad k_f = \frac{(1/\sqrt{2})(V_{1m}^2 + V_{3m}^2 + V_{5m}^2)^{1/2}}{\left(\frac{2}{\pi}\right)\left(V_{1m} \pm \frac{V_{3m}}{3} \pm \frac{V_{5m}}{5}\right)}$$

(*ii*) **Cosine Series.** Consider the following cosine series :

$$v = V_{1m}\cos\omega t \pm V_{3m}\cos 3\omega t \pm V_{5m}\cos 5\omega t$$

$$= V_{1m}\cos\theta \pm V_{3m}\cos 3\theta \pm V_{5m}\cos 5\theta$$

Obviously, in this case, zeros occur at $\theta = \pm\pi/2$ or 90°. Moreover, positive and negative half-cycles are symmetrical.

$$\therefore \quad V_{av} = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}(V_{1m}\cos\theta \pm V_{3m}\cos 3\theta \pm V_{5m}\cos 5\theta)\,d\theta = \frac{2}{\pi}\left(V_{1m} \pm \frac{V_{3m}}{3} \pm \frac{V_{5m}}{5}\right)$$

$$\therefore \quad k_f = \frac{(1/\sqrt{2})(V_{1m}^2 + V_{3m}^2 + V_{5m}^2)^{1/2}}{\left(\frac{2}{\pi}\right)\left(V_{1m} \pm \frac{V_{3m}}{3} \mp \frac{V_{5m}}{5}\right)}$$

Example 20.1. *A voltage given by $v = 50 + 24 \sin \omega t - 20 \sin 2\omega t$ is applied across the circuit shown in Fig. 20.5. What would be the readings of the instruments if $\omega = 10{,}000$ rad/s. A_1 is thermoelectric ammeter, A_2 a moving-coil ammeter an V an electrostatic voltmeter.*

Solution. It may be noted that the thermoelectric ammeter and the electrostatic voltmeter record the r.m.s. values of the current and voltage respectively. But the moving coil ammeter records the average values. Since the average values of the sinusoidal waves are zero, hence the moving coil ammeter reads the d.c. component of the current only. The d.c. will pass only through the inductive branch and not through the capacitive branch.

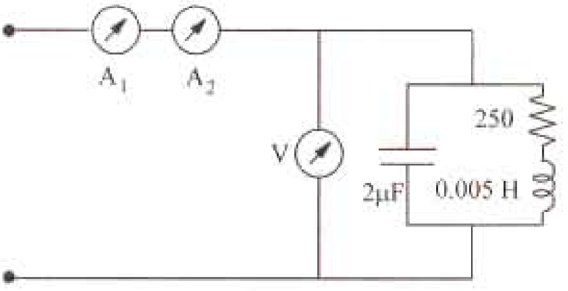


Fig. 20.5

$$\therefore \quad I_{DC} = \frac{V_{DC}}{R} = \frac{50}{250} = \mathbf{0.2\ A}$$

Equivalent impedance of the circuit at fundamental frequency is

$$\mathbf{Z}_1 = \frac{(R + jX_{L1})(-jX_{C1})}{R + jX_{L1} - jX_{C1}} = \frac{(250 + j50)(-j50)}{250 + j(50-50)} = \frac{2{,}500 - j12{,}500}{250} = 10 - j50 = 51\angle -78°42'$$

$\therefore$ r.m.s. fundamental current $I_1 = I_{1m}/\sqrt{2} = 24/51 \times \sqrt{2} = 0.33$ A

Equivalent impedance of the circuit at the second harmonic is

$$\mathbf{Z}_2 = \frac{(R + jX_{L2})(-jX_{C2})}{R + jX_{L2} - jX_{C2}} = \frac{(250 + 100)(-j25)}{200 + j75} = 31.5\angle -88°43'$$

$\therefore$ r.m.s. value of second harmonic current

$$I_2 = I_{2m}/\sqrt{2} = 20/31.5 \times \sqrt{2} = 0.449 \text{ A}$$

r.m.s. current in the circuit is

$$I = \sqrt{I_{DC}^2 + I_1^2 + I_2^2} = \sqrt{0.2^2 + 0.33^2 + 0.449^2} = 0.593 \text{ A}$$

Hence, the reading of the thermoelectric ammeter is **0.593 A**

The voltmeter reading $V = \sqrt{50^2 + (24/\sqrt{2})^2 + (20/\sqrt{2})^2} = \mathbf{54.66}$

Example 20.2. *Draw one complete cycle of the following wave*

$i = 100 \sin \omega t + 40 \sin 5\ \omega t$

Determine the average value, the r.m.s. value and form factor of the wave.

(Elect. Engineering, Osmania Univ. 1985)

Solution. $I_{av} = \frac{2}{\pi}\left(\frac{I_{2m}}{1} + \frac{I_{5m}}{5}\right) = \frac{2}{\pi}\left(\frac{100}{1} + \frac{40}{5}\right) = \mathbf{68.7\ A}$

$$I = \sqrt{\frac{I_{1m}^2}{2} + \frac{I_{5m}^2}{2}} = \frac{1}{\sqrt{2}}(100^2 + 40^2)^{1/2} = \mathbf{76.2\ A}$$

Form factor $= \dfrac{I}{I_{av}} = \dfrac{76.2}{68.7} = \mathbf{1.109}$

20.6. Power Supplied by a Complex Wave

Let the complex voltage be represented by the equation

$$e = E_{1m} \sin \omega t + E_{2m} \sin 2\omega t + \ldots.. E_{nm} \sin n\omega t$$

be applied to a circuit. Let the equation of the resultant current wave be

$$i = I_{1m} \sin(\omega t + \phi_1) + I_{2m} \sin(2\omega t + \phi_2) + \ldots.. I_{nm} \sin(n\omega t + \phi_n)$$

The instantaneous value of the power in the circuit is $p = ei$ watt

For obtaining the value of this product, we will have to multiply every term of the voltage wave, in turn, by every term in the current wave. The average power suplied during a cycle would be equal to the sum of the average values over one cycle of each individual product term. However, as proved in Art. 20.4 earlier, the average value of all product terms involving harmonics of different frequencies will be zero over one cycle, so that we need consider only the products of current and voltage harmonics of the same frequency.

Let us consider a general term of this nature *i.e.* E_{nm} sin $n\omega t \times I_{nm}$ sin $(n\omega t - \phi_n)$ and find its average value over one cycle of the fundamental.

$$\text{Average value of power } = \frac{1}{2\pi}\int_0^{2\pi} E_{nm}I_{nm} \sin n\omega t \sin(n\omega t - \phi_n)\, d(\omega t)$$

$$= \frac{E_{nm}I_{nm}}{2\pi}\int_0^{2\pi} \sin n\theta \sin(n\theta - \phi_n)\, d\theta$$

$$= \frac{E_{nm}I_{nm}}{2\pi}\int_0^{2\pi} \frac{\cos\phi_n - \cos(2n\theta - \phi_n)}{2}\, d\theta$$

$$= \frac{E_{nm}I_{nm}\cos\phi}{2} = \frac{E_{nm}}{\sqrt{2}}\cdot\frac{I_{nm}}{\sqrt{2}}\cdot\cos\phi_n = E_nI_n\cos\phi_n$$

where E_n and I_n are the r.m.s. values of the voltage and current respectively. Hence, total average power supplied by a complex wave is *the sum of the average power supplied by each harmonic component acting independently.*

$\therefore$ Total power is $P = E_1I_1\cos\phi_1 + E_2I_2\cos\phi_2 + \ldots\ldots E_nI_n\cos\phi_n$

The overall power factor is given by

$$\text{p.f.}^{*} = \frac{\text{total watts}}{\text{total voltamperes}} = \frac{E_1I_1\cos\phi_1 + E\ I_2\cos\phi_2 + \ldots}{E\times I}$$

when E = r.m.s. value of the complex voltage wave

I = r.m.s. value of the complex current wave

Example 20.3. *A single-phase voltage source 'e' is given by*

$$e = 141\sin\omega t + 42.3\sin 3\omega t + 28.8\sin 5\omega t$$

The corresponding current in the load circuit is given by

$$i = 16.5\sin(\omega t + 54.5^\circ) + 8.43\sin(3\omega t - 38^\circ) + 4.65\sin(5\omega t - 34.3^\circ)$$

Find the power supplied by the source. **(Electrical Circuits, Nagpur Univ. 1991)**

Solution. In problems of such type, it is best to deal with each harmonic separately

Power at fundamental

$$= E_1I_1\cos\phi_1 = \frac{E_{1m}}{\sqrt{2}}\cdot\frac{I_{1m}}{\sqrt{2}}\cos\phi_1 = \frac{E_{1m}I_{1m}}{2}\cos\phi_1 = \frac{141\times16.5}{2}\cos54.5^\circ = 675.5\text{ W}$$

$$\text{Power at 3rd harmonic } = \frac{E_{3m}I_{3m}}{2}\cos\phi_3 = \frac{42.3\times8.43}{2}\cos38^\circ = 140.5\text{ W}$$

$$\text{Power at 5th harmonic } = \frac{28.8\times4.65}{2}\cos34.3^\circ = 55.5\text{ W}$$

Total power supplied = 675.5 + 140.5 + 55.5 = **871.5 W**

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* When harmonics are present, it is obvious that the overall p.f. of the circuit cannot be stated lagging or leading. It is simply the ratio of power in watts of voltamperes.

Example 20.4. *A complex voltage is given by $e = 60 \sin \omega t + 24 \sin (3\omega t + \omega/6) + 12 \sin (5\omega t + \pi/3)$ is applied across a certain circuit and the resulting current is given by*

$$i = 0.6 \sin(\omega t - 2\pi/10) + 0.12\sin(\omega t - 2\pi/24) + 0.1\sin(5\pi - 3\pi/4)$$

Find (i) r.m.s value of current and voltage (ii) total power supplied and (iii) the overall power factor.

Solution. In such problems where harmonics are involved, it is best to deal with each harmonic separately.

$$\text{Power at fundamental} = E_1 I_1 \cos\phi_1 = \frac{E_{1m} I_{1m}}{2}\cos\phi_1 = \frac{60 \times 0.6}{2} \times \cos 36° = 14.56 \text{ W}$$

$$\text{Power at 3rd harmonic} = \frac{E_{3m} I_{3m}}{2}\cos 45° = \frac{24 \times 0.12}{2} \times 0.707 = 1.02 \text{ W}$$

$$\text{Power at 5th harmonic} = \frac{E_{5m} I_{5m}}{2}\cos 75° = \frac{12 \times 0.1}{2} \times 0.2588 = 0.16 \text{ W}$$

(*i*) R.M.S. current R.M.S. current $I = \sqrt{I_1^2 + I_3^2 + I_5^2} = \sqrt{\left(\frac{I_{1m}}{\sqrt{2}}\right)^2 + \left(\frac{I_{3m}}{\sqrt{2}}\right)^2 + \left(\frac{I_{5m}}{\sqrt{2}}\right)^2}$

(*ii*) Total power $= 14.56 + 1.02 + 0.16 =$ **15.74 W** $= \sqrt{\frac{0.6^2}{2} + \frac{0.12^2}{2} + \frac{0.1^2}{2}} =$ **0.438 A**

R.M.S. volts, $E = \sqrt{\frac{60^2}{2} + \frac{24^2}{2} + \frac{12^2}{2}} =$ **46.5 V**

(*iii*) Overall p.f. $= \frac{\text{watts}}{\text{voltamperes}} = \frac{15.74}{46.5 \times 0.438} =$ **0.773**

20.7. Harmonics in Single-phase A.C. Circuits

If an alternating voltage, containing various harmonics, is applied to a single-phase circuit containing linear circuit elements, then the current so produced also contains harmonics. Each harmonic voltage will produce its own current independent of others. By the principle of superposition, the combined current can be found. We will now consider some of the well-known elements like pure resistance, pure inductance and pure capacitance and then various combinations of these. In each case, we will assume that the applied complex voltage is represented by

$$e = E_{1m}\sin\omega t + E_{2m}\sin 2\omega t + \ldots + E_{nm}\sin n\omega t$$

(*a*) Pure Resistance

Let the circuit have a resistance of R which is independent of frequency.

The instantaneous current i_1 due to fundamental voltage is

$$i_1 = \frac{E_{1m}\sin\omega t}{R}$$

Similarly, $\quad i_2 = \frac{E_{2m}\sin 2\omega t}{R}$ for 2nd harmonic

and $\quad i_n \frac{E_{nm}\sin n\omega t}{R}$ ... for nth harmonic

total current $\quad i = i_1 + i_2 + \ldots + i_n$

$$= \frac{E_{1m}\sin\omega t}{R} + \frac{E_{2m}\sin 2\omega t}{R} + \ldots + \frac{E_{nm}\sin n\omega t}{R}$$

$$= I_{1m}\sin\omega t + I_{2m}\sin 2\omega t + \ldots + I_{nm}\sin n\omega t$$

It shows that

(*i*) the waveform of the resulting current is similar to that of the applied voltage *i.e.* the two waves are identical.

(*ii*) the percentage of harmonic content in the current wave is the same as in the applied voltage.

(*b*) Pure Inductance

Let the inductance of the circuit be L henry whose reactance varies directly as the frequency of the applied voltage. Its reactance for the fundamental would be $X_1 = \omega L$: for the second harmonic, $X_2 = 2\omega L$, for the third harmonic, $X_3 = 2\omega L$ and for the nth harmonic $X_n = n\omega L$. However, for every harmonic term, the current will lag behind the voltage by 90°.

Current due to fundamental, $i_1 = \dfrac{E_{1m}}{\omega L}\sin(\omega t - \pi/2)$

Current due to 2nd harmonic, $i_2 = \dfrac{E_{2m}}{2\omega L}\sin(2\omega t - \pi/2)$

Current due to 3rd harmonic, $i_3 = \dfrac{E_{3m}}{3\omega L}\sin(3\omega t - \pi/2)$

Current due to nth harmonic, $i_n = \dfrac{E_{nm}}{n\omega L}\sin(n\omega t - \pi/2)$

$\therefore$ Total current $i = i_1 + i_2 + ...i_n$

$$= \frac{E_{1m}}{\omega L}\sin(\omega t - \pi/2) + \frac{E_2 m}{2\omega L}\sin(2\omega t - \pi/2) + \ldots + \frac{E_{nm}}{n\omega L}\sin(n\omega t - \pi/2)$$

It can be seen from the above equation that

(*i*) the waveform of the current differs from that of the applied voltage.

(*ii*) for the nth harmonic, the percentage harmonic content in the current-wave is $1/n$ of the corresponding harmonic content in the voltage wave. It means that in an inductive circuit, the current waveform shows less distortion that the voltage waveform. In this case, current more nearly approaches a sine wave than it does in a circuit containing resistance.

(*c*) Pure Capacitance

In this case,

$X_1 = \dfrac{1}{\omega C}$ – for fundamental ; $X_2 = \dfrac{1}{2\omega C}$ – for 2nd harmonic

$X_3 = \dfrac{1}{3\omega C}$ – for 3rd harmonic ; $X_n = \dfrac{1}{n\omega C}$ – for nth harmonic

$$i_1 = \frac{E_{1m}}{1/\omega C}\sin(\omega t + \pi/2) = \omega C E_{1n}\sin(\omega t + \pi/2)$$

$$i_2 = \frac{E_{2m}}{1/2\omega C}\sin(2\omega t + \pi/2) = 2\omega C E_{2m}\sin(2\omega t + \pi/2)$$

$$i_n = \frac{E_{mn}}{1/n\omega C}\sin(n\omega t + \pi/2) = n\omega C E_{nm}\sin(n\omega t + \pi/2)$$

For every harmonic term, the current will lead the voltage by 90°.

Now $i = i_1 + i_2 + ... + i_n$

$$= \omega C E_{1m}\sin(\omega t + \pi/2) + 2\omega C E_{2m}\sin(2\omega t + \pi/2) + \ldots + n\omega C E_{nm}\sin(n\omega t + \pi/2)$$

This equation shows that

(*i*) the current and voltage waveforms are dissimilar.

(*ii*) percentage harmonic content of the current is larger than that of the applied voltage wave. For example, for nth harmonic, it would be n time larger.

(*iii*) as a result, the current wave is more distorted than the voltage wave.

(*iv*) effect of capacitor on distortion is just the reverse of that of inductance.

Example 20.5. *A complex wave of r.m.s. value 240 V has 20% 3rd harmonic content, 5% 5th harmonic content and 2% 7th harmonic content. Find the r.m.s. value of the fundamental and of each harmonic.* **(Elect. Circuits, Gujarat Univ. 1985)**

Solution. Let V_1, V_3, V_5 and V_7 be the r.m.s. values of the fundamental and harmonic voltages. Then

$$V_3 = 0.2\ V_1\ ;\ \ V_5 = 0.05\ V_1 \text{ and } V_7 = 0.02\ V_1$$

$$240 = (V_1^{\,2} + V_3^{\,2} + V_5^{\,2} + V_7^{\,2})^{1/2}$$

$$\therefore \quad 240 = [V_1^{\,2} + (0.2\ V_1)^2 + (0.05\ V_1)^2 + (0.02\ V_1)^2]^{1/2}$$

$$\therefore \quad V_1 = \mathbf{235\ V}\ ;\ V_3 = 0.2 \times 235 = \mathbf{47\ V}$$

$$V_5 = 0.05 \times 235 = \mathbf{11.75\ V}\ ;\ V_7 = 0.02 \times 235 = \mathbf{4.7\ V}$$

Example 20.6. *Derive an expression for the power, power factor and r.m.s. value for a complex wave.*

A voltage $e = 250 \sin \omega t + 50 \sin(3\omega t + \pi/3) + 2 \sin(\omega t + 5\pi/6)$ *is applied to a series circuit of resistance* 20 Ω *and inductance 0.05 H. Derive (a) an expression for the current (b) the r.m.s. value of the current and for the voltage (c) the total power supplied and (d) the power factor. Take* ω = *314 rad/s.* **(Electrical Circuits, Nagpur Univ. 1991)**

Solution. For Fundamental

$$X_1 = \omega L = 314 \times 0.05 = 15.7\ \Omega;\ \mathbf{Z}_1 = 20 + j15.7 = 25.4\angle 38.1°\ \Omega$$

For Third Harmonic

$$X_3 = 3\omega L = 3 \times 15.7 = 47.1\,\Omega;\ \mathbf{Z}_3 = 20 + j47.1 = 51.2\angle 67°\ \Omega$$

For Fifth Harmonic

$$X_5 = 5\,\omega L = 5 \times 15.7 = 78.5\,\Omega;\ \mathbf{Z}_5 = 20 + j78.5 = 81\angle 75.7°\ \Omega$$

(*a*) Expression for the current is

$$i = \frac{250}{25.4}\sin(\omega t - 38.1°) + \frac{50}{51.2}\sin(3\omega t + 60° - 67°) + \frac{20}{81}\sin(5\omega t + 150° - 75.7°)$$

$$\therefore \quad i = 9.84\sin(\omega t - 38.1°) + 0.9\sin(3\omega t - 7°) + 0.25\sin(5\omega t + 74.3°)$$

(*b*) *R.M.S. current* $\quad I = \sqrt{\dfrac{I_{1m}^{\;2}}{2} + \dfrac{I_{3m}^{\;2}}{2} + \dfrac{I_{5m}^{\;2}}{2}}$

$$I^2 = \frac{9.84^2}{2} + \frac{0.9^2}{2} + \frac{0.25^2}{2} = 48.92$$

$$\therefore \quad I = \sqrt{48.92} = \mathbf{6.99\ A}$$

R.M.S. voltage $\quad V = \sqrt{\dfrac{250^2}{2} + \dfrac{50^2}{2} + \dfrac{20^2}{2}} = \mathbf{180.8\ V}$

(*c*) Total power $\quad = I^2R = 48.92 \times 20 = \mathbf{978\ W}$

(*d*) Power factor $= \dfrac{\text{Watts}}{\text{VA}} = \dfrac{978}{180.8 \times 6.99} = \mathbf{0.773}$

Example 20.7. *An r.m.s. current of 5 A, which has a third harmonic content, is passed through a coil having a resistance of 1 Ω and an inductance of 10 mH. The r.m.s. voltage across the coil is 20 V. Calculate the magnitude of the fundamental and harmonic components of current if the fundamental frequency is 300/2 π Hz. Also, find the power dissipated.*

Solution. (*i*) **Fundamental Frequency**

$\omega = 300$ rad/s ; $X_L = 300 \times 10^{-2} = 3\ \Omega$ $\therefore$ $\mathbf{Z}_1 = 1 + j\,3 = 3.16\ \angle\ 71.6°$ ohm

If V_1 is the r.m.s. value of the fundamental voltage across the coil, then

$$V_1 = I_1 Z_1 = 3.16\ I_1$$

(*ii*) **Third Harmonic**

$X_3 = 3 \times 3 = 9\ \Omega$; $\mathbf{Z}_3 = 1 + j\,9 = 9.05\ \angle 83.7°$ ohm ; $V_3 = I_3 Z_3 = 9.05\ I_3$

Since r.m.s. current of the complex wave is 5 A and r.m.s. voltage drop 20 V

$$5 = \sqrt{I_1^{\,2} + I_3^{\,2}} \text{ and } 20 = \sqrt{V_1^{\,2} + V_3^{\,2}}$$

Substituting the values of V_1 and V_3, we get, $20 = [(3.16\ I_1)^2 + (9.05\ I_3)^2]^{1/2}$

Solving for I_1 and I_3, we have $I_1 =$ **4.8 A** and $I_3 =$ **1.44 A**

Power dissipated $= I^2R = 5^2 \times 1 =$ **25 W**

Example 20.8. *An e.m.f. represented by the equation $e = 150 \sin 314\,t + 50 \sin 942\,t$ is applied to a capacitor having a capacitance 20 μF. What is the r.m.s. value of the charging current ?*

Solution. For Fundamental

$$X_{C1} = 1/\omega C = 10^6/20 \times 314 = 159\Omega;\ I_{1m} = E_{1m}/X_{C1} = 150/159 = 0.943 \text{ A}$$

For Third Harmonic

$X_{C3} = 1/3\ \omega C = 159/3 = 53\ \Omega$ $\therefore$ $I_{3m} = E_{3m}/X_{C3} = 50/53 = 0.943$ A

r.m.s. value of charging current,

$$I = \sqrt{\frac{I_{1m}^{\ \ 2}}{2} + \frac{I_{3m}^{\ \ 2}}{2}} = \sqrt{\frac{0.943^2}{2} + \frac{0.943^2}{2}} \quad \text{or} \quad \mathbf{I = 0.943\ A}$$

Example 20.9. *The voltage given by $v = 100 \cos 314\,t + 50 \sin (1570t - 30°)$ is applied to a circuit consisting of a 10 Ω resistance, a 0.02 H inductance and a 50 μF capacitor. Determine the instantaneous current through the circuit. Also find the r.m.s. value of the voltage and current.*

Solution. For Fundamental

$\omega = 314$ rad/s; $X_L = 314 \times 0.02 = 6.28\ \Omega$

$X_C = 10^6/314 \times 50 = 63.8\ \Omega$; $X = X_L - X_C = 6.28 - 63.8 = -\,57.32\ \Omega$

$Z = \sqrt{10^2 + (-57.32)^2} = 58.3\ \Omega$; $I_{1m} = 100/58.3 = 1.71$ A

$\phi_1 = \tan^{-1}(-\,57.32/10) = -\,80.2°$ (lead) ; $i_1 = 1.71 \cos(314t + 80.2°)$

For Fifth Harmonic

Inductive reactance $= 5\ X_L = 5 \times 6.28 = 31.4\ \Omega$

Capacitive reactance $= X_C/5 = 63.8/5 = 12.76\ \Omega$

Net reactance $= 31.4 - 12.76 = 18.64\ \Omega$

$$Z = \sqrt{10^2 + 18.64^2} = 21.2\Omega$$

$I_{5m} = 50/21.2 = 2.36$ A; $\phi_5 = \tan^{-1}(18.64/10) = 61.8°$ (lag)

$i_5 = 2.36 \sin(1570\,t - 30° - 61.8°) = 2.36 \sin(1570t - 91.8°)$

Hence, total instantaneous current is

$$i = i_1 + i_5 = 1.71 \cos(314\,t + 80.2°) + 2.36 \sin(1570t - 91.8°)$$

$$R.M.S.\ volt = \sqrt{\frac{100^2}{2} + \frac{50^2}{5}} = \mathbf{79.2\ V}$$

$$R.M.S.\ current = \sqrt{\frac{1.71^2}{2} + \frac{2.36^2}{2}} = \mathbf{2.06\ A}$$

Example 20.10. *A 6.36 μF capacitor is connected in parallel with a resistance of 500 Ω and the combination is connected in series with a 500-Ω resistor. The whole circuit is connected across an a.c. voltage given by e = 300 sin ωt + 100 sin (3ωt + π / 6).*

If ω = 314 rad/s, find

(i) power dissipated in the circuit

(ii) an expression for the voltage across the series resistor

(iii) the percentage harmonic content in the resultant current.

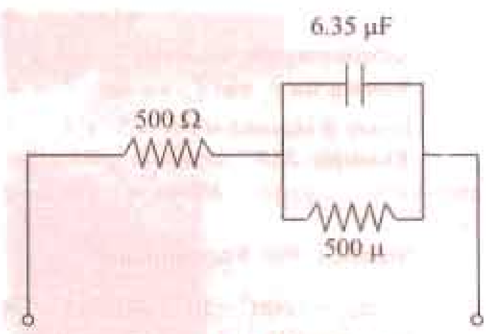


Fig. 20.6

Solution. For Fundamental

$$X_{C1} = \frac{1}{\omega C} = \frac{10^6}{314 \times 6.36} = 500\ \Omega$$

The impedance of the whole series-Parallel circuit is given by

$$\mathbf{Z}_1 = 500 + \frac{500(-j500)}{500 - j500} = 750 - j250 = 791\angle -18.4°$$

For Third Harmonic

$$X_{C3} = 1/3\omega\,C = 500/3 = 167\ \Omega$$

$$\therefore\quad \mathbf{Z}_3 = 500 + \frac{500(-j167)}{500 - j167} = 550 - j150 = 570\angle -15.3°$$

$$\therefore\ i = \frac{300}{791}\sin(\omega t + 18.4°) + \frac{100}{570}\sin(3\omega t + 45.3°)$$

$$= 0.397\ \sin(\omega t + 18.4°) + 0.175 \sin(3\omega t + 45.3)$$

$$(i)\ \text{Power dissipated} = \frac{E_{1m}I_{1m}}{2}\cos\phi_1 + \frac{E_{3m}I_{3m}}{2}\cos\phi_3$$

$$= \frac{300 \times 0.379}{2} \times \cos 18.4° + \frac{100 \times 0.175}{2}\cos 15.3 = \mathbf{62.4\ W}$$

(ii) The voltage drop across the series resistor would be

$$E_R = iR = 500[0.379\ \sin(\omega t + 18.4°) + 0.175 \sin(3\omega t + 45.3°)]$$

$$e_R = \mathbf{189.5\ sin\ (\omega t + 18.4°) + 87.5 sin(3\omega t + 45.3°)}$$

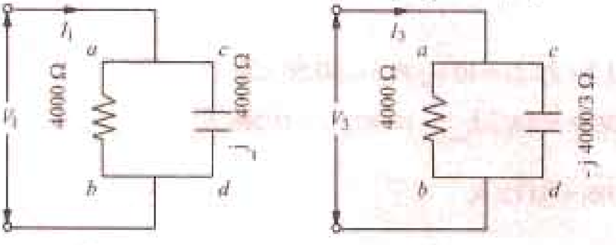
(iii) The percentage harmonic content of the current is = 87.5/189.5 × 100 = 46.2%

Example 20.11. *An alternating voltage of v = 1.0 sin 500t + 0.5 sin 1500t is applied across a capacitor which can be represented by a capacitance of 0.5 μF shunted by a resistance of 4,000 Ω. Determine*

(i) the r.m.s. value of the current (ii) the r.m.s. value of the applied voltage

(iii) the p.f. of the circuit. **(Circuit Theory and Components, Madras Univ. 1981)**

Solution. For Fundamental [Fig. 20.7 (*a*)]



(*a*) (*b*)

Fig. 20.7

$V_1 = 1.0/\sqrt{2} = 0.707$ V

Let, $V_1 = (0.707 + j0)$

Capacitive reactance

$= jX_{C1} = -j\ 10^6/500 \times 0.5$

$= j\ 4000\ \Omega\ ;\ R = 4{,}000\ \Omega$

$\therefore\ I_{ab1} = 0.707/4{,}000 = 0.177$ mA

$I_{cd1} = 0.707/-j4{,}000 = j0.177$ m/A

$\therefore\ I_1 = 0.177 + j0.177$

$= 0.25\ \angle 45^\circ$ mA

Hence, I_1 lead the fundamental voltage by 45°.

$P_{ab1} = 0.707 \times 0.177 = 0.125$ mW ; $P_{cd1} = 0$

For Third Harmonic [Fig. 20.7 (*b*)]

$\mathbf{V}_3 = 0.5/\sqrt{2} = 0.3535\ \angle 0^\circ\ ;\ R = 4{,}000\ \Omega\ ;\ \mathbf{X}_{C3} = -j4{,}000/3\ \Omega$

$\mathbf{I}_{ab3} = 0.3535/4000 = 0.0884\ \angle 0^\circ$ mA ; $\mathbf{I}_{cd3} = 0.3535/-j(4{,}000/3) = j\ 0.265$ mA

$\mathbf{I}_1 = 0.0884 + j0.265 = 0.28\ \angle 71.6^\circ$ mA

$P_{ab3} = 0.3535 \times 0.0884 = 0.0313$ mW ; $P_{cd3} = 0$

(*i*) *R.M.S.* current $= \sqrt{I_1^{\,2} + I_3^{\,2}} = \sqrt{0.25^2 + 0.28^2} =$ **0.374 mA**

(*ii*) *R.M.S.* voltage $= \sqrt{(1/\sqrt{2})^2 + (0.5/\sqrt{2})^2} =$ **0.79 V**

(*iii*) Power factor = watts/voltamppers

Wattage = $(0.125 + 0.0313) \times 10^{-3} = 0.1563 \times 10^{-3}$ W

Volt-amperes = $0.79 \times 0.374 = 0.295$; p.f. = $0.1563 \times 10^{-3}/0.295 =$ **0.0005**

20.8. Selective Resonance due to Harmonics

When a complex voltage is applied across a circuit containing both inductance and capacitance, it may happen that the circuit resonates at one of the harmonic frequencies of the applied voltage. This phenomenon is known as *selective resonance.*

If it is a series circuit, then large currents would be produced at resonance, even though the applied voltage due to this harmonic may be small. Consequently, it would result in large harmonic voltage appearing across both the capacitor and the inductance.

If it is a parallel circuit, then at resonant frequency, the resultant current drawn from the supply would be minimum.

It is because of the possibility of such selective resonance happening that every effort is made to eliminate harmonics in supply voltage.

However, the phenomenon of selective resonance has been usefully employed in some wave analyses for determining the harmonic content of alternating waveforms. For this purpose, a variable inductance, a variable capacitor, a variable non-inductive resistor and a fixed non-inductive resistance or shunt for an oscillograph are connected in series and connected to show the waveform of the voltage across the fixed non-inductive resistance. The values of inductance and capacitance are adjusted successively to give resonance for the first, third, first and seventh harmonics and a record of the waveform is obtained by the oscillograph. A quick inspection of the shape of the waveform helps to detect the presence of absence of a particular harmonic.

Example 20.12. *An e.m.f. $e = 200 \sin \omega t + 40 \sin 3\ \omega t + 10 \sin 5\ \omega t$ is impressed on a circuit comprising of a resistance of 10 Ω, a variable inductor and a capacitance of 30 μF, all connected in series. Find the value of the inductance which will given resonance with triple frequency component of the pressure and estimate the effective p.f. of the circuit, $\omega = 300$ radian/second.* **(Elect. Engg. I, Bombay Univ. 1985)**

Solution. For resonance at third harmonic

$3\omega L = 1/3\omega C \quad \therefore\ L = 1/9\,\omega^2 C = 10^6/9 \times 300^2 \times 30 =$ **0.041 H**

$Z_1 = 10 + j\left(300 \times 0.041 - \frac{10^6}{\ldots}\right) = 10 + j(12.3 - 111.1) = 10 - j98.8 = 99.3 \angle -84.2^\circ$

$$Z_3 = 10 + j\left(3\omega L - \frac{1}{3\omega C}\right) = 10 + j(36.9 - 37.0) = 10\angle 0^\circ$$

$$Z_5 = 10 + j\left(5\omega L - \frac{1}{5\omega L}\right) = 10 + j(61.5 - 22.2) = 10 + j39.3 = 40.56\angle 75.7^\circ$$

$I_{1m} = 200/99.3 = 2.015$ A ; $I_{3m} = 40/10 = 4$A ; $I_{5m} = 10/40.56 = 0.246$ A

$$I = \sqrt{\frac{2.015^2}{2} + \frac{4^2}{2} + \frac{0.246^2}{2}} = \sqrt{10.06} = 3.172 \text{ A}$$

$$V = \sqrt{\frac{200^2}{2} + \frac{40^2}{2} + \frac{10^2}{2}} = 144.5 \text{ V} \text{ ; Power} = I^2R = 10.06 \times 10 = 100.6 \text{ W}$$

Volt-amperes $VI = 144.5 \times 3.172 = 458$ VA ; Power factor = 100.6/458 = **0.22**

Example 20.13. *A coil having $R = 100\ \Omega$ and $L = 0.1$ H is connected in series with a capacitor across a supply, the voltage of which is given by $e = 200 \sin 314t + 5 \sin 3454t$. What capacitance would be required to produce resonance with the 11th harmonic. Find (a) the equation of the current and (b) the r.m.s. value of the current, if this capacitance is in circuit.*

Solution. For series resonance, $X_L = X_C$

Since resonance is required for 11th harmonic whose frequency is 3454 rad/s, hence

$$3454\,L = \frac{1}{3454\,C} ; C = \frac{1}{3454^2 \times 0.1} \text{ farad} = \mathbf{0.838\ \mu F}$$

(a) **For Fundamental**

Inductive reactance = $\omega L = 314 \times 0.1 = 31.4\ \Omega$

Capacitive reactance = $1/\omega C = 10^6/0.836 \times 314 = 3796\,\Omega$

$\therefore$ Net reactance = $3796 - 31.4 = 3765\ \Omega$; Resistance = $100\ \Omega$

$\therefore$ $Z_1 = \sqrt{100^2 + 3765^2} = 3767\ \Omega$; $\tan\ \phi_1 = 3765/100 = 37.65$

$\therefore$ $\phi_1 = 88^\circ 28'$ (leading) = 1.546 radian

Now $E_{1m} = 200\ V$; $Z_1 = 3767\ \Omega$ $\therefore$ $I_{1m} = 200/3767 = 0.0531$ A

Eleventh Harmonic

New reactance = 0; Impedance $Z_{11} = 100\ \Omega$

$\therefore$ Current $I_{11m} = 5/100 = 0.05$ A ; $\phi_{11} = 0$ ... at resonance

Hence, the equation of the current is

$$i = \frac{200}{3767}(\sin 314t + 1.546) + \frac{5}{100}\sin(3454t + 0)$$

$$\mathbf{i = 0.0531 \sin (314\ t + 1.546) + 0.05 \sin 3454\ t}$$

(b) $I = \sqrt{(0.0531)^2/2 + (0.05)^2/2} = \mathbf{0.052\ A}$

20.9. Effect of Harmonics on Measurement of Inductance and Capacitance

Generally, with the help of ammeter and voltmeter readings, the value of impedance, inductance and capacitance of a circuit can be calculated. But while dealing with complex voltages, the use of instrument readings does not, in general, give correct values of inductance and capacitance except in the case of a circuit containing only pure resistance. It is so because, in the case of resistance, the voltage and current waveforms are similar and hence the values of r.m.s. volts and r.m.s amperes (as read by the voltmeter and ammeter respectively) would be the same whether they ware sinusoidal or non-sinusoidal (*i.e.* complex).

(*i*) **Effect on Inductances**

Let L be the inductance of a circuit and E and I the r.m.s. values of the applied voltage and current as read by the instruments connected in the circuit. For a complex voltage

$$E = 0.707\sqrt{(E_{1m}{}^2 + E_{3m}{}^2 + E_{5m}{}^2 + ...)}$$

Hence $$I = 0.707\sqrt{\left[\left(\frac{E_{1m}}{\omega L}\right)^2 + \left(\frac{E_{3m}}{3\omega L}\right)^2 + \left(\frac{E_{5m}}{5\omega L}\right) + ...\right]}$$

$$= \frac{0.707}{\omega L}\sqrt{\left(E_{1m}{}^2 + \frac{1}{9}E_{3m}{}^2 + \frac{1}{25}E_{5m}{}^2 + ...\right)}$$

$$\therefore \quad L = \frac{0.707}{\omega I}\sqrt{\left(E_{1m}{}^2 + \frac{1}{9}E_{3m}{}^2 + \frac{1}{25}E_{5m}{}^2 + ...\right)}$$

For calculating the value of L from the above expression, it is necessary to known the absolute value of the amplitudes of several harmonic voltages. But, in practices, it is more convenient to deal with relative values than with absolute values. For this purpose, let us multiply and divide the right-hand side of the above expression by E but write the E *in the denominator* in its form $0.707 \sqrt{(E_{1m}{}^2 + E_{3m}{}^2 + E_{5m}{}^2 + ...)}$

$$\therefore \quad L = \left[\frac{0.707}{\omega I}\sqrt{E_{1m}{}^2 + 1/9.E_{3m}{}^2 + 1/25.E_{5m}{}^2 + ...}\right] \times \left[\frac{E}{0.707\sqrt{(E_{1m}{}^2 + E_{3m}{}^2 + E_{5m}{}^2 + ...)}}\right]$$

or $$L = \frac{E}{\omega I}\sqrt{\left[\frac{E_{1m}{}^2 + 1/9 \cdot E_{3m}{}^2 + 1/25 \cdot E_{5m}{}^2 + ...)}{(E_{1m}{}^2 + E_{3m}{}^2 + E_{5m}{}^2 + ...)}\right]} = \frac{E}{\omega I}\sqrt{\left[\frac{1 + 1/9 \cdot (E_{3m}/E_{1m})^2 + 1/25 \cdot (E_{5m}/E_{1m})^2 + ...}{1 + (E_{3m}/E_{1m})^2 + (E_{5m}/E_1)^2 + ...}\right]}$$

If the effect of harmonics were to be neglected, then the value of the inductance would appear to be $E/\omega I$ but the true or actual value is less than this. The apparent value has to be multiplied by the quantity under the radical to get the true value of inductance when harmonics are present.

The quantity under the radical is called the correction factor *i.e.*

True inductance (L) = Apparent inductance (L') × correction factor

(*ii*) Effect on Capacitance

Let the capacitance of the circuit be C farads and E and I the instrument readings for voltage and current. Since the instruments read r.m.s. values, hence, as before,

$$E = 0.707\sqrt{(E_{1m}{}^2 + E_{3m}{}^2 + E_{5m}{}^2 + ...)}$$

Hence $$I = 0.707\sqrt{\left[\left(\frac{E_{1m}}{1/\omega C}\right)^2 + \left(\frac{E_{3m}}{1/3\omega C}\right)^2 + \left(\frac{E_{5m}}{1/5\omega C}\right) + ...\right]}$$

$$= 0.707\sqrt{(\omega C E_{1m})^2 + (3\omega C E_{3m})^2 + (5\omega C E_{5m})^2 + ...}$$

$$= 0.707\ \omega C\sqrt{E_{1m}{}^2 + 9E_{3m}0^2 + 25E_{sm}{}^2 + ...}$$

$$\therefore \quad C = \frac{I}{0.707\omega\sqrt{(E_{1m}{}^2 + 9E_{3m}{}^2 + 25E_{5m}{}^2 + ...)}}$$

Again, we will multiply and divide the right-hand side E but in this case, we will write E in the *numerator* in its form $[0.707\sqrt{(E_{1m}{}^2 + E_{3m}{}^2 + E_{5m}{}^2 + ...)}]$

$$\therefore \quad C = \left[\frac{1}{0.707\,\omega E\sqrt{(E_{1m}{}^2 + 9E_{3m}{}^2 + 25E_{5m}{}^2 + ...)}}\right] \times \left[0.707\sqrt{(E_{1m}{}^2 + E_{3m}{}^2 + E_{5m}{}^2 + ...)}\right]$$

$$= \frac{1}{\omega E}\sqrt{\left[\frac{E_{1m}^{\;2}+E_{3m}^{\;2}+E_{5m}^{\;2}.....}{E_{1m}^{\;2}+9E_{3m}^{\;2}+25E_{5m}^{\;2}+.....}\right]} = \frac{1}{\omega E}\sqrt{\left[\frac{1+(E_{3m}/E_{1m})^2+(E_{5m}/E_{1m})^2.....}{1+9(E_{3m}/E_{1m})^2+25(E_{5m}/E_{1m})^2+.....}\right]}$$

Again, if the effects of harmonics were neglected, the value of capacitance would appear to be $I/\omega E$ but its true value is less than this. For getting the true value, this apparent value will have to be multiplied by the quantity under the radical (which, therefore, is referred to as correction factor).*

$\therefore$ True capacitance (C) = Apparent capacitance (C') × correction factor

Example 20.14. *A current of 50-Hz containing first, third and fifth harmonics of maximum values 100, 15 and 12 A respectively, is sent through an ammeter and an inductive coil of negligibly small resistance. A voltmeter connected to the terminals shows 75 V. What would be the current indicated by the ammeter and what is the exact value of the inductance of the coil in henrys ?*

Solution. The r.m.s. current is

$$I = 0.707\sqrt{I_{1m}^{\;2}+I_{3m}^{\;2}+I_{5m}^{\;2}} = 0.707\sqrt{(100^2+15^2+12^2)} = 72 \text{ A}$$

Hence, current indicated by the ammeter is **72 A**

Now $\quad E = 0.707\sqrt{(E_{1m}^{\;2}+E_{3m}^{\;2}+E_{5m}^{\;2})}$

Also $\quad I_{1m} = \dfrac{E_{1m}}{\omega L};\; I_{3m} = \dfrac{E_{3m}}{3\omega L};\; I_{5m} = \dfrac{E_{5m}}{5\omega L}$

$\therefore \quad E_{1m} = I_{1m}\cdot\omega L;\; E_{3m} = I_{3m}3\omega L;\; E_{5m} = I_{5m}\cdot 5\omega L$

$$\therefore \quad E = 0.707\sqrt{(I_{1m}\omega L)^2+(I_{3m}3\omega L)^2+(I_{5m}5\omega L)^2} = 0.707\omega L\sqrt{I_{1m}^{\;2}+9I_{3m}^{\;2}+25I_{5m}^{\;2}}$$

$$\therefore \quad 75 = 0.707L\times 2\pi\times 50\sqrt{100^2+9\times 15^2+25\times 12^2} \qquad \therefore \qquad L = \mathbf{0.0027\ H}$$

Note. Apparent inductance $L' = \dfrac{E}{\omega I} = \dfrac{75}{2\pi\times 50\times 72} = 0.00331$ H

Example 20.15. *The capacitance of a 20 μF capacitor is checked by direct connection to an alternating voltage which is supposed to be sinusoidal, an electrostatic voltmeter and a dynamometer ammeter being used for measurement. If the voltage actually follows the law,*

$$e = 100 \sin 250\, t + 20 \sin (500 - \phi) + 10 \sin (750\, t - \phi)$$

Calculate the value of capacitance as obtained from the direct ratio of the instrument readings.

Solution. True value, $\quad C = 20\ \mu F$

Apparent value $\quad C'$ = value read by the instruments

Now, $\quad C = C' \times$ correction factor.

Let us find the value of correction factor.

Here $\quad E_{1m} = 100$; $E_{2m} = 20$ and $E_{3m} = 10$

$$\therefore \text{ Correction factor } = \sqrt{\left[\frac{E_{1m}^{\;2}+E_{2m}^{\;2}+E_{3m}^{\;2}}{E_{1m}+4E_{2m}^{\;2}+9E_{5m}^{\;2}}\right]} = \sqrt{\left[\frac{100^2+20^2+10^2}{100^2+4\times 20^2+9\times 10^2}\right]} = 0.9166$$

$$20 = C'\times 0.9166 \qquad \therefore \qquad C' = 20/0.9166 = \mathbf{21.82\ \mu F.}$$

21.10 Harmonics in Different Three-phase Systems

In tree-phase systems, harmonics may be produced in the same way as in single-phase systems.

* It may be noted that this correction factor is different from that in the case of pure inductance.

Hence, for all calculation they are treated in the same manner *i.e.* each harmonic is treated separately. Usually, even harmonics are absent in such systems. But care must be exercised when dealing with odd, especially, third harmonics and all multiples of 3rd harmonic (also called the triple-*n* harmonics).

(*a*) Expressions for Phase E.M.Fs.

Let us consider a 3-phase alternator having identical phase windings (*R*, *Y* and *B*) in which harmonics are produced. The three phase e.m.fs. would be represented in their proper phase sequence by the equation.

$$e_R = E_{1m}(\omega t + \Psi_1) + E_{3m}(3\omega t + \Psi_3) + E_{5m}\sin(5\omega t + \Psi_5) + ...$$

$$e_Y = E_{1m}\sin\left(\omega t - \frac{2\pi}{3} + \Psi_1\right) + E_{3m}\sin\left\{3\left(\omega t - \frac{2\pi}{3}\right) + \Psi_3\right\} + E_{5m}\sin\left\{5\left(\omega t - \frac{2\pi}{3}\right) + \Psi_5\right\}$$

$$e_B = E_{1m}\sin\left(\omega t - \frac{4\pi}{3} + \Psi_1\right) + E_{3m}\sin\left\{3\left(\omega t - \frac{4\pi}{3}\right) + \Psi_3\right\} + E_{5m}\sin\left\{5\left(\omega t - \frac{4\pi}{3}\right) + \Psi_5\right\}$$

On simplification, these become

$$e_R = E_{1m}\sin(\omega t + \Psi_1) + E_{3m}(3\omega t + \Psi_3) + E_{5m}\sin(5\omega t + \Psi_5) + ... \quad \text{– as before}$$

$$e_Y = E_{1m}\sin\left(\omega t - \frac{2\pi}{3} + \Psi_2\right) + E_{3m}\sin(3\omega t - 2\pi + \Psi_3) + E_{5m}\sin\left(5\omega t - \frac{10\pi}{3} + \Psi_5\right) + ...$$

$$= E_{1m}\sin\left(\omega t - \frac{2\pi}{3} + \Psi_1\right) + E_{3m}\sin(3\omega t + \Psi_3) + E_{5m}\sin\left(5\omega t - \frac{4\pi}{3} + \Psi_5\right) + ...$$

$$e_B = E_{1m}\sin\left(\omega t - \frac{4\pi}{3} + \Psi_1\right) + E_{3m}\sin(3\omega t + \Psi_3) + E_{5m}\sin\left(5\omega t - \frac{2\pi}{3} + \Psi_5\right) + ...$$

From these expressions, it is clear that

(*i*) All third harmonics are equal in all phases of the circuit *i.e.* they are in time phase.

(*ii*) Fifth harmonics in the three phases have a negative phase sequence of *R*, *B*, *Y* because the fifth harmonic of blue phase reaches its maximum value before that in the yellow phase.

(*iii*) All harmonics which are not multiples of three, have a phase displacement of 120° so that they can be dealt with in the usual manner.

(*iv*) At any instant, all the e.m.fs. have the same direction which means that in the case of a *Y*-connected system they are directed either away from or towards the neutral point and in the case of Δ-connected system, they flow in the same direction.

Main points can be summarized as below :

(*i*) all triple-*n* harmonics *i.e.* 3rd, 9th, 15th etc. are in phase,

(*ii*) the 7th, 13th and 19th harmonics have positive phase rotation of *R*, *Y*, *B*.

(*iii*) the 5th, 11th and 17th harmonics have a negative phase sequence of *R*, *B*, *Y*.

(*b*) Line Voltage for a Star-connected System

In this system, the line voltages will be the *difference* between successive phase voltages and hence will contain no third harmonic terms because they, being identical in each phase, will cancel out. The fundamental will have a line voltage $\sqrt{3}$ times the phase voltage. Also, fifth harmonic has line voltage $\sqrt{3}$ its phase voltage.

But it should be noted that in this case the r.m.s. value of the line voltage will be less than $\sqrt{3}$ times the r.m.s. value of the phase voltage due to the absence of third harmonic term from the line voltage. It can be proved that for any line voltage.

$$\text{Line value} = \sqrt{3}\sqrt{\frac{E_1^{\,2} + E_5^{\,2} + E_7^{\,2}}{E_1^{\,2} + E_3^{\,2} + E_5^{\,2} + E_7^{\,2}}}$$

where E_1, E_3 etc. are r.m.s. values of the phase e.m.fs.

(c) **Line Voltage for a Δ-connected System**

If the winding of the alternator are delta-connected, then the resultant e.m.f. acting round the closed mesh would be the sum of the phase e.m.fs. The sum of these e.m.fs. is zero for fundamental, 5th, 7th, 11th etc. harmonics. Since the third harmonics are in phase, there will be a resultant third harmonic e.m.f. of three times the phase value acting round the closed mesh. It will produce a circulating current whose value will depend on the impedance of the windings at the third harmonic frequency. It means that the third harmonic e.m.f. would be short-circuited by the windings with the result that there will be no third harmonic voltage across the lines. The same is applicable to all triple-*n* harmonic voltages. Obviously, the line voltage will be the phase voltage but without the triple-*n* terms.

Example 20.16. *A 3-ϕ generator has a generated e.m.f. of 230 V with 15 per cent third harmonic and 10 per cent fifth harmonic content. Calculate*

(i) the r.m.s. value of line voltage for Y-connection;

(ii) the r.m.s. value of line voltage for Δ-connection.

Solution. Let E_1, E_3, E_5 be the r.m.s. values of the phase e.m.fs. Then

$$E_3 = 0.15\ E_1 \text{ and } E_5 = 0.1\ E_1$$

$$\therefore \quad 230 = \sqrt{E_1^{\ 2} + (0.15E_1)^2 + (0.1\,E_1)^2}$$

$$E_1 = 226 \text{ V} \qquad \therefore \quad E_3 = 0.15 \times 226 = 34 \text{ V and } E_5 = 0.1 \times 226 = 22.6 \text{ V}$$

(*i*) r.m.s. value of the fundamental line voltage = $\sqrt{3} \times 226 = 392$ V

r.m.s. value of third harmonic line voltage = 0

r.m.s. value of 5th harmonic line voltage $\sqrt{3} \times 22.6 =$ **39.2 V**

$\therefore$ r.m.s. value of line voltage $V_L = \sqrt{392^2 + 39.2^2} =$ **394 V**

(*ii*) In Δ-connection, again the third harmonic would be absent from the line voltage

$\therefore$ r.m.s. value of line voltage $V_L = \sqrt{226^2 + 22.6^2} =$ **227.5 V**

(*d*) **Circulating Current in Δ-connected Alternator**

Let the three symmetrical phase e.m.fs. of the alternator be represented by the equations,

$$e_R = E_{1m}\sin(\omega t + \Psi_1) + E_{3m}(3\omega t + \Psi_3) + E_{5m}\sin(5\omega t + \Psi_5) + \ldots\ldots$$

$$e_Y = E_{1m}\sin(\omega t + \Psi_1 - 2\pi/3) + E_{3m}\sin(3\omega t + \Psi_3) + E_{5m} + \sin(5\omega t + \Psi_5 - 4\pi/3)\ldots\ldots$$

$$e_B = E_{1m}\sin(\omega t + \Psi_1 - 4\pi/3) + E_{3m}\sin(3\omega t + \Psi_3) + E_{5m}\sin(5\omega t + \Psi_5 - 2\pi/3) + \ldots\ldots$$

The resultant e.m.f. acting round the Δ-connected windings of the armature is the sum of these e.m.fs. Hence it is given by $\qquad e = e_R + e_Y + e_B$

$$\therefore \quad e = 3E_{3m}\sin(3\omega t + \phi_3) + 3E_{9m}\sin(9\omega t + \Psi_9) + 3E_{15}\sin(15\omega t + \Psi_{15}) + \ldots\ldots$$

If *R* and *L* represent respectively the resistance and inductance per phase of the armature winding, then the circulating current due to the resultant e.m.f. is given by

$$i_c = \frac{3E_{3m}\sin(3\omega t + \Psi_3)}{3\sqrt{(R^2 + 9L^2\omega^2)}} + \frac{3E_9 m\sin(9\omega t + \Psi_9)}{3\sqrt{(R^2 + 81L^2\omega^2)}} + \frac{3E_{15m}\sin(15\omega t + \Psi_{15})}{3\sqrt{(R^2 + 225L^2\omega^2)}} + \ldots$$

$$= \frac{E_{3m}\sin(3\omega t + \Psi_3)}{\sqrt{(R^2 + 9L^2\omega^2)}} + \frac{E_{9m}\sin(9\omega t + \Psi_9)}{\sqrt{(R^2 + 81L^2\omega^2)}} + \frac{E_{15m}\sin(15\omega t + \Psi_{15})}{\sqrt{(R^2 + 225L^2\omega^2)}}$$

The r.m.s. value of the current is given by

$$I_C = 0.707\sqrt{[E_{3m}{}^2/(R^2+9L^2\omega^2)+E_{9m}{}^2/(R^2+81L^2\omega^2)+(E_{15m}{}^2/(R^2+225L^2\omega^2)+......]}$$

(c) Three-phase four-wire System

In this case, there will be no third harmonic component in line voltage. For the 4-wire system, each phase voltage (*i.e.* line to neutral) may contain a third harmonic component. If it is actually present, then current will flow in the *Y*-connected load. In case load is balanced, the resulting third harmonic line currents will all be in phase so that neutral wire will have to carry three times the third harmonic line current. There will be no current in the neutral wire either at fundamental frequency, or any harmonic frequency other than the triple-*n* frequency.

20.11. Harmonics in Single and 3-phase Transformers

The flux density in transformer course is usually maintained at a fairly high value in order to keep the required volume of iron to the minimum. However, due to the non-linearity of magnetisation curve, some third harmonic distortion is always produced. Also, there is usually a small percentage of fifth harmonic. The magnetisation current drawn by the primary contains mainly third harmonic whose proportion depends on the size of the primary applied voltage. Hence, the flux is sinusoidal.

In the case of three-phase transformers, the production of harmonics will be affected by the method of connection and the type of construction employed.

(*a*) Primary Windings Δ-CONNECTED

Each primary phase can be considered as separately connected across the sinusoidal supply.

(*i*) The core flux will be sinusoidal which means that magnetizing current will contain 3rd harmonic component in addition to relatively small amounts of other harmonics of higher order.

(*ii*) In each phase, these third harmonic currents will be in phase and so produce a circulating current round the mesh with the result that there will be no third harmonic component in the line current.

(*b*) Primary Winding Connected in 4-wire Star

Each phase of the primary can again be considered as separately connected across a sinusoidal supply.

(*i*) The flux in the transformer core would be sinusoidal and so would be the output voltage.

(*ii*) The magnetizing current will contain 3rd harmonic component. This component being in phase in each winding will, therefore, return through the neutral wire.

(*c*) Primary Windings Connected 3-wire Star

Since there is no neutral wire, there will be no return path for the 3rd harmonic component of the magnetizing current. Hence, there will exist a condition of forced magnetization so that core flux must contain third harmonic component which is in phase in each limb of the transformer core. Although there will be a magnetic path for these fluxes in the case of shell type 3-phase transformer, yet in the case of three-limb core type transformer, the third harmonic component of the flux must return *via* the air. Because of the high reluctance magnetic path in such transformers, the third harmonic flux is reduced to a very small value. However, if the secondary of the transformer is delta-connected, then a third harmonic circulating current would be produced. This current would be in accordance with Lenz's law tend to oppose the very cause producing it *i.e.* it would tend to minimize the third harmonic component of the flux.

Should the third and fifth secondary be *Y*-connected, then provision of an additional Δ-connected winding, in which this current can flow, becomes necessary. This tertiary winding additionally served the purpose of preserving magnetic equilibrium of the transformer in the case of unbalanced loads. In this way, the output voltage from the secondary can be kept reasonably sinusoidal.

Example 20.17. *Determine whether the following two waves are of the same shape*

$v = 10 \sin(\omega t + 30°) - 50 \sin(3\omega t - 60°) + 25 \sin(5\omega t + 40°)$

$i = 1.0 \sin(\omega t - 60°) + 5 \sin(3\omega t - 150°) + 2.5 \cos(5\omega t - 140°)$

(Principles of Elect. Engg-II Jadavpur Univ. 1989)

Solution. Two waves possess the same waveshape

(*i*) if they contain the same harmonics

(*ii*) if the ratio of the corresponding harmonics to their respective fundamentals is the same

(*iii*) if the harmonics are similarly spaced with respect to their fundamentals.

In other words,

(*a*) the ratio of the magnitudes of corresponding harmonics must be constant and

(*b*) with fundamentals in phase, the corresponding harmonics of the two waves must be in phase.

The test is applied first by checking the ratio of the corresponding harmonics and then coinciding the fundamentals by shifting one wave. If the phase angles of the corresponding harmonics are the same, then the two waves have the same shape.

In the present case, condition (*i*) is fulfilled because the voltage and current waves contain the same hermonics, *i.e.* third and fifth.

Secondly, the ratio of the magnitude of corresponding current and voltage harmonics is the same *i.e.* 1/10.

Now, let the fundamental of the current wave be shifted ahead by 90° so that it is brought in phase with the fundamental of the voltage wave. It may be noted that the third and fifth harmonics of the current wave will be shifted by 3 × 90° = 270° and 5 × 90° = 450° respectively. Hence, the current wave becomes

$$i' = 1.0 \sin(\omega t - 60° + 90°) + 5 \sin(3\omega t - 150° + 270°) + 2.5 \cos(5\omega t - 140° + 450°)$$

$$= 1.0 \sin(\omega t + 30°) + 5 \sin(3\omega t + 120°) + 2.5 \cos(5\omega t + 310°)$$

$$= 1.0 \sin(\omega t + 30°) - 5 \sin(3\omega t - 60°) + 2.5 \sin(5\omega t + 40°)$$

It is seen that now the corresponding harmonics of the voltage and current waves are in phase.

Since all conditions are fulfilled, the two waves are of the same waveshape.

Tutorial Problem No. 20.1

1. A series circuit consist of a coil of inductance 0.1 H and resistance 25 Ω and a variable capacitor. Across this circuit is applied a voltage whose instantaneous value is given by

$$v = 100 \sin \omega t + 20 \sin(3\omega t - 45) + 5 \sin(5\omega t - 30) \text{ where } \omega = 314 \text{ rad/s}$$

Determine the value of *C* which will produce response at third harmonic freqpency and with this value of *C*, find

(*a*) an expression for the current in the circuit (*b*) the r.m.s. value of this current (*c*) the total power absorbed.

[11.25μF, (*a*) i = 0.398 sin (ωt + 84.3) + 0.8 sin (3 ωt + 45) + 0.485 sin (5 ωt + 106) (*b*) 0.633 A (*c*) 10 W]

2. A voltage given by $v = 200 \sin 314\, t + 520 \sin(942\, t + 45°)$ is applied to a circuit consisting of a resistance of 20 Ω, and inductance of 20 mH and a capacitance of 56.3 μF all connected in series.

Calculate the r.m.s. values of the applied voltage and current. Find also the total power absorbed by the circuit. **[146 V ; 3.16 A 200 W]**

3. A voltage given by $v = 100 \sin \omega t + 8 \sin 3\omega t$ is applied to a circuit which has a resistance of 1 Ω, an inductance of 0.02 H and a capacitance of 60μF. A hot-wire ammeter is connected in series with the circuit and a hot-wire voltmeter is connected to the terminals. Calculate the ammeter and voltameter readings and the power supplied to the circuit. **[71 V ; 5.18 A ; 26.8 W]**

4. A certain coil has a resistance of 20 Ω and an inductance of 0.04 H. If the instantaneous current flowing in it is represented by $i = 5 \sin 300\,t + 0.8 \sin 900\,t$ amperes, derive an expression for the instantaneous value of the voltage applied across the ends of the coil and calculate the r.m.s. value of that voltage.

[V = 117 sin (300 t + 0.541) + 33 sin (900 t + 1.06) ; 0.86 V]

5. A voltage given by the equation $v = \sqrt{2}\,100 \sin 2\pi \times 50t + \sqrt{2.20} \sin 2\pi.\ 150t$ is applied to the terminals of a circuit made up of a resistance of 5 Ω, an inductance of 0.0318 H and a capacitor of 12.5 μF all in series. Calculate the effective current and the power supplied to the circuit. **[0.547 A ; 1.5 W]**

6. An alternating voltage given by the expression $v = 1{,}000 \sin 314t + 100 \sin 942t$ is applied to a circuit having a resistance of 100 Ω and an inductance of 0.5 H. Calculate r.m.s. value of the current and p.f. **[3.81 A ; 0.535]**

7. The current in a series circuit consisting of a 159 μF capacitor, a reactor with a resistance of 10 Ω and an inductances of 0.0254 H is given $i = \sqrt{2}(8 \sin \omega t + 2 \sin 3\omega t)$ amperes. Calculate the power input and the power factor. Given $\omega = 100\pi$ radian/second. **[680 W : 0.63]**

8. If the terminal voltage of a circuit is $100 \sin\ \omega t + 50 \sin(3\omega t + \pi/4)$ and the current is $10 \sin(\omega t + \pi 3)\ \ +5 \sin 3\omega t$, calculate the power consumption. **[522.6 W]**

9. A single-phase load takes load takes a current of $4 \sin\ (\omega t + \pi/6) + 1.5 \sin(3\omega t + \pi/3)$ A from source represented by $360 \sin\ \omega t$ volts. Calculate the power dissipated by the circuit and the circuit power factor.

[623.5 W ; 0.837]

10. An e.m.f. given by $e = 100 \sin\ \omega t + 40 \sin(\omega t - \pi/6) + 10 \sin(5\omega t - \pi 3)$ volts is applied to a series circuit having a resistance of 100 Ω, an inductance of 40.6 mH and a capacitor of 10 μF. Derive an expression for the current in the circuit. Also, find the r.m.s. value of the current and the power dissipated in the circuit. Take ω = 314 rad/second. **[0.329 A, 10.8 W]**

11. A p.d. of the form $V = 400 \sin \omega t + 30 \sin 3\omega t$ is applied to a rectifier having a resistance of 50Ω in one direction and 200 in the reverse direction. Find the average and effective values of the current and the p.f. of the circuit. **[1.96 A, 4.1 A, 0.51]**

12. A coil having $R = 2$ Ω and $L = 0.01$ H carries a current given by $i = 50 + 20 \sin 300\,t$

A moving-iron ammeter, a moving-coil voltmeter and a dynameter wattmeter, are used to indicate current, voltage and power respectively. Determine the readings of the instruments and equation for the p.d. **[121.1 V ; 52 A ; 5.4 kW, V = 100 + 72 sin (300 t + 0.982)]**

13. Two circuits having impedances at 50 Hz of $(10 + j6)$ Ω and $(10 - j6)$ respectively are connected in parallel across the terminals of an a.c. system, the waveform of which is represented by $v = 100 \sin \omega t + 35\ \sin\ 3\omega t + 10 \sin 5\omega t$, the fundamental frequency being 50 Hz. Determine the ratio of the readings of two ammeters, of negligible resistance, connected one in each circuit. **[6.35 ; 6.72]**

14. Explain what is meant by harmonic resonance in a.c. circuits.

A current having an instantaneous value of $2\ (\sin \omega t + \sin 3\omega t)$ amperes is passed through a circuit which consists of a coil of resistance R and inductance L in series with a capacitor C. Derive an expression for the value of ω at which the r.m.s. circuit voltage is a minimum. Determine the voltage if the coil has inductance 0.1 H and resistance 150 Ω and the capacitance is 10 μ F. Determine also the circuit voltage at the fundamental resonant frequency. **[$\omega = 1/\sqrt{(LC)}$; 378 V ; 482 V]**

15. An r.m.s. current of 5 A which has a third-harmonic content, is passed through a coil having a resistance of 1Ω and an inductance of 10 mH. The r.m.s. voltage across the coil is 20 V. Calculate the magnitudes of the fundamental and harmonic components of current if the fundamental frequency is $300/2\pi$ Hz Also, find the power dissipated. **[4.8 A ; 1.44 A ; 25 W]**

16. Derive a general expression for the form factor of a complex wave containing only odd-order harmonics. Hence, calculate the form factor of the alternating current represented by

$i = 2.5 \sin 157\,t + 0.7 \sin 471\,t + 0.4 \sin 785\,t$ **[1.038]**

OBJECTIVE TESTS – 20

1. Non-sinusoidal waveforms are made up of
   (*a*) different sinusoidal waveforms
   (*b*) fundamental and even harmonics
   (*c*) fundamental and odd harmonics
   (*d*) even and odd harmonics only.
2. The positive and negative halves of a complex wave are symmetrical when
   (*a*) it contains even harmonics
   (*b*) phase difference between even harmonics and fundamental is 0 or π
   (*c*) it contains odd harmonics
   (*d*) phase difference between even harmonics and fundamental is either $\pi/2$ or $3\pi/2$.
3. The r.m.s. value of the complex voltage given by $v = 16\sqrt{2}\sin\omega t + 12\sqrt{2}\sin 3\omega t$ is
   (*a*) $20\sqrt{2}$ (*b*) 20
   (*c*) $28\sqrt{2}$ (*d*) 192
4. In a 3-phase system, ___th harmonic has negative phase sequence of RBY.
   (a) 9 (*b*) 13
   (*c*) 5 (*d*) 15
5. A complex current wave is given by the equation $i = 14\sin\omega t + 2\sin 5\omega t$. The r.m.s. value of the current is ___ ampere.
   (*a*) 16 (*b*) 12
   (*c*) 10 (*d*) 8
6. When a pure inductive coil is fed by a complex voltage wave, its current wave
   (*a*) has larger harmonic content
   (*b*) is more distorted
   (*c*) is identical with voltage wave
   (*d*) shows less distortion.
7. A complex voltage wave is applied across a pure capacitor. As compared to the fundamental voltage, the reactance offered by the capacitor to the third harmonic voltage would be
   (*a*) nine time (*b*) three times
   (*b*) one-third (*d*) one-ninth
8. Which of the following harmonic voltage components in a 3-phase system would be in phase with each other ?
   (*a*) 3rd, 9th, 15th etc.
   (*b*) 7th, 13th, 19th etc
   (*c*) 5th, 11th, 17th etc.
   (*d*) 2nd, 4th, 6th etc.

ANSWERS

1. (*a*) 2. (*c*) 3. (*b*) 4. (*c*) 5. (*c*) 6. (*d*) 7. (*c*) 8. (*d*)

21 FOURIER SERIES

21.1. Harmonic Analysis

By harmonic analysis is meant the process of determining the magnitude, order and phase of the several harmonics present in a complex periodic wave.

For carrying out this analysis, the following methods are available which are all based on Fourier theorem:

(*i*) Analytical Method–the standard Fourier Analysis

(*ii*) Graphical Method– (*a*) by Superposition Method (Wedgemore' Method) (*b*) Twenty four Ordinate Method

(*iii*) Electronic Method–by using a special instrument called 'harmonic analyser'

We will consider the first and third methods only.

21.2. Periodic Functions

A function $f(t)$ is said to be periodic if $f(t+T)=f(t)$ for all values of t where T is some positive number. This T is the interval between two successive repetitions and is called the period of $f(t)$. A sine wave having a period of $T=2\pi/\omega$ is a common example of periodic function.

21.3. Trigonometric Fourier Series

Suppose that a given function $f(t)$ satisfies the following conditions (known as Dirichlet conditions):

1. $f(t)$ is periodic having a period of T.
2. $f(t)$ is single-valued everywhere.
3. In case it is discontinuous, $f(t)$ has a finite number of discontinuties in any one period.
4. $f(t)$ has a finite number of maxima and minima in any one period.

The function $f(t)$ may represent either a voltage or current waveform. According to Fourier theorem, this function $f(t)$ may be represented in the trigonometric form by the infinite series.

$$f(t)=a_0+a_1\cos\omega_0 t+a_2\cos 2\omega_0 t+a_3\cos 3\omega_0 t+...+a_n\cos n\omega_0 t$$
$$+\ b_1\sin\omega_0 t+b_2\sin 2\omega_0 t+b_3\sin 3\omega_0 t+...+b_n\sin n\omega_0 t$$
$$=a_0+\sum_{n=1}^{\infty}(a_n\cos n\omega_0 t+b_n\sin n\omega_0 t) \qquad ...(i)$$

Putting $\omega_0 t=\theta$, we can write the above equation as under

$$f(\theta)=a_0+a_1\cos\theta+a_2\cos 2\theta+a_3\cos 3\theta+...+a_n\cos n\theta+b_1\sin\theta+b_2\sin 2\theta+b_3\sin 3\theta+...+b_n\sin n\theta$$
$$=a_0+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta) \qquad ...(ii)$$

Since $\omega_0=2\pi/T$, Eq. (*i*) above can be written as

$$f(t)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2\pi n}{T}t+b_n\sin\frac{2\pi n}{T}t\right) \qquad ...(iii)$$

where ω_0 is the fundamental angular frequency, T is the period and a_0, a_n and b_n are constants which depend on n and $f(t)$. The process of determining the values of the constants a_0, a_n and b_n is called Fourier Analysis. Also. $\omega_0=2\pi T=2\pi f_0$ where f_0 is the fundamental frequency.

It is seen from the above Fourier Series that the periodic function consists of sinusoidal components of frequency 0, ω_0, $2\omega_0 \ldots n\omega_0$. This representation of the function $f(t)$ is in the frequency domain. The first component a_0 with zero frequency is called the *dc* component. The sine and cosine terms represent the harmonics. The number n represents the order of the harmonics.

When $n = 1$, the component $(a_1 \cos\omega_0 t + b_1 \sin\omega_0 t)$ is called the first harmonic or the fundamental component of the waveform.

When $n = 2$, the component $(a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t)$ is called the second harmonic of the waveform.

The nth harmonic of the waveform is represented by $(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$. It has a frequency of $n\omega_0$ *i.e.* n times the frequency of the fundamental component.

21.4. Alternate Forms of Trigonometric Fourier Series

Eq. (*i*) given above can be written as

$f(t) = a_0 + (a_1 \cos\omega_0 t + b_1 \sin\omega_0 t) + (a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t) + \ldots + (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$

Let. $a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = A_n \cos(n\omega_0 t - \phi_n)$

$= A_n \cos n\omega_0 t \cos\phi_n + A_n \sin n\omega_0 t \sin\phi_n$

$\therefore \quad a_n = A_n \cos\phi_n$ and $b_n = A_n \sin\phi_n$

$\therefore \; A_n \sqrt{a_n^2 + b_n^2}$ and $\phi_n = \tan^{-1} b_n / a_n$

Similarly, let $(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = A_n \sin(n\omega_0 t + \Psi_n))$

$= A_n \sin n\omega_0 t \cos\Psi_n + A_n \cos n\omega_0 t \sin\phi_n$

As seen from Fig. 21.1, $b_n = A_n \cos\psi_n$ and $a_n = A_n \sin\psi_n$

$\therefore \quad A_n = \sqrt{a_n^2 + b_n^2}$ and $\psi = \tan^{-1} a_n / b_n$

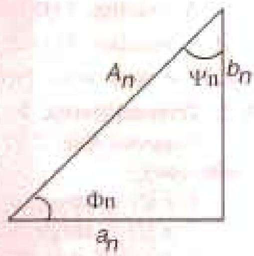


Fig. 21.1

The two angles ϕ_n and ψ_n are complementary angles.

Hence, the Fourier series given in Art. 21.2 may be put in the following two alternate forms

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t - \phi_n)$$

or

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \sin(n\omega_0 t + \psi_n)$$

21.5. Certain Useful Integral Calculus Theorems

The Fourier coefficients or constants $a_0, a_1, a_2 \ldots a_n$ and $b_1, b_2 \ldots\ldots b_n$ can be evaluated by integration process for which purpose the following theorems will be used.

(*i*) $\displaystyle\int_0^{2\pi} \sin n\theta d\theta = \frac{1}{b} |\cos n\theta|_0^{2\pi} = \frac{1}{n}(1-1) = 0$

(*ii*) $\displaystyle\int_0^{2\pi} \sin n\theta d\theta = -\frac{1}{n} | \cos n\theta |_0^{2\pi} = -\frac{1}{n}(0-0) = 0$

(*iii*) $\displaystyle\int_0^{2\pi} \sin n\theta d\theta heta = \frac{1}{2}\int_0^{2\pi} (1 - \cos 2n\theta) d\theta = \frac{1}{2} | \theta - \frac{1}{2n} \sin 2n\theta |_0^{2\pi} = \pi$

(*iv*) $\displaystyle\int_0^{2\pi} \cos^2 n\theta d\theta = \frac{1}{2}\int_0^{2\pi} (\cos 2n\theta + 1) d\theta = \frac{1}{2}\left| \frac{1}{2n} \sin 2n\theta + \theta \right|_0^{2\pi} = \pi$

(v) $\int_0^{2\pi} \sin m\theta \cos n\theta d\theta = \frac{1}{2}\int_0^{2\pi} \{\sin (m+n)^{\theta} + \sin (m-n)\theta\} d\theta$

$$= \frac{1}{2}\left| -\frac{1}{m+n}\cos(m+n)\theta - \frac{1}{m-n}\cos(m-n)\theta \right|_0^{2\pi} = 0$$

(vi) $\int_0^{2\pi} \cos m\theta \cos n\theta d\theta = \frac{1}{2}\int_0^{2\pi} \{\cos(m+n)\theta + \cos(m-n)\theta\} d\theta$

$$= \frac{1}{2}\left| \frac{1}{m+n}\sin(m+n)\theta + \frac{1}{m-n}\sin(m-n)\theta \right|_0^{2\pi} = 0 \ldots \text{for } n \neq m$$

(vii) $\int_0^{2\pi} \sin m\theta \sin n\theta d\theta = \frac{1}{2}\int_0^{2\pi} \{\cos(m-n)\theta - \cos(m+n)\theta\} dthda$

$$= \frac{1}{2}\left| \frac{1}{m-n}\sin(m-n)\theta - \frac{1}{m+n}\sin(m+n)\theta \right|_0^{2\pi} = 0 \ldots \text{for } n \neq m$$

where m and n are any positive integers.

21.6. Evaluation of Fourier Constants

Let us now evaluate the constants a_0, a_n and b_n by using the above integral calculus theorems

(i) Value of a_0

For this purpose we will integrate both sides of the series given below over one period *i.e.* for $\theta = 0 \text{ to } \theta = 2\pi$.

$$f(\theta) = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \ldots + a_n \cos n\theta$$
$$+ \; b_1 \sin\theta + b_2 \sin 2\theta + \ldots + b_n \sin n\theta$$

$$\therefore \int_0^{2\pi} f(\theta) d\theta = \int_0^{2\pi} a_0 d\theta + a_1\int_0^{2\pi} \cos\theta d\theta + a_2\int_0^{2\pi} \cos 2\theta d\theta + \ldots + a_n\int_0^{2\pi} \cos n\theta d\theta$$
$$+ \; b_1\int_0^{2\pi} \sin\theta d\theta + b_2\int_0^{2\pi} \sin 2\theta d\theta + \ldots + b_n\int_0^{2\pi} \sin n\theta d\theta$$

$$= a_0|\theta|_0^{2\pi} + 0 + 0 + \ldots 0 + 0 + 0 + \ldots + 0 = 2\pi a_0$$

$$\therefore \qquad a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) d\theta \text{ or } = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) d\theta$$

= mean value of $f(\theta)$ between the limits 0 to 2π *i.e.* over one cycle or period.

Also, $a_0 = \frac{1}{2\pi}(\text{net area})_0^{2\pi}$

If we take the periodic function as $f(t)$ and integrate over period T (which corresponds to 2π),

we get $a_0 = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{1}{T}\int_{t_1}^{t_1+T} f(t)dt$

where t_1 can have any value.

(ii) Value of a_n

For finding the value of a_n, multiply both sides of the Fourier Series by cos $n\theta$ and integrate between the limits $\theta = 0 \text{ to } / 2\pi$

$$\therefore \int_0^{2\pi} f(\theta)\cos n\theta d\theta = a_0\int_0^{2\pi}\cos n\theta d\theta + a_1\int_0^{2\pi}\cos\theta\cos n\theta d\theta + a_2\int_0^{2\pi}\cos 2\theta\cos n\theta d\theta$$

$$+a_n\int_0^{2\pi}\cos^2 n\theta d\theta + b_1\int_0^{2\pi}\cos\theta\cos n\theta d\theta + b_2\int_0^{2\pi}\sin 2\theta\cos n\theta d\theta + ... + b_n\int_0^{2\pi}\sin n\theta\cos n\theta d\theta$$

$$= 0+0+0+...+a_n\int_0^{2\pi}\cos^2 n\theta d\theta + 0+0+...+0 = a_n\int_0^{2\pi}\cos^2 n\theta d\theta = \pi a_n$$

$$\therefore a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta = 2\times\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta$$

= 2 × average value of $f(\theta)$ cos $n\theta$ over one cycle of the fundamental.

Also, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n d\theta = 2\times\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta$

If we take periodic function as $f(t)$, then different expressions for a_n are as under.

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{2\pi n}{T}t\,dt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi n}{T}t\,dt$$

Giving different numerical values to n, we get

a_1 = 2 × average of $f(\theta)$ cos θ over one cycle ....$n = 1$

a_2 = 2 × average value of $f(\theta)$ cos 2θ over one cycle etc. .... $n = 2$

(iii) Value of b_n

For finding its value, multiply both sides of the Fourier Series of Eq. (*i*) by sin $n\theta$ and integrate between limits $\theta = 0$ to $\theta = 2\pi$.

$$\therefore \int_0^{2\pi} f(\theta)\sin n\theta d\theta = a_0\int_0^{2\pi}\sin n\theta d\theta + a_1\int_0^{2\pi}\cos\theta\sin n\theta d\theta$$

$$+ a_2\int_0^{2\pi}\cos 2\theta\sin n\theta d\theta + ... + a_n\int_0^{2\pi}\cos n\theta\sin n\theta d\theta$$

$$+ b_1\int_0^{2\pi}\sin\theta\sin n\theta d\theta + b_2\int_0^{2\pi}\sin 2\theta\sin n\theta d\theta + ... + b_n\int_0^{2\pi}\sin^2 n\theta d\theta$$

$$= 0+0+0+...+0+0+...+b_n\int_0^{2\pi}\sin^2 n\theta d\theta = b_n\int_0^{2\pi}\sin^2 n\theta d\theta = b_n\pi$$

$$\therefore \int_0^{2\pi} f(\theta)\sin\theta d\theta = b_n\times\pi$$

$$\therefore b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta = 2\times\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta$$

= 2 × average value of $f(\theta)$ sin $n\theta$ over one cycle of the fundamental.

$\therefore b_1$ = 2 × average value of $f(\theta)\sin\theta$ over one cycle ... $n = 1$

b_2 = 2 × average value of $f(\theta)\sin 2\theta$ over one cycle ... $n = 2$

Also, $b_n = \frac{2}{t}\int_0^T f(t)\sin\frac{2\pi n}{T}t\,dt + \frac{2}{T}\int_{T/2}^{T/2} f(t)\sin\frac{2\pi n}{T}t\,dt$

$$= \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t\,dt = \frac{2}{T}\int_{T/2}^{T/2} f(t)\sin n\omega_0 t dt$$

Hence, for Fourier analysis of a periodic function, the following procedure should be adopted:

(*i*) Find the term a_0 by integrating both sides of the equation representing the periodic function between limits 0 to 2π or 0 to T or $-T/2$ to $T/2$ or t_1 to $(t_1 + T)$.

$$\therefore\ a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta = \frac{1}{T}\int_0^T f(t)dt = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{1}{T}\int_{t_1}^{t_1+T} f(t)dt$$

= average value of the function over one cycle.

(*ii*) Find the value of a_n by multiplying both sides of the expression for Fourier series by $\cos n\theta$ and then integrating it between limits 0 to 2π or 0 to T or $-T/2$ to $T/2$ or t_1 to $(t_1 + T)$.

$$\therefore\qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta = 2\times\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta$$

Since $\pi = T/2$, we have

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{2\pi n}{T}t\,dt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2\pi n}{T}t\,dt = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\cos\frac{2\pi n}{T}t\,dt$$

$$= \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t dt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t dt = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\cos n\omega_0 t dt$$

= 2 × average value of $f(\theta)\cos n\theta$ over one cycle of the fundamental.

Values of a_1, a_2, a_3 etc. can be found from above by putting n = 1, 2, 3 etc.

(*iii*) Similarly, find the value of b_n by multiplying both sides of Fourier series by $\sin n\theta$ and integrating it between the limits 0 to 2π or 0 to T or $-T/2$ to $T/2$ or t_1 to $(t_1 + T)$

$$\therefore\ b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta = 2\times\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta$$

$$= \frac{2}{T}\int_0^T f(t)\sin\frac{2\pi n}{T}t\,dt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2\pi n}{T}t\,dt = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\sin\frac{2\pi n}{t}t\,dt$$

$$= \frac{2}{T}\int_0^T f(t)\sin\ n\omega_0 t dt = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\,dt = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\sin n\omega_0 t\,dt$$

= 2*x* average value of f (θ) sin $n\theta$ of $f(t)$ sin $\frac{2\pi n}{T}t$ or $f(t)$ sin $n\omega_0 t$ over one cycle of the fundamental.

Values of b_1, b_2, b_3 etc. can be found from above by putting n = 1, 2, 3 etc.

21.7. Different Types of Functional Symmetries

A non-sinusoidal wave can have the following types of symmetry:

1. Even Symmetry

The function $f(t)$ is said to possess even symmetry if $f(t) = f(-t)$.

It means that as we travel equal amounts in time to the left and right of the origin (*i.e.* along the + *X*-axis and −*X*-axis), we find the function to have the same value. For example in Fig. 21.2 (*a*), points A and B are equidistant from point O. Here the two function values are equal and positive. At points C and D, the two values of the function are again equal, though negative. Such a function is symmetric with respect to the vertical axis. Examples of even function are: t^2, $\cos 3t$, $\sin^2 5t$ $(2 + t^2 + t^4)$ and a constant A because the replacement of t by $(-t)$ does not change the value of any of these functions. For example, $\cos \omega t = \cos(-\omega t)$.

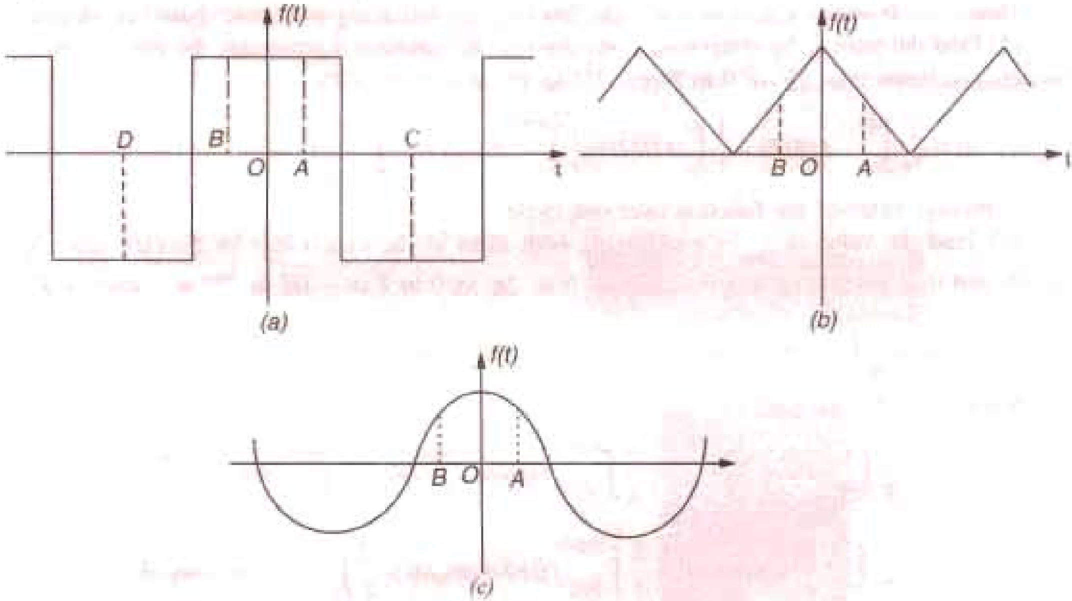


Fig. 21.2

This type of symmetry can be easily recognised graphically because mirror symmetry exists about the vertical or $f(t)$ axis. The function shown in Fig. 21.2 has even symmetry because when folded along vertical axis, the portions of the graph of the function for positive and negative time fit exactly, one on top of the other.

The effect of the even symmetry on Fourier series is that the constant $b_n = 0$ *i.e.* the wave has no sine terms. In general, $b_1, b_2, b_3 \ldots b_n = 0$. The Fourier series of an even function contains only a constant term and cosine terms *i.e.*

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{2\pi n}{T} t$$

The value of a_n may be found be integrating over any half-period.

$$\therefore \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta d\theta = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\, dt$$

2. Odd Symmetry

A function $f(t)$ is said to possess odd symmetry if $f(-t) = -f(t)$.

It means that as we travel an equal amount in time to the left or right from the origin, we find the function to be the same except for a reversal in sign. For example, in Fig. 21.3 the two points A and B are equidistant from point O. The two function values at A and B are equal in magnitude but opposite in sign. In other words, if we replace t by $(-t)$, we obtained the negative of the given function. The X-axis divides an odd function into two halves with equal areas above and below the X-axis. Hence, $a_0 = 0$.

Examples of odd functions are: t, $\sin t$, $t \cos 50$ $t(t + t^3 + t^5)$ and $t\sqrt{(1+t^2)}$

A sine function is an odd function because $\sin(-\omega t) = -\sin \omega t$.

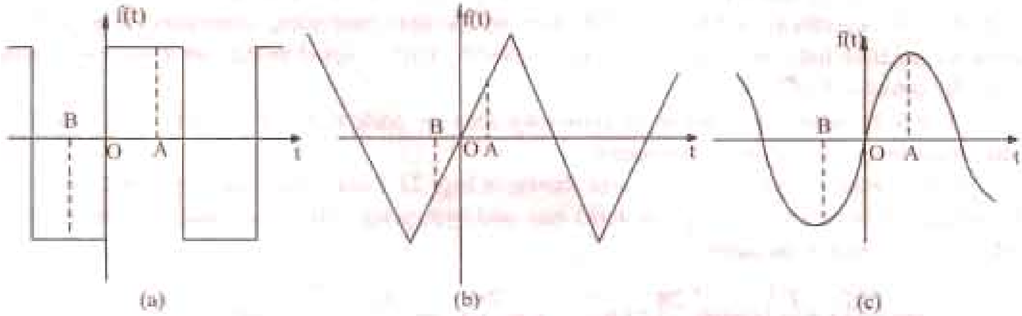


Fig. 21.3

An odd function has symmetry about the origin rather than about the $f(t)$ axis which was the case for an even function. The effect of odd symmetry on a Fourier series is that it contains no constant term or consine term. It means that $a_0 = 0$ and $a_n = 0$ *i.e.* $a_1, a_2, a_3 \ldots a_n = 0$. The Fourier series expansion contains only sine terms.

$$\therefore \quad f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

The value of b_n may be found by integrating over any half-period.

$$\therefore \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta d\theta = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\, dt$$

3. Half-wave Symmetry or Mirror-Symmetry or Rotational Symmetry

A function $f(t)$ is said to possess half-wave symmetry if $f(t) = -f(t \pm T/2) \text{or} - f(t) = f(t \pm T/2)$.

It means that the function remains the same if it is shifted to the left or right by half a period and then flipped over (*i.e.* multiplied by – 1) in respect to the *t*-axis or horizontal axis. It is called mirror symmetry because the negative portion of the wave is the mirror image of the positive portion of the wave displaced horizontally a distance *T*/2.

In other words, a waveform possesses half symmetry only when we invert its negative half-cycle and get an exact duplicate of its positive half-cycle. For example, in Fig. 21.4 (*a*) if we invert the negative half-cycle, we get the dashed *ABC* half-cycle which is exact duplicate of the positive half-cycle. Same is the case with the waveforms of Fig. 21.4 (*b*) and Fig. 21.4 (*c*). In case

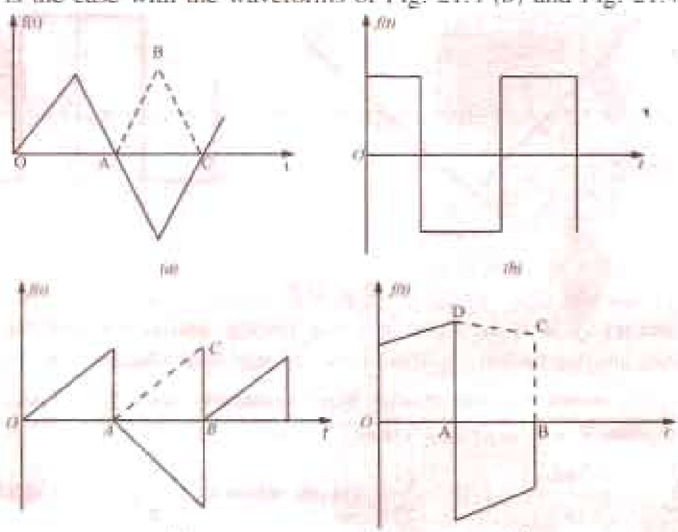


Fig. 21.4

of doubt, it is helpful to shift the inverted half-cycle by a half-period to the left and see if it superimposes the positive half-cycle. If it does so, there exists half-wave symmetry otherwise not. It is seen that the waveform of Fig. 21.4 (*d*) does not possess half-wave symmetry. It is so because when its negative half-cycle is inverted and shifted by half a period to the left it does not superimpose the positive half-cycle.

It may be noted that half-wave symmetry may be present in a waveform which also shows either odd symmetry or even symmetry:

For example, the square waveform shown in Fig. 21.4 (*a*) possesses even symmetry whereas the triangular waveform of Fig. 24.4 (*b*) has odd symmetry. All cosine and sine waves possess half-wave symmetry because

$$\cos\frac{2\pi}{T}\left(t\pm\frac{T}{2}\right)=\cos\left(\frac{2\pi}{T}t\pm\pi\right)=-\cos\frac{2\pi}{T}t;\sin\frac{2\pi}{T}\left(t\pm\frac{T}{2}\right)=\sin\left(\frac{2\pi}{T}t\pm\pi\right)=-\sin\frac{2\pi}{T}t$$

It is worth noting that the Fourier series of any function which possesses half-wave symmetry has zero average value and contains only odd harmonics and is given by

$$f(t)=\sum_{\substack{n=1\\odd}}^{\infty}\left(a_n\cos\frac{2\pi n}{T}t+b_n\sin\frac{2\pi n}{T}t\right)=\sum_{\substack{n=1\\odd}}^{\infty}(a_n\cos n\omega_0 t+b_n\sin n\omega_0 t)=\sum_{\substack{n=1\\odd}}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)$$

where, $$a_n=\frac{4}{T}\int_0^{T/2}f(t)\cos\frac{2\pi n}{T}dt=\frac{2}{\pi}\int_0^{\pi}f(\theta)\cos n\theta d\theta \qquad \text{... } n \text{ odd}$$

$$b_n=\frac{4}{T}\int_0^{T/2}f(t)\sin\frac{2\pi n}{T}dt=\frac{2}{\pi}\int_0^{\pi}f(\theta)\sin n\theta d\theta \qquad \text{... } n \text{ odd}$$

4. Quarter-wave Symmetry

An odd or even function with rotational symmetry is said to possess quarter-wave symmetry.

Fig. 21.5 (*a*) possesses half-wave symmetry as well as odd symmetry. The wave shown in Fig. 21.5 (*b*) has both half-wave symmetry and even symmetry.

The mathematical test for quarter-wave symmetry is as under:

Odd quarter-wave $f(t)=-f(t+T/2)$ and $f(-t)=-f(t)$

Even quarter-wave $f(t)=-f(t+T/2)$ and $f(t)=f(-t)$

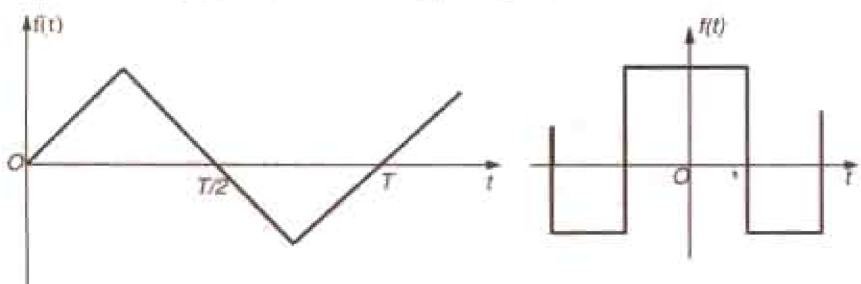


Fig. 21.5

Since each quarter cycle is the same in a way having quarter-wave symmetry, it is sufficient to integrate over one quarter period *i.e.* from 0 to *T*/4 and then multiply the result by 4.

(*i*) If $f(t)$ or $f(\theta)$ is odd and has quarter-wave symmetry, then a_0 is 0 and a_n is 0. Hence, the Fourier series will contain only odd sine terms.

$$\therefore\ f(t)=\sum_{\substack{n=1\\odd}}^{\infty}b_n\sin\frac{2\pi nt}{T}\ \text{ or }\ f(\theta)=\sum_{\substack{n=1\\odd}}^{\infty}b_n\sin n\theta,\ \text{where } b_n=\frac{1}{\pi}\int_0^{2\pi}f(\theta)\sin n\theta d\theta$$

It may be noted that in the case of odd quarter-wave symmetry, the integration may be carried over a quarter cycle.

$$\therefore\ a_n = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\cos n\theta d\theta \qquad \text{... } n \text{ odd}$$

$$= \frac{8}{T}\int_0^{T/4} f(t)\sin n\omega t dt \qquad \text{... } n \text{ odd}$$

(*ii*) If $f(t)$ or $f(\theta)$ is even and, additionally, has quarter-wave symmetry, then a_0 is 0 and b_n is 0. Hence, the Fourier series will contain only odd cosine terms.

$$\therefore\quad f(\theta) = \sum_{\substack{n=1\\ \text{odd}}}^{\infty} a_n \cos n\theta d\theta = \sum_{\substack{n=1\\ \text{odd}}}^{\infty} a_n \cos n\omega_0 t dt;\ \text{where } a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta$$

In this case a_n may be found by integrating over any quarter period.

$$a_n = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\sin n\theta d\theta \qquad \text{... } n \text{ odd}$$

$$= \frac{8}{T}\int_0^{T/4} f(t)\cos n\omega t\, dt \qquad \text{... } n \text{ odd}$$

21.8. Line or Frequency Spectrum

A plot which shows the amplitude of each frequency component in a complex waveform is called the line spectrum or frequency spectrum (Fig. 21.6). The amplitude of each frequency component is indicated by the length of the vertical line located at the corresponding frequency. Since the spectrum represents frequencies of the harmonics as discrete lines of appropriate height, it is also called a discrete spectrum. The lines decrease rapidly for waves having convergent series. Waves with discontinuties such as the sawtooth and square waves have spectra whose amplitudes decrease slowly because their series have strong high harmonics. On the other hand, the line spectra of waveforms without discontinuties and with a smooth appearance have lines which decrease in height very rapidly.

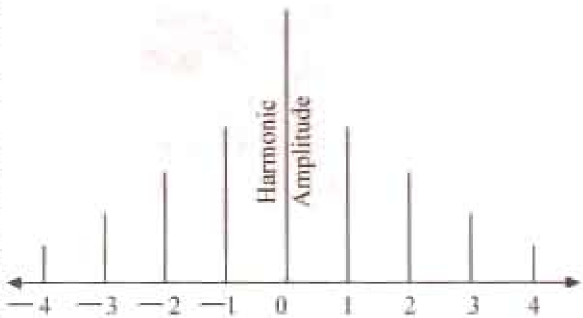


Fig. 21.6

The harmonic content and the line spectrum of a wave represent the basic nature of that wave and never change irrespective of the method of analysis. Shifting the zero axis changes the symmetry of a given wave and gives its trigonometric series a completely different appearance but the same harmonics always appear in the series and their amplitude remains constant.

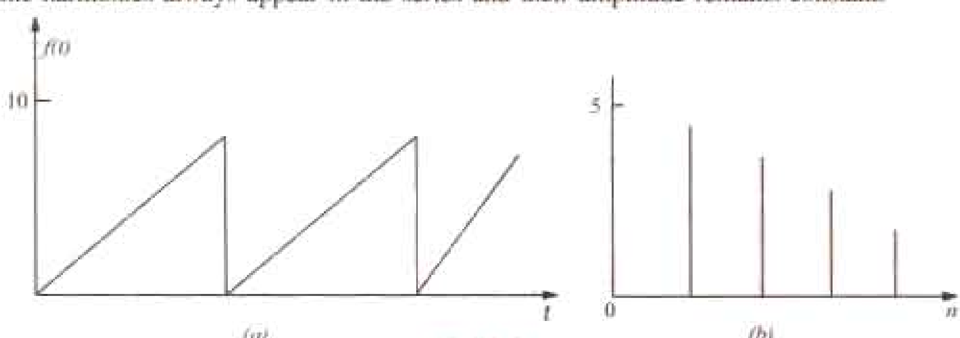


Fig. 21.7

Fig. 21.7 shows a smooth wave alongwith its line spectrum. Since there are only sine terms in its trigonometric series (apart from $a_0 = \pi$), the harmonic amplitudes are given by b_n.

21.9 Procedure for Finding the Fourier Series of a Given Function

It is advisable to follow the following steps:

1. Step No. 1

If the function is defined by a set of equations, sketch it approximately and examine for symmetry.

2. Step No. 2

Whatever be the period of the given function, take it as 2π (Ex. 20.6) and find the Fourier series in the form

$$f(\theta) = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + \ldots + a_n \cos n\theta + b_1 \sin\theta + b_2 \sin 2\theta + \ldots + b_n \sin n\theta$$

Step No. 3

The value of the constant a_0 can be found in most cases by inspection. Otherwise it can be found as under:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)d\theta$$

4. Step No. 4

If there is no symmetry, then a_0 is found as above whereas the other two fourier constants can be found by the relation.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta d\theta$$

$$b_n = \int_0^{2\pi} f(\theta)\sin n\theta d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta d\theta$$

5. Step No. 5

If the function has even symmetry *i.e.* $f(\theta) = f(-\theta)$, then $b_n = 0$ so that the Fourier series will have no sine terms. The series would be given by

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta d\theta \text{ where } a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta d\theta$$

6. Step No. 6

If the given function has odd symmetry *i.e.* $f(-\theta) = -f(\theta)$ then $a_0 = 0$ and $a_n = 0$. Hence, there would be no cosine terms in the Fourier series which accordingly would be given by

$$f(\theta) = \sum_{n=1}^{\infty} b_n \sin\omega_0 t \text{ ; where } b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta d\theta$$

7. Step No. 7

If the function possesses half-wave symmetry *i.e.* $f(\theta) = -f(\theta \pm \pi)$ or $f(t) = -f(t \pm T/2)$, then a_0 is 0 and the Fourier series contains only odd harmonics. The Fourier series is given by

$$f(\theta) = \sum_{\substack{n=1 \\ odd}}^{\infty} a_n(\cos n\theta + b_n \sin n\theta)$$

where $\quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta \ldots n \text{ odd}, \; b_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta d\theta \quad \ldots n \text{ odd}$

8. Step No. 8

If the function has even quarter-wave symmetry then $a_0 = 0$ and $b_n = 0$. It means the Fourier series will contain no sine terms but only odd cosine terms. It would be given by

$$f(\theta)=\sum_{\substack{n=1\\ odd}}^{\infty} a_n \cos n\theta, \text{ where } a_n=\frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta=\frac{2}{\pi}\int_0^{\pi} f(\theta)\cos n\theta d\theta=\frac{4}{\pi}\int_0^{\pi/2} f(\theta)\cos n\theta d\theta \text{ ... n odd}$$

9. Step No. 9

If the function has odd quarter-wave symmetry, then $a_0 = 0$ and $a_n = 0$. The Fourier series will contain only odd sine terms (but no cosine terms).

$$\therefore \; f(\theta)=\sum_{\substack{n=1\\ odd}}^{\infty} b_n \sin n\theta, \text{ where } b_n=\frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta=\frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta d\theta=\frac{4}{\pi}\int_0^{\pi/2} f(\theta)\sin n\theta d\theta \text{ ... n odd}$$

10. Step No. 10

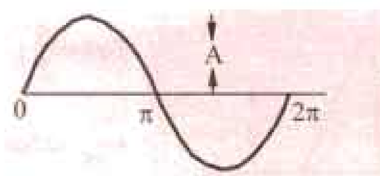
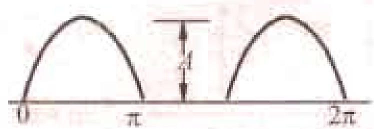
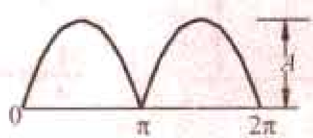
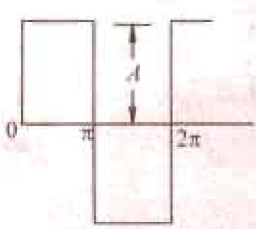
Having found the coefficients, the Fourier series as given in step No. 2 can be written down.

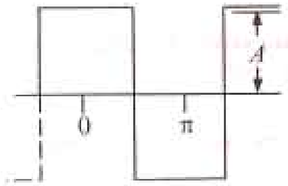
11. Step No. 11

The different harmonic amplitudes can be found by combining similar sine and cosine terms *i.e.* $A_n=\sqrt{a_n^2+b_n^2}$

where A_n is the amplitude of the n_{th} harmonic.

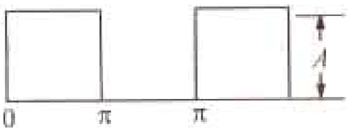
Table No. 21.1

	Wave form	*Appearance*	*Equation*
A.	Sine wave		$f(t) = A = A \sin \omega t$
B.	Half-wave rectified sine wave		$f(t)=A\left(\frac{1}{\pi}+\frac{1}{2}\sin\omega t-\frac{2}{3\pi}\cos 2\omega t\right)$ $-\frac{2}{15\pi}\cos 4\omega t-\frac{2}{35\pi}\cos 6\omega t$ $-\frac{2}{63\pi}\cos 8\omega t\ldots.$
C	Full-wave rectified sine wave		$f(t)=\frac{2A}{\pi}(1-\frac{2}{3}\cos 2\omega t$ $-\frac{2}{15}\cos 4\omega t-\frac{2}{35}\cos 6\omega t\ldots.)$
D.	Rectangular or square wave		$f(t)=\frac{44}{\pi}(\sin\omega t+\frac{1}{3}\sin 3\omega t \ldots..)$ $+\frac{1}{5}\sin 5\omega t+\frac{1}{7}\sin 7\omega t+\ldots.)$



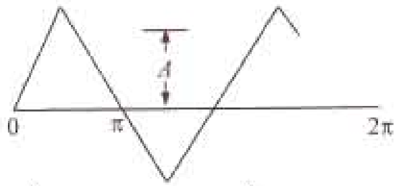
$$f(t) = \frac{4A}{\pi}\left(\cos\omega t - \frac{1}{3}\cos 3\omega t + \frac{1}{5}\cos 5\omega t........\right)$$

E. Rectangular or square wave pulse

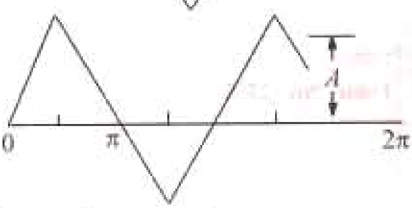


$$f(t) = \frac{A}{2} + \frac{2A}{\pi}\left(\sin\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \frac{1}{7}\sin 7\omega t + ...\right)$$

F. Triangular wave

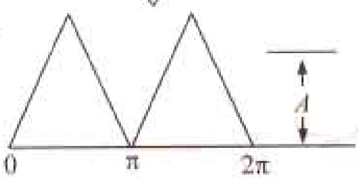


$$f(t) = \frac{8A}{\pi^2}\left(\sin\omega t - \frac{1}{9}\sin 3\omega t + \frac{1}{25}\sin 5\omega t - \frac{1}{49}\sin 7\omega t + ...\right)$$



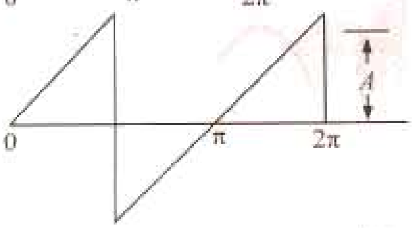
$$f(t) = \frac{8A}{\pi^2}\left(\cos\omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t + \frac{1}{49}\cos 7\omega t + ...\right)$$

G. Triangular pulse



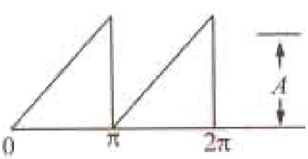
$$f(t) = \frac{A}{2} + \frac{4A}{\pi^2}\left(\sin\omega t - \frac{1}{9}\sin 3\omega t + \frac{1}{25}\sin 5\omega t - \frac{1}{49}\sin 7\omega t + ...\right)$$

H. Sawtooth wave



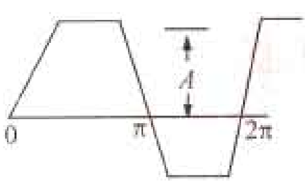
$$f(t) = \frac{2A}{\pi}\left(\sin olegat - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t - \frac{1}{4}\sin 4\omega t + ...\right)$$

I Sawtooth pulse



$$f(t) = \frac{A}{\pi}\left(\frac{\pi}{2} - \sin\omega_0 t - \frac{1}{2}\sin 2\omega t - \frac{1}{3}\sin 3\omega t - \frac{1}{4}\sin 4\omega t\right)$$

J. Trapezoidal wave



$$f(t) = \frac{3\sqrt{3}A}{\pi^2}\left(\sin\omega t - \frac{1}{25}\sin 5\omega t + \frac{1}{49}\sin 7\omega t......\right)$$

21.10. Wave Analyzer

A wave analyzer is an instrument designed to measure the individual amplitude of each harmonic

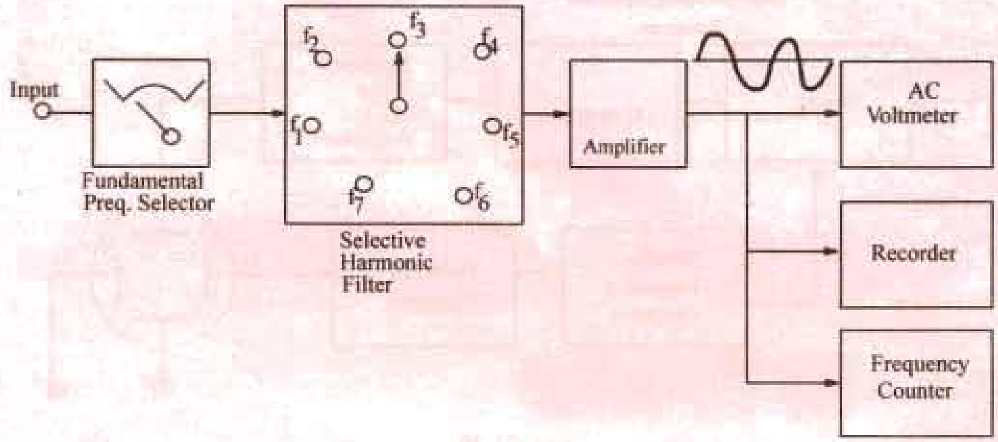


Fig. 21.8.

component in a complex waveform. It is the simplest form of analysis in the frequency domain and can be performed with a set of tuned filters and a voltmeter. That is why such analyzers are also called frequency-selective voltmeters. Since such analyzers sample the frequency spectrum successively, *i.e.* one after the other, they are called non-real-time analyzers.

The block diagram of a simple wave analyzer is shown in Fig. 21.8. It consists of a tunable fundamental frequency selector that detects the fundamental frequency f_1 which is the lowest frequency contained in the input waveform.

Once tuned to this fundamental frequency, a selective harmonic filter enables switching to multiples of f_1. After amplification, the output is fed to an a.c. voltmeter, a recorder and a frequency counter. The voltmeter reads the r.m.s amplitude of the harmonic wave, the recorder traces its waveform and the frequency counter gives its frequency. The line spectrum of the harmonic component can be plotted from the above data.

For higher frequencies (MHz) heterodyne wave analyzers are generally used. Here, the input complex wave signal is heterodyned to a higher intermediate frequency (IF) by an internal local oscillator. The output of the IF amplifier is rectified and is applied to a dc voltmeter called heterodyned tuned voltmeter.

The block diagram of a wave analyzer using the heterodyning principle is shown in Fig. 21.9.

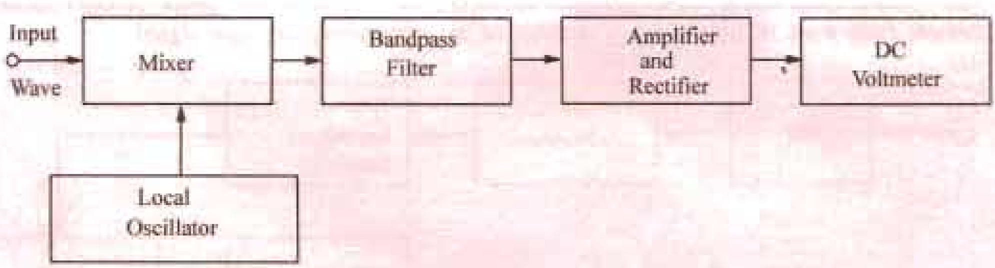


Fig. 21.9

The signal from the internal, variable-frequency oscillator heterodynes with the input signal in a mixer to produce output signal having frequencies equal to the sum and difference of the oscillator frequency f_0 and the input frequency f_1. Generally, the bandpass filter is tuned to the 'sum frequency' which is allowed to pass through. The signal coming out of the filter is amplified, rectified and then applied to a dc voltmeter having a decibel-calibrated scale. In this way, the peak amplitudes of the fundamental component and other harmonic components can be calculated.

21.11. Spectrum Analyzer

It is a real-time instrument *i.e.* it simultaneously displays on a CRT, the harmonic peak

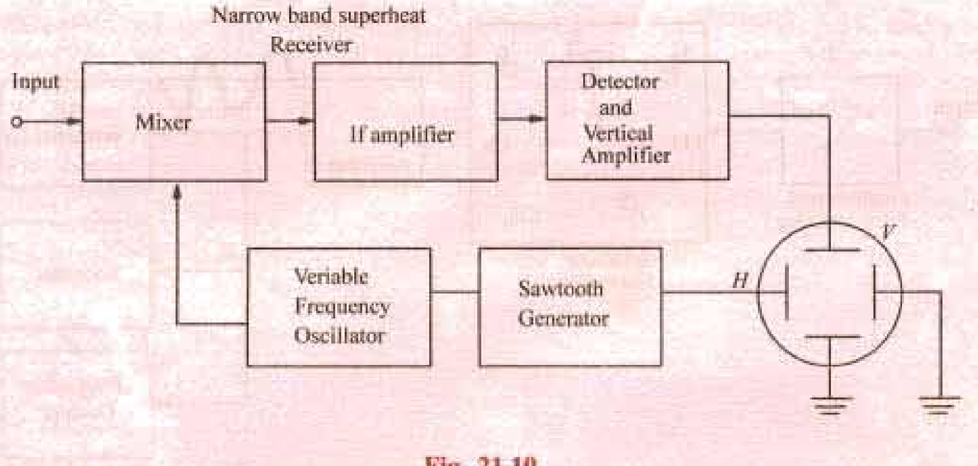


Fig. 21.10

values versus frequency of all wave components in the frequency range of the analyzer. The block diagram of such an analyzer is shown in Fig. 21.10.

As seen, the spectrum analyzer uses a CRT in combination with a narrow-band superheterodyne receiver. The receiver is tuned by varying the frequency of the voltage-tuned variable-frequency oscillator which also controls the sawtooth generator that sweeps the horizontal time base of the CRT deflection plates. As the oscillator is swept through its frequency band by the sawtooth generator, the resultant signal mixes and beats with the input signal to produce an intermediate frequency (IF) signal in the mixer. The mixer output occurs only when there is a corresponding harmonic component in the input signal which matches with the sawtooth generator signal. The signals from the IF amplifier are detected and further amplified before applying them to the vertical deflection plates of the CRT. The resultant output displayed on the CRT represents the line spectrum of the input complex or nonsinusoidal waveform.

21.12. Fourier Analyzer

A Fourier analyzer uses digital signal processing technique and provides information regarding the contents of a complex wave which goes beyond the capabilities of a spectrum analyzer. These analyzers are based on the calculation of the discrete Fourier transform using an algorithm called the fast Fourier transformer. This algorithm calculates the amplitude and phase of each harmonic component from a set of time domain samples of the input complex wave signal.

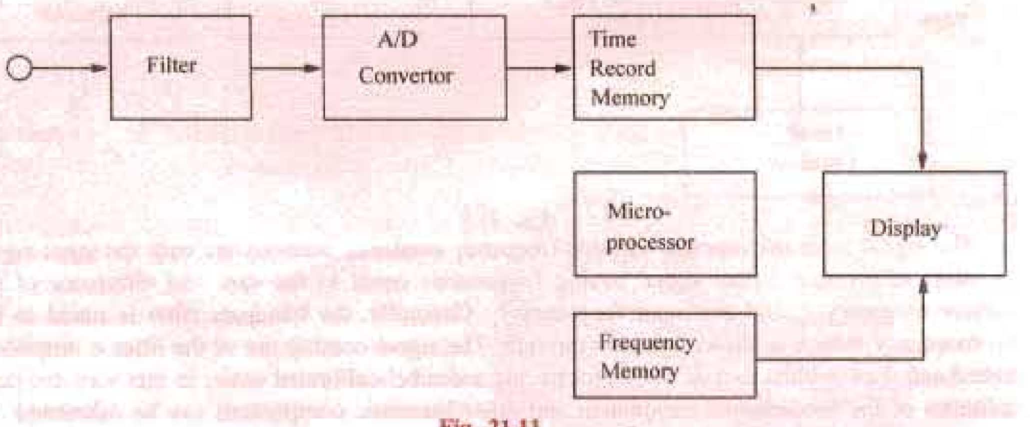


Fig. 21.11

A basic block diagram of a Fourier analyzer is shown in Fig. 21.11. The complex wave signal applied to the instrument is first filtered to remove out-of-band frequency components. Next, the signal is applied to an analog/digital (A/D) convertor which samples and digitizes it at regular time intervals until a full set of samples (called a time record) has been collected. The microprocessor then performs a series of computations on the time data to obtain the frequency-domain results *i.e.* amplitude versus frequency relationships. These results which are stored in memory can be displayed on a CRT or recorded permanently with a recorder or plotter.

Since Fourier analyzers are digital instruments, they can be easily interfaced with a computer or other digital systems. Moreover, as compared to spectrum analyzers, they provide a higher degree of accuracy, stability and repeatability.

21.13. Harmonic Synthesis

It is the process of building up the shape of a complex waveform by adding the instantaneous values of the fundamental and harmonics. It is a graphical procedure based on the knowledge of the different components of a complex waveform.

Example 21.1. *A complex voltage waveform contains a fundamental voltage of r.m.s. value 220 V and frequency 50 Hz alongwith a 20% third harmonic which has a phase angle lagging by* $3\pi/4$ *radian at* $t = 0$. *Find an expression representing the instantaneous complex voltage v. Using harmonic synthesis, also sketch the complex waveform over one cycle of the fundamental.*

Solution. The maximum value of the fundamental voltage is = $200 \times \sqrt{2}$ = 310 V. Its angular velocity is $\omega = 2\pi \times 50$ = 100π rad/s. Hence, the fundamental voltage is represented by 310 sin 100π t.

The amplitude of the third harmonic = 20% of 310 = 62V. Its frequency is 3 × 50 = 150 Hz. Hence, its angular frequency is = $2\pi \times 150 = 300\pi$ rad/s. Accordingly, the third harmonic voltage can be represented by the equation 62 sin $(300\pi t - 3\pi/4)$. The equation of the complex voltage is given by υ = 310 sin $100\pi t + 62\sin(300\pi t - 3\pi/4)$

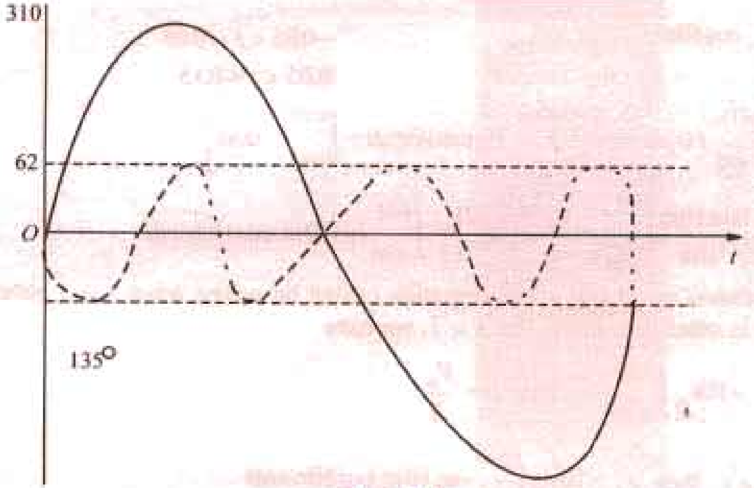


Fig 21.12

In Fig. 21.12 are shown one cycle of the fundamental and three cycles of the third harmonic component initially lagging by $3\pi/4$ radian or 135°. By adding ordinates at different intervals, the complex voltage waveform is built up as shown.

Incidentally, it would be seen that if the negative half-cycle is reversed, it is identical to the positive half-cycle. This is a feature of waveforms possessing half-wave symmetry which contains the fundamental and odd harmonics.

Example 21.2. *For the nonsinusoidal wave shown in Fig. 21.13, determine (a) Fourier coefficients* a_0 a_1 *and* b_4 *and (b) frequency of the fourth harmonic if the wave has a period of 0.02 second.*

Solution. The function $f(\theta)$ has a constant value from $\theta = 0$ to $\theta = 4\pi/3$ radian and 0 value from $\theta = 4\pi/3$ radian to $\theta = 2\pi$ radian.

(a) $a_0 = \frac{1}{2\pi}$ (net area per cycle)$_0^{2\pi} = \frac{1}{2\pi}\left(6 \times \frac{4\pi}{3}\right) = 4$

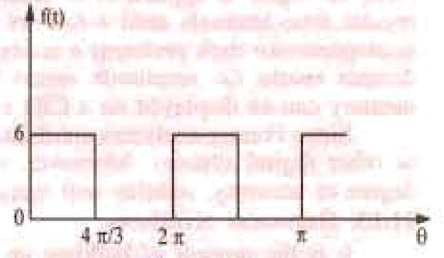


Fig. 21.13

$$a_3 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos 3\theta d\theta = \frac{1}{\pi}\int_0^{4\pi/3} 6\cos 3\theta d\theta$$

$$= \frac{6}{\pi}\left|\frac{\sin 3\theta}{3}\right|_0^{4\pi/3} = \frac{2}{\pi}(\sin 4\pi) = 0$$

$$b_4 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin 4\theta d\theta = \frac{1}{\pi}\int_0^{4\pi/3} 6\sin 4\theta d\theta$$

$$= \frac{6}{\pi}\left|\frac{-\cos 4\theta}{4}\right|_0^{4\pi/3} = -\frac{3}{2\pi}\left(\frac{\cos 16\pi}{3} - \cos 0\right) = \frac{-3}{2\pi}(-0.5-1) = \frac{9}{4\pi}$$

(b) Frequency of the fourth harmonic $= 4f_0 = 4/T = 4/0.02 = 200$ Hz.

Example 21.3. *Find the Fourier series of the "half sinusoidal" voltage waveform which represents the output of a half-wave rectifier. Sketch its line spectrum.*

Solution. As seen from Fig. 21.14 (a), $T = 0.2$ second, $f_0 = 1/T = 1/0.2 = 5$ Hz and $\omega_0 = 2\pi, f_0 = 10\pi$ rad/s. Moreover, the function has even symmetry. Hence, the Fourier series will contain no sine terms because $b_n = 0$.

The limits of integration would not be taken from $t = 0$ to $t = 0.2$ second, but from $t = -0.5$ to $t = 0.15$ second in order to get fewer equations and hence fewer integrals. The function can be written as

$$f(t) = V_m \cos 10\pi t \qquad -0.05 < t < 0.05$$
$$= 0 \qquad 0.05 < t < 0.15$$

$$a_0 = \frac{1}{T}\int_{-0.05}^{0.15} f(t)dt = \frac{1}{0.2}\left[\int_{-0.05}^{0.05} V_m \cos 10\pi t dt + \int_{0.05}^{0.15} 0.dt\right]$$

$$= \frac{V_m}{0.2}\left|\frac{\sin 10\pi t}{10\pi}\right|_{-0.05}^{0.05} = \frac{V_m}{\pi}\ a_n = \frac{2V_m}{0.2}\int_{-0.05}^{0.05} \cos 10\pi t.\cos 10\pi nt\, dt$$

The expression we obtain after integration cannot be solved when $n = 1$ although it can be solved when n is other than unity. For $n = 1$, we have

$$a_1 = 10V_m\int_{-0.05}^{0.05} \cos^2 10\pi t\, dt = \frac{V_m}{2}$$

When $n \neq 1$, then $a_n = 10V_m\int_{-0.05}^{0.05} \cos 10\pi t.\cos 10\pi ntdt$

$$= \frac{10V_m}{2}\int_{-0.05}^{0.05} [\cos 10\pi(1+n)t \cos 10\pi(1-n)t]dt = \frac{2V_m}{\pi}.\frac{\cos(\pi n/2)}{(1-n^2)} \quad ...n \neq 1$$

$$a_2 = \frac{2V_m}{\pi}.\frac{\cos\pi}{1-4} = \frac{2V_m}{\pi}.\frac{-1}{-3} = \frac{2V_m}{2\pi}; a_3 = \frac{2V_m}{\pi}.\frac{\cos 3\pi/2}{1-3^2} = 0; a_4 = \frac{2V_m}{\pi}.\frac{\cos 2\pi}{1-4^2} = -\frac{2V_m}{15\pi}$$

$$a_5 = 0; a_6 = \frac{2V_m}{\pi}.\frac{\cos 3\pi}{1-6^2} = \frac{2V_m}{35\pi} \text{ and so on}$$

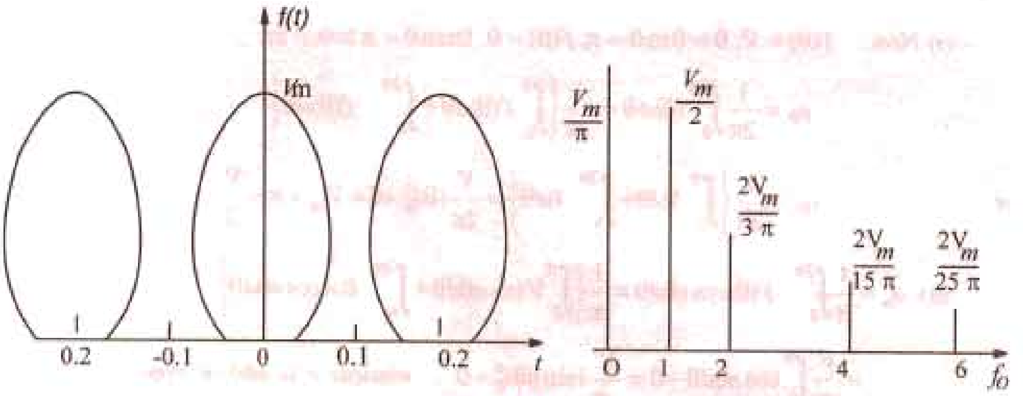


Fig. 21.14

Substituting the values of a_0, a_1, a_2, a_4 etc. in the standard Fourier series expression given in Art.

20.3. we have

$$f(t) = a_0 + a_1 \cos 2\omega_0 t + a_2 \cos_2 \omega 0t + a_4 \cos 4\omega_0 t + a_6 \cos 6\omega_0 t + \ldots$$

$$= \frac{V_m}{\pi} + \frac{V_m}{2}\cos 10\pi t + \frac{2V_m}{3\pi}\cos 20\pi t - \frac{2V_m}{15\pi}\cos 40\pi t + \frac{2V_m}{35}\cos 60\pi t, \ldots$$

$$= V_m\left(\frac{1}{\pi} + \frac{1}{2}\cos\omega_0 t + \frac{2}{3\pi}\cos 2\omega_0 t\pi - \frac{2}{15\pi}\cos 4\omega_0 t + \frac{2}{35\pi}\cos 6\omega_0 t \ldots\right)$$

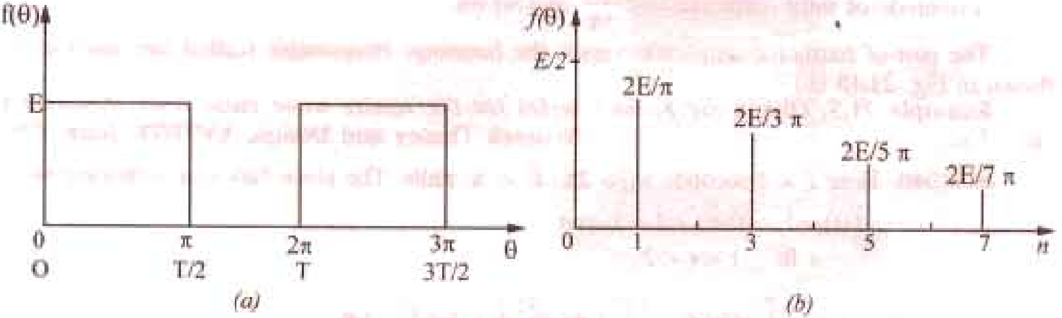
The line spectrum which is a plot of the harmonic amplitudes versus frequency is given in Fig. 21.14(*b*).

Example 21.4. *Determine the Fourier series for the square voltage pulse shown in Fig. 21.15 (a) and plot its line spectrum.* **(Network Theory, Nagpur Univ. 1992)**

Solution. The wave represents a periodic function of θ or ωt or $\left(\frac{2\pi t}{T}\right)$ having a period extending over 2π radians or T seconds. The general expression for this wave can be written as

$$f(\theta) = a_0 + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + \ldots$$

$$\ldots + b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta + \ldots$$



(a) (b)

Fig. 21.15

(*i*) Now, $f(\theta) = V;\ \theta = 0 \text{ to } \theta = \pi;\ f(\theta) = 0,\ \text{from } \theta = \pi \text{ to } \theta = 2\pi$

$$\therefore \qquad a_0 = \frac{1}{2\pi}\int_0^{2\pi}(\theta)d\theta = \frac{1}{2\pi}\left\{\int_0^{\pi} f(\theta)d\theta + \int_{\pi}^{2\pi} f(\theta)d\theta\right\}$$

or

$$a_0 = \frac{1}{2\pi}\left\{\int_0^{\pi} V d\theta + \int_{\pi}^{2\pi} 0 d\theta\right\} = \frac{V}{2\pi}|\,\theta\,|_0^{\pi} + 0 = V_{2\pi} \times \pi = \frac{V}{2}$$

$$(ii)\ a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta = \frac{1}{\pi}\left\{\int_0^{\pi} V\cos n\theta d\theta + \int_{\pi}^{2\pi} 0\times\cos n\theta d\theta\right\}$$

$$= \frac{V}{\pi}\int_0^{\pi}\cos n\theta d\theta + 0 = \frac{V}{n\pi}|\sin n\theta|_0^{\pi} = 0 \text{ ... whether } n \text{ is odd or even}$$

$$(iii)\ b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta = \frac{1}{\pi}\left\{\int_0^{\pi} V\sin n\theta d\theta + \int_{\pi}^{2\pi} 0\times\sin n\theta d\theta\right\}$$

$$= \frac{V}{\pi}\int_0^{\pi}\sin n\theta d\theta + 0 = \frac{V}{\pi}\left|\frac{-\cos n\theta}{n}\right|_0^{\pi} = \frac{V}{n\pi}(-\cos n\pi + 1)$$

Now, when n is odd, $(1 - \cos n\pi) = 2$ but when n is even, $(1 - \cos n\pi) = 0$.

$$\therefore\ b_1 = \frac{2V}{\pi} \text{ ...} n = 1;\ b_2 = \frac{V}{2\pi}\times 0 = 0 \text{ ... } n = 2;\ b_3 = \frac{V}{3\pi}\times 2 = \frac{2V}{3\pi} \text{ ...} n = 3 \text{ and so on.}$$

Hence, substituting the values of a_0, a_1, a_2 etc. and b_1, b_2 etc. in the above given Fourier series, we get

$$f(\theta) = \frac{V}{2} + \frac{2V}{\pi}\sin\theta + \frac{2V}{3\pi}\sin 3\theta + \frac{2V}{5\pi}\sin 5\theta + \ldots = \frac{E}{2} + \frac{2V}{\pi}\left(\sin\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \ldots\right)$$

It is seen that the Fourier series contains a constant term $V/2$ and odd harmonic components whose amplitudes are as under:

Amplitude of fundamental or first harmonic $= \dfrac{2V}{\pi}$

Amplitude of second harmonic $= \dfrac{2V}{3\pi}$

Amplitude of third harmonic $= 2\dfrac{EV}{5\pi}$ and so on.

The plot of harmonic amplitude versus the harmonic frequencies (called line spectrum) is shown in Fig. 21.15 (*b*).

Example 21.5. *Obtain the Fourier series for the square wave pulse train indicated in Fig. 21.16.* **(Network Theory and Design, AMIETE June, 1990)**

Solution. Here $T = 2$ second, $\omega_0 = 2\pi/T = \pi$ rad/s. The given function is defined by

$$f(t) = 1 \qquad 0 < t < 1 = 0 \text{ and}$$
$$= 0; \qquad 1 < t < 2$$

$$a_0 = \frac{1}{T}\int_0^{T} f(t)dt = \frac{1}{2}\int_0^{2} 1.dt = \frac{1}{2}\left[\int_0^{1} 1.dt + \int_1^{2} 0.dt\right] = \frac{1}{2}$$

Even otherwise by inspection $a_0 = (1 + 0)/2 = 1/2$

$$a_n = \frac{2}{T}\int_0^{T} f(t)\cos n\omega_0 t dt = \frac{2}{2}\int_0^{2} 1.\cos n\pi t dt = \left[\int_0^{1} 1.\cos n\pi t dt + \int_1^{2}(0).\cos n\pi t dt\right]$$

$$= \int_0^1 \cos n\pi t dt = \left| \frac{\sin n\pi t}{n\pi} \right|_0^1 = 0$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t dt = [\int_0^1 1.\sin n\pi t\, dt] + \int_1^2 (0)\sin n\pi t dt] = \int_0^1 \sin n\pi t\, dt$$

$$= \left| \frac{-\cos n\pi t}{n\pi} \right|_0^1 = \frac{1-\cos n\pi}{n\pi}$$

$\therefore$ $b_n = 2/n\pi$ ..... when n is odd; $= 0$ ... when n is even

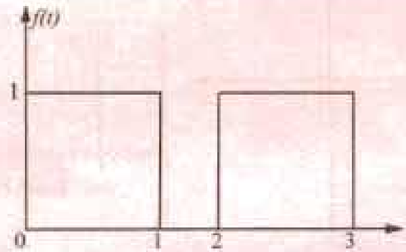


Fig. 21.16

$$\therefore \quad f(t) = a_0 + \frac{2}{\pi}\sum_{\substack{n=1\\ odd}}^{\infty} \frac{1}{n}\sin 3n\pi t = \frac{1}{2} + \frac{2}{\pi}\left(\sin \pi t + \frac{1}{3}\sin 3\pi t + \frac{1}{5}\sin 5\pi\, t \text{ etc.}\right)$$

Example 21.6. *Find the trigonometric Fourier series for the square voltage waveform shown in Fig. 21.17(a) and sketch the line spectrum.*

Solution. The function shown in Fig. 21.17 (*a*) is an odd function because at any time $f(-t) = -f(t)$. Hence, its Fourier series will contain only sine terms *i.e.* $a_n = 0$. The function also possesses half-wave symmetry, hence, it will contain only odd harmonics.

As seen from Art. 21.7 (2) the Fourier series,for the above wave is given by

$$f(\theta) = \sum_{\substack{n=1\\ odd}}^{\infty} b_n \sin n\theta t \quad \text{where } b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} V\sin n\theta d\theta + \int_{\pi}^{2\pi} -V\sin n\theta d\theta\right\} = \frac{V}{\pi n}\left|-\cos n\theta\right|_0^{\pi} + \frac{V}{\pi n}\left|\cos n\theta\right|_{\pi}^{2\pi}$$

$$= \frac{V}{\pi n}\{(-\cos n\pi + \cos 0) - \cos \pi n\}$$

$$= \frac{V}{\pi n}\{(1-\cos n\pi) + (1-\cos n\pi)\} = \frac{2V}{\pi n}(1-\cos n\pi)$$

Now, $1 - \cos n\pi = 2$ when n is odd

and $= 0$ when n is even

$$\therefore \qquad b_1 = \frac{2V}{\pi} \times 2 = \frac{4V}{\pi} \text{ ... putting } n = 1;\ b_2 = 0 \text{ ... putting } n = 2$$

$$b_3 = \frac{2V}{\pi 3} \times 2 = \frac{4V}{3\pi} \text{ ... putting } n = 3;\ b_4 = 0\text{... putting } n = 4$$

$$b_5 = \frac{2V}{\pi .5} \times 2 = \frac{4V}{5\pi} \text{ .. putting } n = 5 \text{ and so on.}$$

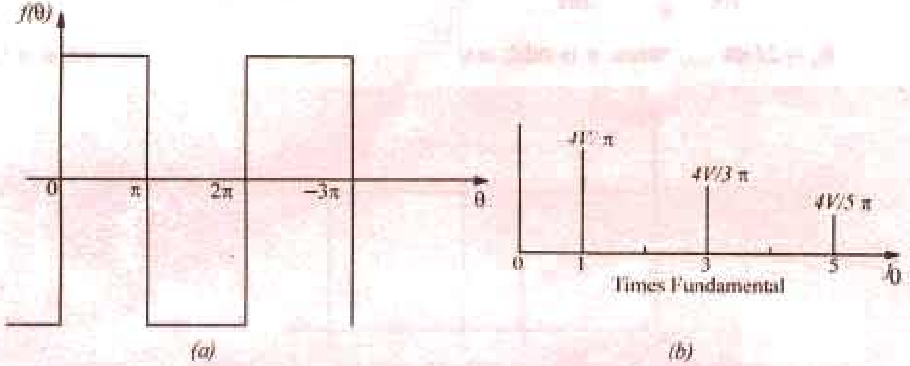


Fig. 21.17

Hence, the Fourier series for the given waveform is

$$f(\theta) = \frac{4V}{\pi}\sin\theta + \frac{4V}{3\pi}\sin 3\theta + \frac{4V}{5\pi}\sin 50 + \ldots$$

$$= \frac{4V}{\pi}\left(\sin\omega_0 t + \frac{1}{3}3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \ldots\right) = \frac{4V}{\pi}\left(\sin\frac{2\pi}{T}t + \frac{1}{3}\sin\frac{6\pi}{T}t + \frac{1}{5}\sin\frac{10\pi}{T}t + \ldots\right)$$

The line spectrum of the function is shown in Fig. 21.17 (*b*). It would be seen that the harmonic amplitudes decrease as $1/n$, that is, the third harmonic amplitude is 1/3 as large as the fundamental, the fifth harmonic is 1/5 as large and so on.

Example 21.7. *Determine the Fourier series for the square voltage waveform shown in Fig. 21.17 (a). Plot its line spectrum.*

Solution. This is the same question as given in Ex. 21.6 but has been repeated to illustrated a singhtly different technique. As seen from Fig. 21.17(*a*) T = 2π, hence, $\omega_0 = 2\pi f_0 = 2\pi / T = 2\pi / \pi = 1$. Over one period the function can be defined as

$$f(t) = V \ 0 < t < 5$$

$$= -V,\ \pi < t < 2\pi$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin n\omega_0 t dt \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt dt = \frac{1}{\pi}\left[\int_0^{\pi} f(t)\sin nt dt + \frac{1}{\pi}\int f(t)\sin nt dt\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} V\sin nt dt + \frac{1}{\pi}\int_{\pi}^{2\pi}(-V)\sin nt dt = \frac{V}{\pi}\left|\frac{-\cos nt}{n}\right|_0^{\pi} + \frac{V}{\pi}\left|\frac{\cos nt}{n}\right|_{\pi}^{2\pi}$$

$$= -\frac{V}{n\pi}(\cos n\pi - \cos 0) + \frac{V}{n\pi}(\cos 2n\pi - \cos n\pi)$$

Since cos 0 is 1 and cos $2n\pi = 1$ $\quad \therefore \; b_n = \frac{2V}{n\pi}(1 - \cos n\pi)$

When n is even, $\cos n\pi = 1$ $\quad \therefore \; b_n = 0$

When n is odd, cos $n\pi = -1$ $\quad \therefore \; b_n = \frac{2V}{n\pi}(1+1) = \frac{4V}{n\pi} = n = 1,3,5...$

Substituting the value of b_n, the Fourier series become

$$f(t) = \sum_{\substack{n=1 \\ odd}}^{\infty} \frac{4V}{n\pi} \sin nt = \frac{4V}{\pi} \sum_{\substack{n=1 \\ odd}}^{\infty} \frac{1}{n} \sin nt = \frac{4V}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + ...\right)$$

Since $\omega_0 = 1$, the above expression in general terms becomes

$$f(t) = \frac{4V}{\pi}\left(\sin \omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + ...\right)$$

The line spectrum is as shown in Fig. 21.17 (*b*).

Example 21.8. *Determine the Fourier series of the square voltage waveform shown in Fig. 20.18.*

Solution. As compared to Fig. 21.117 (*a*) given above, the vertical axis of figure has been shifted by $\pi/2$ radians. Replacing t by $(t + \pi/2)$ in the above equation, the Fourier series of the waveform shown in Fig. becomes

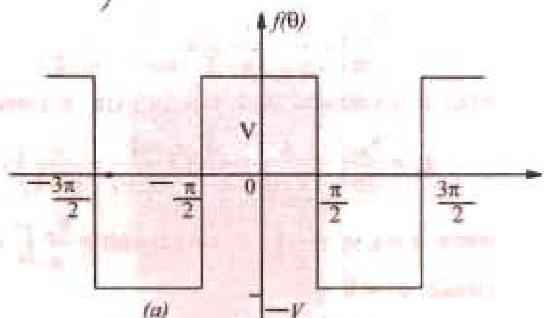


Fig. 21.18

$$f(t) = \frac{4V}{\pi}\left[\sin\left(t + \frac{\pi}{2}\right) + \frac{1}{3}\sin 3\left(t + \frac{\pi}{2}\right) + \frac{1}{5}\sin 5\left(t + \frac{\pi}{2}\right) + ...\right]$$

$$= \frac{4V}{\pi}\left(\cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t ...\right)$$

Example 21.9. *Determine the trigonometric Fourier series for the half-wave rectified sine wave form shown in Fig. 21.19 (a) and sketches line spectrum.*

Solution. The given waveform shows no symmetry, hence its series would contain both sine and cosine terms. Moreover, its average value is not obtainable by inpsection, hence it will have to be found by integration.

Here, $T = 2\pi, \omega_0 = 2\pi / T = 1$. Hence, equation of the given waveform is $V = V_m \sin \omega t = V_m \sin t$.

The given waveform is defined by $f(t) = V_m \sin t, 0 < t < \pi = 0, \pi < t < 2\pi$

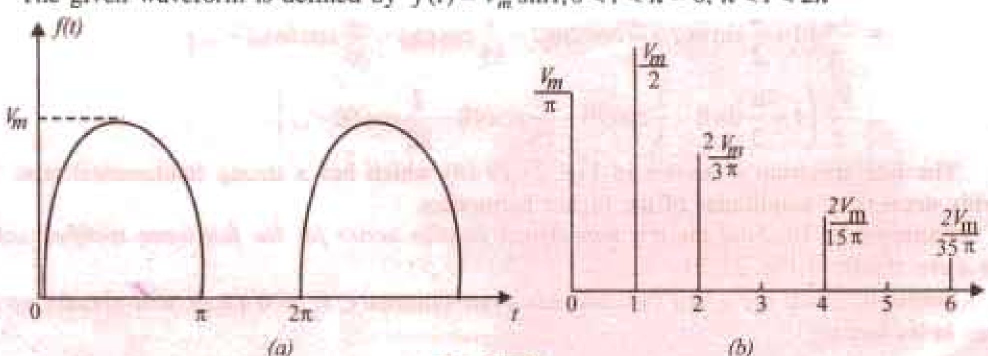


Fig. 21.19

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt = \frac{1}{2\pi}\left[\int_0^{\pi} V_m \sin t dt + \int_{\pi}^{2\pi} dt = \frac{1}{2\pi}\int_0^{\pi} V_m \sin t dt\right]$$

$$= \frac{V_m}{2\pi}|-\cos t|_0^{\pi} = -\frac{V_m}{2\pi}(\cos\pi - \cos 0) = \frac{V_m}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos nt dt = \frac{1}{\pi}\int_0^{\pi} V_m \sin t \cos nt dt$$

$$= \frac{V_m}{2\pi}\int_0^{\pi} [\sin(n+1)t - \sin(n-1)t]dt = \frac{V_m}{2\pi}\left|\frac{-\cos(n+1)t}{n+1} + \frac{\cos(n-1)t}{n-1}\right|_0^{\pi}$$

$$= \frac{V_m}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

when n is even, $\cos\ (n+1)\pi = -1$ and $\cos(n-1)\pi = -1$

$$\therefore \quad a_n = \frac{V_m}{2\pi}\left[\frac{1}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right] = -\frac{2V_m}{\pi(n^2-1)} \qquad \ldots n = 2, 4, 6 \text{ etc.}$$

when n is odd and $\neq 1$, $\cos\ (n+1)\pi = 1$ and $\cos\ (n-1)\pi = 1$

$$\therefore \quad a_n = \frac{V_m}{2\pi}\left(-\frac{1}{n+1} + \frac{1}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right) = 0 \qquad \ldots n = 3, 5, 7 \text{ etc.}$$

when $n = 1$, $a_1 = \frac{1}{\pi}\int_0^{\pi} V_m \sin t.\cot t dt = \frac{V_m}{\pi}\int_0^{\pi} \sin t \cos t dt = \frac{V_m}{2\pi}\int_0^{\pi} \sin 2t dt = 0$

Hence, $a_n = 0$ $\qquad \ldots n = 1, 3, 5\ldots$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin nt dt = \frac{1}{\pi}\left[\int_0^{\pi} V_m \sin t \sin nt dt + \int_{\pi}^{2\pi} (0)\sin nt dt\right]$$

$$= \frac{V_m}{\pi}\int_0^{\pi} \sin t \sin nt dt = 0 \text{ for } n = 2, 3, 4, 5 \text{ etc.}$$

However, the expression indeterminate for $n = 1$ so that b_1 has to be evaluated separately.

$$b_1 = \frac{1}{\pi}\int_0^{\pi} V_m \sin t.\sin t dt = \frac{V_m}{\pi}\int_0^{\pi} \sin^2 t dt = \frac{V_m}{2}$$

The required Fourier series for the half-wave rectified voltage waveform is

$$f(t) = \frac{V_m}{\pi} + \frac{V_m}{2}\sin t - \frac{2V_m}{\pi}\sum_{\substack{n=1 \\ even}}^{\infty}\left(\frac{\cos nt}{n^2-1}\right)$$

$$= \frac{V_m}{\pi}\left(1 + \frac{\pi}{2}\sin t - \frac{2}{3}\cos 2t - \frac{2}{15}\cos 4t - \frac{2}{35}\cos 6t\ldots\right)$$

$$= \frac{V_m}{\pi}\left(1 + \frac{\pi}{2}\sin \omega_0 t - \frac{2}{3}\cos 2\omega_0 t - \frac{2}{15}\cos \omega_0 t - \frac{2}{35}\cos 6\omega_0 t - \ldots\right)$$

$$= \frac{V_m}{T}\left(1 + \frac{\pi}{2}\sin \theta - \frac{2}{3}\cos 2\theta - \frac{2}{15}\cos 4\theta - \frac{2}{35}\cos 6\theta - \ldots\right)$$

The line spectrum is shown in Fig. 21.19 (*b*) which has a strong fundamental term with rapdily decreasing amplitudes of the higher harmonics.

Example 21.10. *Find the trigonometrical Fourier series for the full wave rectified voltage sine wave shown in Fig. 21.20.*

Solution. Since the given function has even symmetry, $b_n = 0$ *i.e.* it will contain no sine terms in its series.

The equation of the sinusoidal sine wave given by $V = V_m \sin\theta$. In other words, $f(\theta) = V_m = \sin\theta$.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta = \frac{1}{2\pi}\int_0^{2\pi} V_m \sin\theta d\theta = \frac{2V_m}{2\pi}\int_0^{\pi} \sin\theta d\theta$$

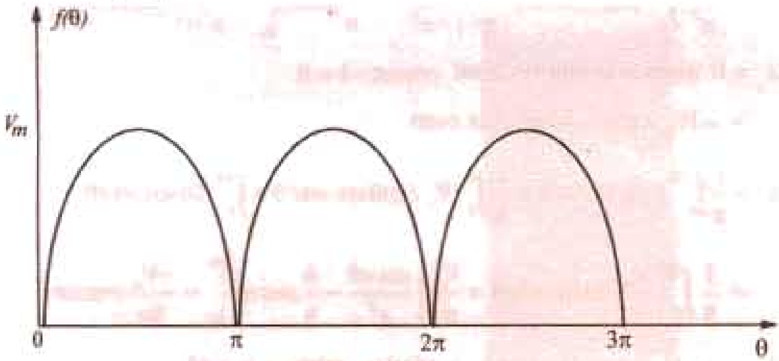
It is so because the two parts $0-\pi$ and $\pi - 2$ are identical.

$$\therefore \qquad a_0 = \frac{V_m}{\pi}|-\cos\theta|_0^{\pi} = \frac{2V_m}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta = \frac{2V_m}{\pi}\int_0^{\pi} \sin\theta\cos n\theta d\theta$$

Now, $\sin A \ \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$$\therefore \qquad a_n = \frac{2V_m}{\pi}\int_0^{\pi}[\sin(1+n)\theta + \sin(1-n)\theta]d\theta$$



Fog. 21.20

$$= -\frac{V_m}{\pi}\frac{\cos(1+n)\theta}{(1+n)} + \frac{\cos(1-n)\theta}{(1-n)}$$

$$= 0 = \frac{-V_m}{\pi}\left[\frac{\cos(1+n)\pi}{1+n} + \frac{\cos(1-n)\pi}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right] \text{ ... when } n \text{ is odd}$$

However, when n is even, then

$$a_n = \frac{-V_m}{\pi}\left[-\frac{1}{1+n} - \frac{1}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right] = \frac{2V_m}{\pi}\left[\frac{1}{1+n} + \frac{1}{1-n}\right] = \frac{-4V_m}{\pi(n^2-1)}$$

$$\therefore \qquad f(\theta) = a_0 - \frac{4V_m}{\pi}\sum_{\substack{n=2 \\ even}}^{\infty}\frac{\cos 2\theta}{(n^2-1)}$$

$$\therefore \qquad f(\theta) = \frac{2V_m}{\pi} - \frac{4V_m}{\pi}\left(\frac{1}{3}\cos^2\theta - \frac{1}{15}\cos 4\theta + \frac{1}{35}\cos 6\theta + ...\right)$$

Example 21.11. *Determine the Fourier series for the waveform shown in Fig. 21.21 (a) and sketch its line spectrum.*

Solution. Its is seen from Fig. 21.21 (*a*) that the waveform equation is $f(\theta) = (V_m/\pi)\theta$. The given function $f(\theta)$ is defined by

$$f(\theta) = \left(\frac{V_m}{\pi}\right)\theta \quad 0 < \theta < \pi$$

$$= 0 \pi < \theta < 2\pi$$

Since the function possesses neither even nor odd symmetry, it will contain both sine and cosine terms.

Average value of the wave over one cycle is $V_m/4$ or $a_0 = V_m/4$. It is so because the average value over the first half cycle is $V_m/2$ and over the second half cycle is 0 hence, the average value for full cycle is $= \dfrac{(V_m/2)+0}{2} = \dfrac{V_m}{4}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta) n\theta d\theta = \frac{1}{\pi}\left[\int_0^{\pi}(V_m/\pi)\theta\cos n\theta d\theta + \int_{\pi}^{2\pi}(0)\cos n\theta d\theta\right]$$

$$= \frac{V_m}{\pi^2}\int_0^{\pi}\theta\cos n\theta d\theta = \frac{V_m}{\pi^2}\left|\frac{\cos n\theta}{n^2} + \frac{\theta}{n}\sin n\theta\right|_0^{\pi} = \frac{V_m}{\pi^2 n^2}(\cos n\pi - 1)$$

$\therefore$ $a_n = 0$ when n is odd because $\cos n\pi - 1 = 0$

$= -2V_m/\pi^2 n^2$ when n is even

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin\theta d\theta = \frac{1}{\pi}\int_0^{\pi}(V_m/\pi)\theta\sin n\theta d\theta + \int_{\pi}^{2\pi}(0)\sin n\theta d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi}\left(\frac{V_m}{\pi}\right)\theta\sin n\theta d\theta = \frac{V_m}{\pi^2}\left|\frac{\sin n\theta}{n^2} - \frac{\theta}{n}\cos n\theta\right|_0^{\pi} = \frac{-V_m}{\pi n}\cos n\pi$$

$\therefore$ $b_n = -V_m | \pi n \, when \, n \, is \, even \, b_n = +V_m | \pi_n$ when n is odd

Substituting the values of various constants in the general expression for Fourier series, we get

$$f(\theta) = \frac{V_m}{4} - \frac{2V_m}{\pi^2}\cos\theta - \frac{2V_m}{(3\pi)^2}\cos 3\theta - \frac{2V_m}{(5\pi)^2}\cos 5\theta ...$$

$$+ \frac{V_m}{\pi}\sin\theta - \frac{V_m}{2\pi}\sin 2\theta + \frac{V_m}{3\pi}\sin 3\theta ....$$

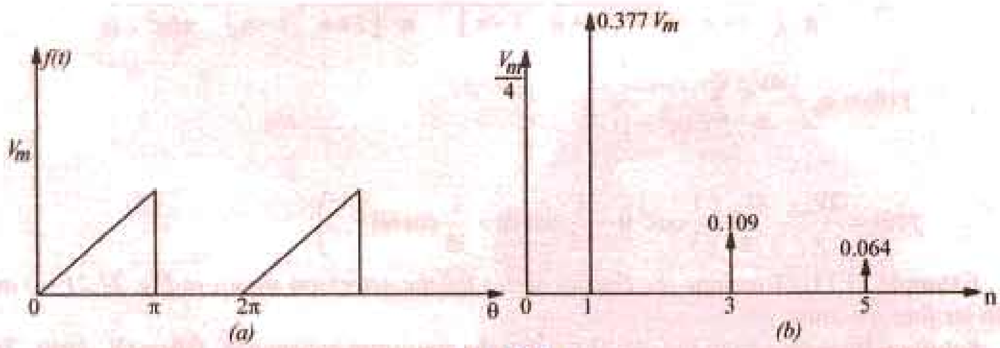


Fig. 21.21

The amplitudes of even harmonics are given directly by b_n but amplitudes of odd harmonics are given by $A_n = \sqrt{a_n^2 + b_n^2}$ (Art. 21.4)

For example,

$$A_1 = \sqrt{(2V_m / \pi^2)^2 + (V_m / \pi)^2} = 0.377 \ V_m$$

$$A_3 = \sqrt{\left(\frac{2V_m}{(3\pi)^2}\right)^2 + \left(\frac{V_m}{2\pi}\right)^2} = 0.109 \ V_m$$

$$A_5 = \left(\frac{2V_m}{(5\pi)^2}\right)^2 + \left(\frac{V_m}{5\pi}\right)^2 = 0.064 V_m \text{ and so on.}$$

The line spectrum is as shown in Fig. 21.21 (*b*).

Example 21.12. *Find the Fourier series for the sawtooth waveform shown in Fig. 21.22 (a). Sketch its line spectrum.*

Solution. Using by the relation $y = mx$, the equation of the function becomes $f(t) = 1.\ t$ or $f(t) = t$.

$$T = 2, \omega_0 = 2\pi / t = 2\pi / 2 = \pi$$

By inspection it is clear that $a_0 = 2/2 = 1$ $\ a_n = \frac{2}{T}\int_0^T f(t) \cos n\omega_0 t = \int_0^2 t.\cos n\pi t.dt$

Since we have to find the integral of two functions, we use the technique of integration by parts *i.e.*

$$\int uvdx = u\int vdx - \int\left(\frac{du}{dx}\int vdx\right)dx$$

$$\therefore \qquad a_n = t.\int_0^2 \cos n\pi t dt - \int_0^2\left(1.\int_0^2 \cos n\pi t dt\right)dt$$

$$= \left|\frac{t}{n\pi}\sin n\pi t\right|_0^2 + \left|\frac{\cos n\pi t}{(n\pi)^2}\right|_0^2 = 0 + \frac{1}{(n\pi)^2}(\cos 2n\pi - \cos 0)$$

Since $\cos 2n\pi = \cos 0$ for all values of n, hence $a_n = 0$

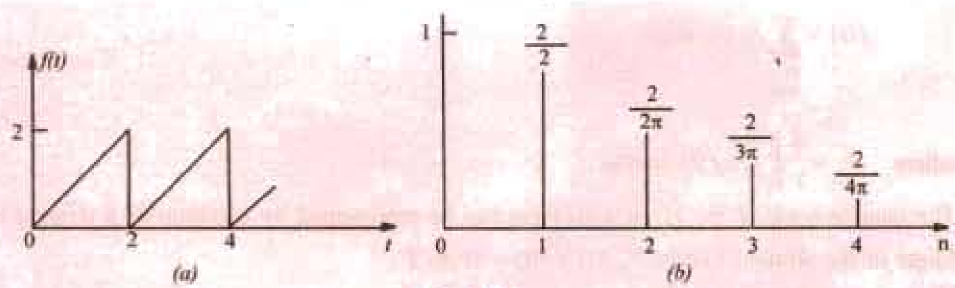


Fig. 21.22

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega_0 t dt = \int_0^T t \sin n\pi t dt$$

Employing integration by parts, we get,

$$b_n = t\int_0^2 \sin n\pi t \, dt - \int_0^2\left(1.\int_0^2 \sin n\pi t dt\right)dt$$

$$\left|t.\frac{-\cos n\pi t}{n\pi}\right|_0^2 dt-\int_0^2\frac{-\cos n\pi t}{n\pi}=\left|\frac{-t}{n\pi}.\cos n\pi t\right|_0^2+\left|\frac{\sin n\pi t}{(n\pi)^2}\right|_0^2=\frac{\sin 2n\pi}{(n\pi)^2}-\frac{2}{n\pi}\cos 2n\pi$$

The sine term is 0 for all values of n because sign of any multiple of 2π is 0. Since value of cosine term is 1 for any multiple of 2π, we have, $b_n = -2/n\pi$.

$$\therefore \qquad f(t)=a_0+\sum_{n=1}^{\infty} b_n \sin n\omega_0 t = a_0-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi t$$

$$= 1-\frac{2}{\pi}\left(\sin \pi t+\frac{1}{2}\sin 2\pi t+\frac{1}{3}\sin 3\pi t+...\right)$$

The line spectrum showing the amplitudes of various harmonics is shown in Fig. 21.22 (*b*).

Example 21.13. *Determine the trigonometric Fourier series of the triangular waveform shown in Fig. 21.23.*

Solution. Since the waveform possesses odd symmetry, hence $a_0 = 0$ and $a_n = 0$ *i.e.* there would be n_o cosine terms in the series. Moreover, the waveform has half-wave symmetry. Hence, series will have only odd harmonics. In the present case, there would be only odd sine terms. Since the waveform possesses quarter-wave symmetry, it is necessary to integrate over only one quarter period of finding the Fourier coefficients.

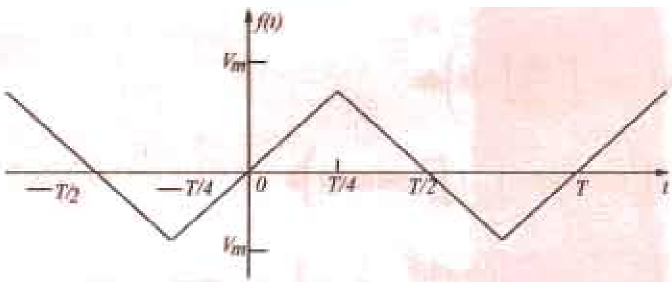


Fig. 21.23

$$f(t)=\sum_{\substack{n=1\\ odd}}^{\infty} b_n \sin n\omega_0 t$$

where $\quad b_n=\frac{8}{T}\int_0^{t/4} f(t)\sin n\omega_0 t$

The quarter-wave of the given waveform can be represented by equation of a straight line.

$\therefore$ Slope of the straight line is $V_m/(T/40) = 4Vm/T$.

Hence, using $Y = mx$, we have

$$f(t)=\left(\frac{4V_m}{T}\right)t0<t<T/4 \;\therefore\; b_n=\frac{8}{T}\int_0^{T/4}\left(\frac{4V_m}{T}\right)t.\sin n\omega_0 t=\frac{32V_m}{T^2}\int_0^{T/4} t.\sin n\omega_0 t dt$$

Using the theorem of integration by parts, we have

$$b_n=\frac{32V_m}{T^2}\left[t\int_0^{t/4}\sin n\omega_0 t\, dt-\int_0^{T/4}\left(1.\int_0^{T/4}\sin n\omega_0 t\, dt\right)dt\right]$$

$$= \frac{32V_m}{T^2}\left[\left(\frac{t}{4}\cdot\frac{-\cos n\omega_0 T/4}{n\omega_0}\right) + \left|\frac{\sin n\omega_0 t}{(n\omega_0)^2}\right|^{T/4} = \frac{32V_m}{T^2}\left[\left(\frac{T}{4}\frac{\cos n\omega_0 T/4}{n\omega_0}\right) + \left(\frac{\sin n\omega_0 T/4}{(n\omega_0)^2}\right)\right]\right.$$

Now, $\omega_0 = 2\pi/T$ or $\omega_0 T = 2\pi$ $\therefore n\omega_0 T/4 = n\pi/2$

$\therefore$ $\cos n\omega_0 T/4 = \cos n\pi/2 = 0$ when n is odd

$$\therefore \quad b_n = \frac{32V_m}{n^2\omega_0^2 T^2}\sin n\omega_0\frac{T}{4} = \frac{32V_m}{n^2(2\pi)^2}\sin\frac{n\pi}{2} = \frac{8V_m}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\therefore \quad b_n = \frac{8V_m}{n^2\pi^2}\ldots n = 1,5,9,13\ldots.b_n = \frac{-8V_m}{n^2\pi^2}\ldots.n = 3,7,11,15,\ldots$$

Substituting this value of b_n, the Fourier series for the given waveform becomes

$$f(t) - \frac{8V_m}{\pi^2}\left(\sin\omega_0 t - \frac{1}{3^2}3\omega_0 t + \frac{1}{5^2}\sin 5\omega_0 t - \frac{1}{7^2}\sin 7\omega_0 t + \ldots\right)$$

Example 21.14. *Determine the Fourier series of the triangular waveform shown in Fig. 21.24.*

Solution. Since the function has even symmetry, $b_n = 0$. Moreover, it also has half-wave symmerty, hence, $a_0 = 0$. The Fourier series can be written as $f(t) = \sum_{n=1}^{\infty} a_n \cos n\omega_0 t$ where

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\, dt.$$

5 1 2 1 3

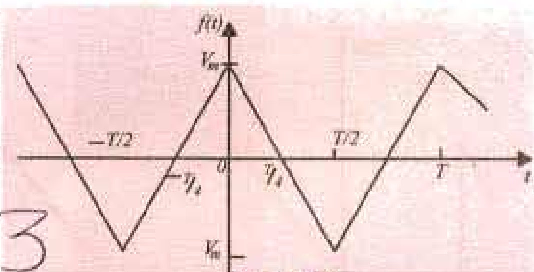


Fig. 21.24

The function is given by the relation $f(t) = \frac{-4V_m}{T}\left(t - \frac{T}{4}\right)$

It is so because for the interval $0 \le t \le T/2$, the slope of the line is $-4V_m \backslash T$.

$$\therefore \quad a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega_0 t\, dt = \frac{4}{T}\int_0^{T/4} f(t)\cos n\omega_0 t\, dt$$

$$= \frac{-16V_m}{T^2}\int_0^{T/2}\left(t - \frac{T}{4}\right)\cos n\omega_0 t\, dt = \frac{-16V_m}{T^2}\int_0^{T/2} t\cos n\omega_0 t\, dt + \frac{4V_m}{T}\int_0^{T/2}\cos n\omega_0 t\, dt$$

$$= \frac{-16V_m}{T^2}\left|\frac{1}{4n^2\omega_0^2}\cdot\cos n\omega_0 t + \frac{t}{n\omega_0}\sin n\omega_0 t\right|_0^{T/2} + \frac{4V}{T}\left|\frac{\sin n\omega_0 t}{n\omega_0}\right|_0^{T/2}$$

Substituting $\omega = 2\pi/T$, we get

$$a_n = \frac{-16V_m}{T^2}\left[\frac{T^2}{4\pi^2 n^2}(\cos n\pi - 1) + \frac{T^2}{4\pi n}(\sin n\pi)\right] + \frac{2V}{\pi n}\sin n\pi$$

Now, $\sin n\pi = 0$ for all values of n, $\cos n\pi = 1$ when n is even add -1 when n is odd.

$$\therefore \quad a_n = \frac{8V_m}{\pi^2 n^2}$$

$$\therefore \; f(t) = \sum_{\substack{n=0 \\ odd}}^{\infty} \frac{8V_m}{\pi^2 n^2} \cos n\omega_0 t = \frac{8V_m}{\pi^2} \sum_{\substack{n=1 \\ odd}}^{\infty} \frac{\cos n\omega_0 t}{n^2}$$

$$\frac{8V_m}{\pi^2}\left(\cos \omega_0 t + \frac{1}{9}\cos 3\omega_0 t + \frac{1}{25}.\cos 5\omega_0 t + \frac{1}{49}\cos 7\omega_0 t + ...\right)$$

Alternative Solution

We can deduce the Fourier series from Fig. of Ex. 21.11 by shifting the vertical axis by $\pi/2$ radians to the right. Replacing t by $(t + \pi/2)$ in the Fourier series of Ex. 21.11, we get

$$f(t) = \frac{8V_m}{\pi^2}\left[\sin \omega_0\left(t + \frac{\pi}{2}\right) - \frac{1}{3^2}\sin 3\omega_0\left(t + \frac{\pi}{2}\right) + \frac{1}{5^2}\sin 5\omega_0\left(t + \frac{\pi}{2}\right) - \frac{1}{7^2}\sin 7\omega_0\left(t + \frac{n}{2}\right) + ...\right]$$

$$= \frac{8V_m}{\pi^2}\left(\cos \omega_0 t + \frac{1}{9}\cos 3\omega_0 t + \frac{1}{25}\cos 5\omega_0 t + \frac{1}{49}\cos \omega_0 t + ...\right)$$

Example 21.15. *Obtain the Fourier series representation of the sawtooth waveform shown in Fig. 21.25 (a) and plot its spectrum.*

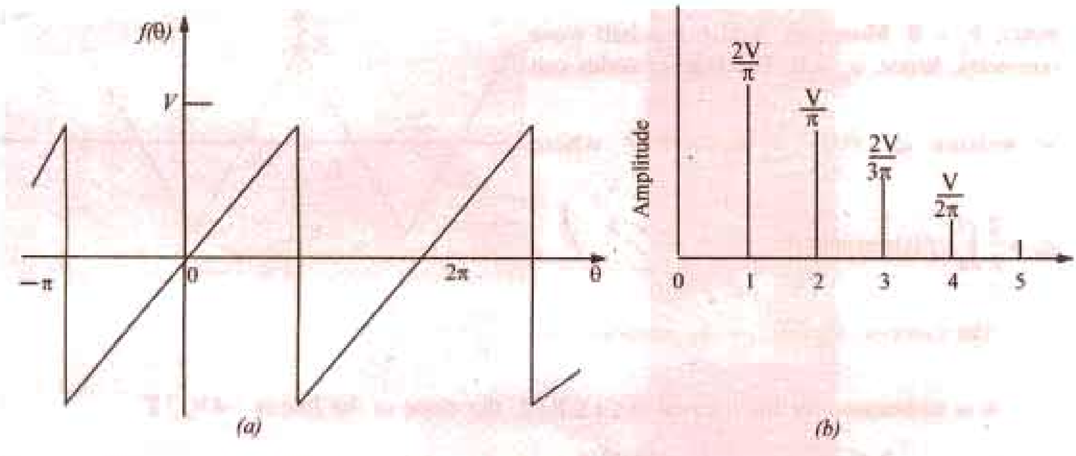


Fig. 21.25

Solution. By inspection, we know that the average value of the wave is zero over a cycle because the height of the curve below and above the X-axis is the same hence, $a_0 = 0$. Moreover, it has odd symmetry so that $a_n = 0$ *i.e.* there would be no cosine terms. The series will contain only sine terms.

$$\therefore \; f(\theta) = \sum_{n=1}^{\infty} b_n \sin n\theta \text{ where } b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta) \sin n\theta \, d\theta$$

The slope of curve is $m = V/\pi$

$\therefore$ we get, $f(\theta) = (V/\pi)\,\theta$.

If we are the limit of integration form $-\pi$ to $+\pi$ then

$$b_n \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{V}{\pi}\right)\theta \sin n\theta \, d\theta = \frac{V}{\pi^2}\left|\frac{1}{n^2}\sin n\theta - \frac{\theta}{n}\cos n\theta\right|_{-\pi}^{+\pi} = \frac{-2V}{n\pi}\cos n\pi$$

The above result has been obtained by making use of integration by parts as explained earlier. $\cos n\pi$ is positive when n is even and is negative when n is odd and thus the signs of the coefficients alternate. The required Fourier series is

$$f(\theta) = \frac{2V}{\pi}\left(\sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta - \frac{1}{4}\sin 4\theta + \ldots\right)$$

or
$$f(t) = \frac{2V}{\pi}\left(\sin\omega_0 t - \frac{1}{2}\sin 2\omega_0 t + \frac{1}{3}\sin 3\omega_0 t - \frac{1}{4}\sin 4\omega_0 t + \ldots\right)$$

As seen, the coefficients decrease as $1/n$ so that the series converges slowly as shown by the line spectrum of Fig. 21.25 (*b*).

The amplitudes of the fundamental of first harmonic, second harmonic, third harmonic and fourth harmonic are $(2/\pi), (2V/2\pi), (2V/3\pi)$ and $(2V/4\pi)$ respectively.

Tutorial Problem. 21.1

1. Determine the Fourier series for the triangular waveform shown in Fig. 21.26 (*a*)

(*Network Theory and Design, AMIETE June 1990*)

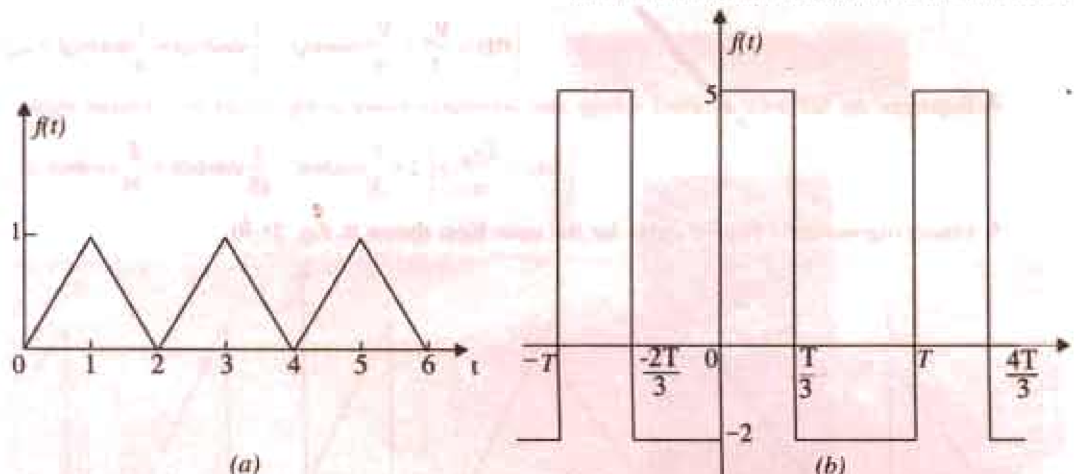


Fig. 21.26

$$\left[\frac{1}{2} - \frac{4}{\pi^2}\cos\omega_0 t + \frac{1}{3^2}\cos 3\omega_0 t + \frac{1}{5^2}\cos 5\omega_0 t + \ldots\right]$$

2. Find the values of the Fourier coefficients a_0, a_n and b_n for the function given in fig. 21.26 (*b*).

$$\left[a_0 = \frac{2}{3}; a_n = \frac{7}{n\pi}\sin\frac{2\pi n}{3}; b_n = \frac{7}{n\pi}\left(1 - \cos\frac{2\pi n}{3}\right)\right]$$

3. Determine the trigonometric series of the triangular waveform shown in Fig. 21.27. Sektch its line spectrum.

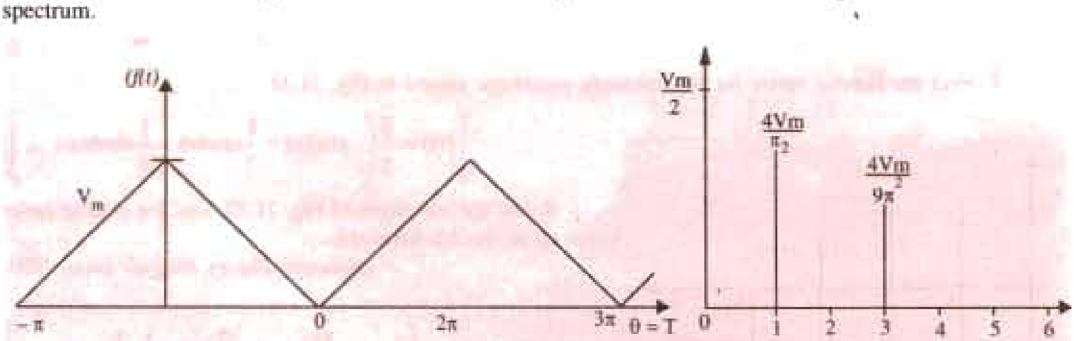


Fig. 21.27

$$\left[\frac{V_m}{2} + \frac{4V_m}{\pi^2}\left(\cos\omega_0 t + \frac{1}{3^2}\cos\omega_0 t + \frac{1}{5^2}\cos 5\omega_0 t + \ldots\right)\right]$$

4. Determine the Fourier series for the sawtooth waveform shown in Fig. 21.28.

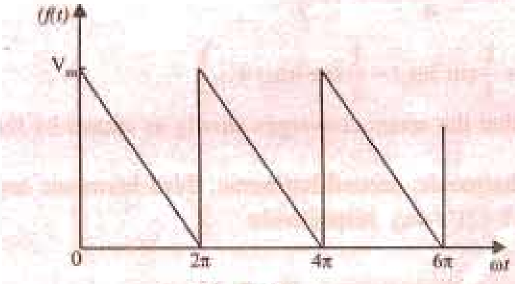


Fig. 21.28

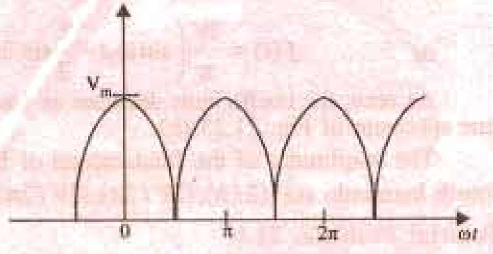


Fig. 21.29

$$\left[f(t) = \frac{V_m}{2} + \frac{V_m}{\pi}(\sin\omega_0 t + \frac{1}{2}\sin 2\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + ...\right]$$

5. Represent the full-wave rectified voltage sine waveform shown in Fig. 21.29 by a Fourier series.

$$\left[f(t) = \frac{2V_m}{\pi} + \left(1 + \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t + \frac{2}{35}\cos 6\omega t ...\right)\right]$$

6. Obtain trigonometric Fourier series for the wave form shown in Fig. 21.30.

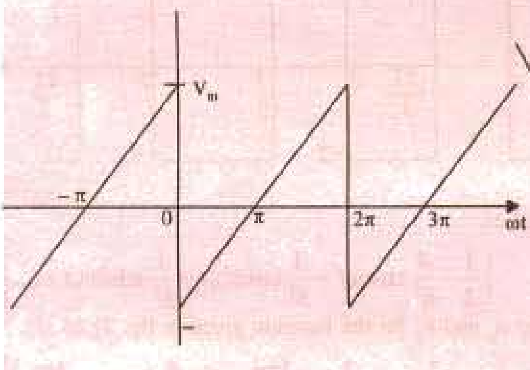


Fig. 21.30

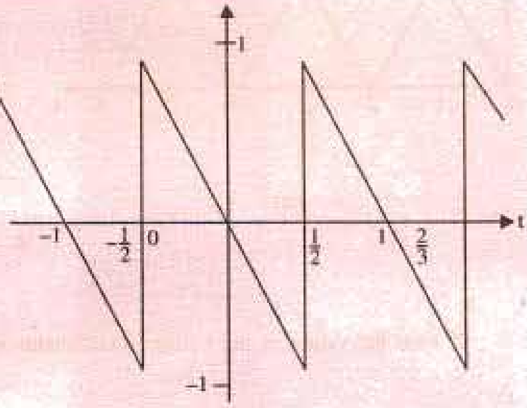


Fig. 21.31

$$\left[f(t) = -\frac{2V_m}{\pi}\left(\sin\omega_0 t + \frac{1}{2}\sin 2\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{4}\sin 4\omega_0 t ...\right)\right]$$

7. Find the Fourier series for the sawtooth waveform shown in Fig. 21.31.

$$\left[f(t) = \frac{2}{\pi}\left(-\sin 2\pi t + \frac{1}{2}\sin 4\pi t - \frac{1}{3}\sin 6\pi t + ...\right)\right]$$

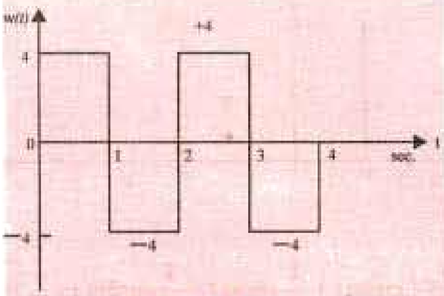


Fig. 21.32

8. For the waveform of Fig. 21.32, find the Fourier series terms up to the 5th harmonic.

(Network Theory Nagpur Univ. 1993)

$$\left[V(t) = \frac{16}{\pi}\left(\sin t + \frac{2}{3}\sin 3t\right) + \frac{1}{5}\sin 5t + ...\right]$$

9. Determine Fourier series of a repetitive triangular wave as shown in Fig. 21.33.
(*a*) What is the magnitude of d.c. component?
(*b*) What is the fundamental frequency?
(*c*) What is the magnitude of the fundamental?
(*d*) Obtain its frequency spectrurm. (*Network Theory Nagpur Univ.1993*)

[(*a*) 5 V (*b*) 1 Hz (*c*) 10 / π volt]

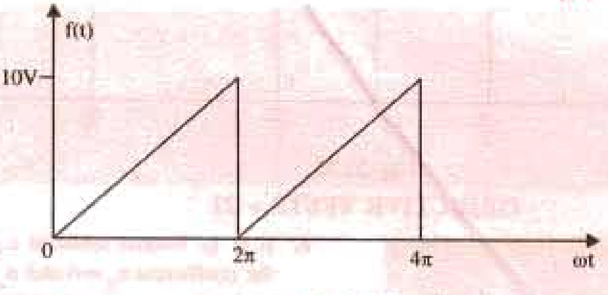


Fig. 21.33

10. Determine the Fourier series of voltage632*1 respones obtained at the o/p of a half wave rectifier shown in Fig. 21.34. Plot the discrete spectrum of the waveform. (*Elect. Network Analysis Nagpur Univ. 1993*)

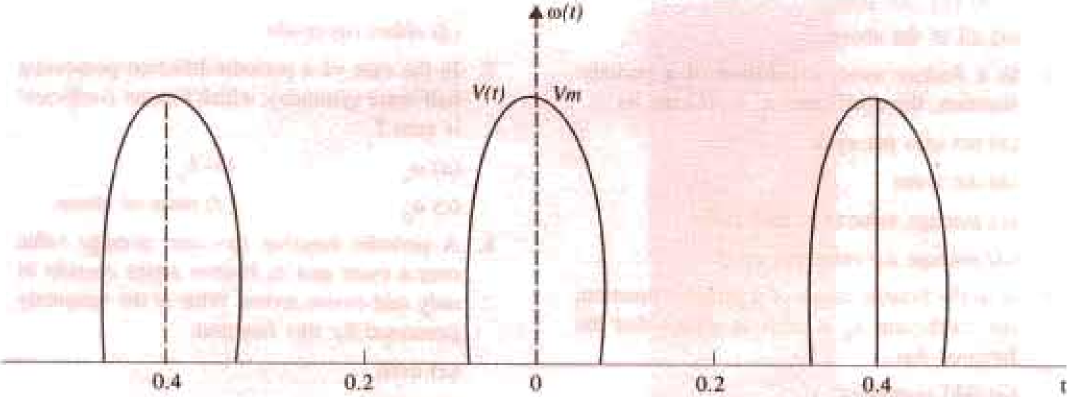


Fig. 21.34

$$\left[V(t) = \frac{V_m}{\pi} + \frac{V_m}{2}\cos\omega_0 t + \frac{2V_m}{3\pi}\cos 10\pi t - \frac{2V_m}{15}\cos 20\pi t + \frac{2V_m}{35\pi}\cos 30\pi t \ldots \right]$$

11. Determine the Fourier coefficients and plot amplitude and phase spectral. (*Network Analysis Nagpur Univ. 1993*)

$$\left[a_0 = 0, b_n = 0, a_1 = \frac{4V_m}{\pi}, a_2 = 0 \right.$$

$$\left. a_3 = \frac{-4V_m}{3\pi}, a_4 = 0, a_5 = \frac{4V_m}{\pi} \right]$$

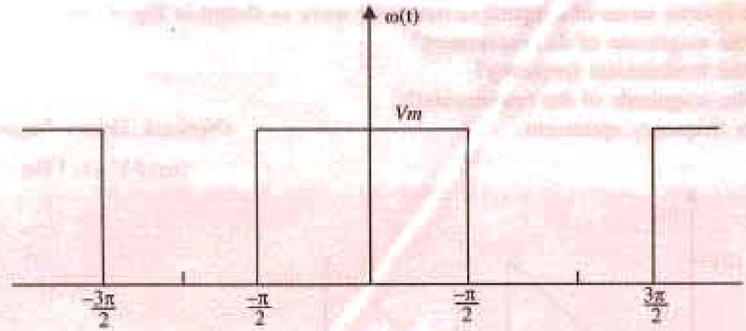


Fig. 21.35

OBJECTIVE TESTS – 21

1. A given function $f(t)$ can be represented by a Fourier series if it
   (*a*) is periodic
   (*b*) is signle valued
   (*c*) has a finite number of maxima and minima in any one period
   (*d*) all of the above.
2. In a Fourier series expanision of a periodic function, the coefficient a_0 represents its
   (*a*) net area per cycle
   (*b*) d.c value
   (*c*) average value over half cycle
   (*d*) average a.c value per cycle
3. If in the Fourier series of a periodic function, the coefficient a_0 is zero, it means that the function has
   (*a*) odd symmetry
   (*b*) even quarter-wave symmetry
   (*c*) odd quarter-wave symmetry
   (*d*) any of the above.
4. A periodic function $f(t)$ is said to possess odd quarter-wave symmetry if
   (*a*) $f(t) = f(-t)$
   (*b*) $f(-t) = -f(t)$
   (*c*) $f(t) = -f(t + T/2)$
   (*a*) both (*b*) and (*c*).
5. If the average value of a periodic function over one period is zero and it consists of only odd harmonics then it must be possessing _______ symmetry.
   (*a*) half-wave
   (*b*) even quarter-wave
   (*c*) odd quarter-wave
   (*d*) odd.
6. If in the Fourier series of a periodic function, the coefficient $a_0 = 0$ and $a_n = 0$, then it must be having _______ symmetry.
   (*a*) odd
   (*b*) odd quarter-wave
   (*c*) even
   (*d*) either (*a*) or (*b*).
7. In the case of a periodic function possessing half-wave symmetry, which Fourier coefficient is zero ?
   (*a*) a_n (*b*) b_n
   (*c*) a_0 (*d*) none of above.
8. A periodic function has zero average value over a cycle and its Fourier series consists of only odd cosine terms. What is the symmetry possessed by this function.
   (*a*) even
   (*b*) odd
   (*c*) even quarter-wave
   (*d*) odd quarter-wave
9. Which of the following periodic function possesses even symmetry ?
   (*a*) cos $3t$ (*b*) sin t
   (*c*) t . cos $50\,t$ (*d*) $(t + t^2 + t^5)$.
10. If the Fourier coefficient b_n of a periodic function is zero, then it must possess ______ symmetry.
    (*a*) even
    (*b*) even quarter-wave
    (*c*) odd
    (*d*) either (*a*) and (*b*).
11. A complex voltage waveform is geien by $V = 120 \sin \omega t + 36(3\omega t + \pi/2) + 12 \sin (5\omega t + \pi)$. It has a time period of T seconds.

The percentage fifth harmonic contents in the waveform is

(*a*) 12 (*b*) 10

(*c*) 36 (*d*) 5

12. In the waveform of *Q*. 11 above, the phase displacement of the third harmonic represents a time interval of ___ seconds.

(*a*) T/12 (*b*) T/3

(*c*) 3T (*d*) T/36

13. When the negative half-cycle of a complex waveform is reversed, it becomes identical to its positive half-cycle. This feature indicates that the complex waveform is composed of

(*a*) fundamental
(*b*) odd harmonics
(*c*) even harmonics
(*d*) both (*a*) and (*b*)
(*e*) both (*a*) and (*c*)

14. A periodic waveform possessing half-wave symmetry has no

(*a*) even harmonics
(*b*) odd harmonics
(*c*) sine terms
(*d*) cosine terms

15. The Fourier series of a wave form possessing even quarter-wave symmetry has only

(*a*) even harmonics
(*b*) odd cosine terms
(*c*) odd sine terms
(*d*) both (*b*) and (*c*).

16. The Fourier series of a waveform possessing odd quarter-wave symmetry contains only

(*a*) even harmonics
(*b*) odd consine terms
(*c*) odd consine terms
(*d*) none of above

ANSWERS

1. (*d*) **2.** (*b*) **3.** (*d*) **4.** (*d*) **5.** (*a*) **6.** (*d*) **7.** (*c*) **8.** (*c*) **9.** (*a*) **10.** (*d*)
11. (*b*) **12.** (*a*) **13.** (*d*) **14.** (*a*) **15.** (*b*) **16.** (*c*)

22 TRANSIENTS

22.1. Introduction

It is quite an easy job to calculate the steady current, which flows in a circuit, when it is connected to a d.c. generator or a battery. Similarly, the alternating current which flows in a circuit when connected to an alternator can also be calculate by the various method discussed in Chapters 13 and 14. These currents are known as *steady* currents because in such cases, it is assumed that (*i*) the circuit components are constant and (*ii*) the circuit has been connected to the generator long enough for any disturbance produced on initial switching, to resolve itself.

In general, transients disturbances are produced whenever

(*a*) an apparatus or circuit is *suddenly* connected to or disconnected from the supply,

(*b*) a circuit is shorted and

(*c*) there is a *sudden* change in the applied voltage from one finite value to another.

We will now discuss the transients produced whenever different circuits are suddenly switched on or off from the supply voltage. In each case, we will assume that the resultant current consists of two parts (*i*) a final steady-stage or normal current and (*ii*) a transient current superimposed on the steady-stage current.

It is essential to remember that the transient currents are not driven by any part of the applied voltage but are entirely associated with the changes in the stored energy in inductors and capacitors. Since there is no stored energy in resistors, *there are no transients in pure resistive circuits.*

22.2. Types of Transients

There are *single-energy* transients and *double-energy* transients. Single-energy transients are those in which only one form of energy, either electromagnetic or electrostatic is involved as in *R-L* and *R-C* circuits.

However, double-energy transients are those in which both electromagnetic or electrostatic is involved as in *R-L-C* circuits Transient disturbances may be further classified as follows :

(*a*) **Initiation Transients** : These are produced when a circuit, which is originally dead, is energised.

(*b*) **Subsidence Transients** : These are produced when an energised circuit is rapidly de-energised and reaches an eventual steady-stage of zero current or voltage, as in the case of short-circuiting an *R-L* or *R-C* circuit suddenly.

(*c*) **Transition Transients** : These are due to sudden but energtic changes from one steady state to another.

(*d*) **Complex Transients** : These are produced in a circuit which is simultaneously subjected to two transients due to two independent disturbances or when the disturbing force producing the transients is itself variable.

(*e*) **Relaxation Transients** : In these transients, the transition occurs cyclically towards states, which when reached, become unstable themselves.

A distinction may also be made between free and forced transients which are produced due to the applied voltage being itself transient.

22.3. Important Differential Equations

Some of the important differential equations, used in the treatment of single and double energy transients, are given below. We will consider both first-order and second-order differential equations.

1. First Order Equations

(*i*) Let $\dfrac{dy}{dx}+ay=0$ where a is a constant.

Its solution is $y = k\,e^{-ax}$ where k is the constant of integration whose value can be found from the boundary conditions *i.e.* conditions prevalent at the instant when the voltage to a circuit is applied or excluded.

(*ii*) If $\dfrac{dy}{dx}+ay=b$ where a and b are constants, then solution is $y=\dfrac{b}{a}+k\,e^{-ax}$

The value of k can again be found from boundary conditions.

(*iii*) If $\dfrac{dy}{dx}+Ay=B$

where A and B are not constants but are functions of x, then the solution is given by

$$y=e^{-\int A\cdot dx}\int e^{\int A\cdot dx}B\,dx+ke^{-\int A\cdot dx}$$

If $A = a =$ constant, then the above equation simplifies to

$$y=e^{-ax}\int e^{ax}\,B\,dx+ke^{-ax}$$

2. Second Order Equations

(*i*) Suppose $\dfrac{d^2y}{dx^2}+a\dfrac{dy}{dx}+by=0$ where a and b are constants, then the solution is

$$y=k_1e^{\lambda 1x}+k_2e^{\lambda 2x}$$

where λ_1 and λ_2 are constants of integration and whose values are,

$$\lambda_1=-\frac{a}{2}-\sqrt{\left(\frac{a^2}{4}-b\right)}\text{ and }\lambda_2=-\frac{a}{2}+\sqrt{\left(\frac{a^2}{4}-b\right)}$$

(*a*) If $a^2/4 > b$, the roots are real and the above solution can be applied without any difficulty.

(*b*) If $a^2/4 < b$, the radicals contain a negative quantity. In that-case, the solution is given by

$$y=e^{-\frac{1}{2}ax}(k_3\sin\lambda_0x+k_4\cos\lambda_0x)$$

where k_3 and k_4 are the new constants of integration and

$$\lambda_0=\sqrt{b-\frac{a^2}{4}}$$

(*c*) If $a^2/4 = b$, then both roots are equal and each is $= -a/2$.

Hence, in this case, the solution becomes $y=k_5\,e^{\lambda t}+k_6t.e^{\lambda t}$

(*ii*) Let $\dfrac{d^2y}{dx^2}+a\dfrac{dy}{dx}+by=c$

where a, b and c are constant. In this case also, the solution will again depend on the root as discussed above.

$$y=k_1e^{\lambda}1^x+k_2e\lambda_2x+c/b$$

(*iii*) (*a*) Let the differential equation be given by

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=u$$

where a, b are c are constants but u is a particular function of the variable x. The solution of such an equation consists of a *particular integral* and a complementary function.

(*b*) Let y be a sinusoidal function of x, then

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=c\sin\omega t$$

In this case, particular integral is

$$y_1=\frac{-c}{\sqrt{a^2\omega^2+(\omega^2-b)^2}}\cos\left[\omega x-\tan^{-1}\left(\frac{\omega^2-b}{a\omega}\right)\right]$$

The complementary function is given by

$$y_2=k_1e^{\lambda_1 x}+k_2e^{\lambda_2 x}\ \text{ where } \lambda_1=-\frac{a}{2}-\sqrt{\left(\frac{a^2}{4}-b\right)}\ \text{ and } \lambda_2=-\frac{a}{2}+\sqrt{\left(\frac{a^2}{4}-b\right)}$$

The complete solution for the above equation is $y = y_1 + y_2$

Further treatment is the same as for case 2 (*i*) above.

22.4. Transients in R-L Circuits (D.C.)

If i, I_s and i_t be the resultant current, find steady-state current and transient current respectively in *R–L* circuit of Fig. 22.1 (*a*), then by superimposition, the equation for the resultant current, for the duration of initiation transient, is

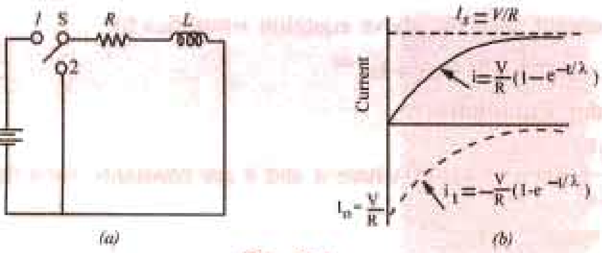


Fig. 22.1

$$i = i_s + i_t \qquad \text{... (i)}$$

Since the applied voltage V drives the steady-stage current, hence

$$I_s = V/R$$

Since the transient current i_t is not associated with any voltage,

$$\therefore \qquad i_tR+L\frac{d_{it}}{dt}=0 \qquad \text{... (ii)}$$

or
$$\frac{di_t}{it}=-\frac{R}{L}dt \qquad \text{... (iii)}$$

or
$$\int\frac{di_t}{i_t}=-\frac{R}{L}\int \text{dt} \qquad \therefore \qquad \text{logh } i_t=-\frac{R}{L}t+K^* \qquad \text{... (iv)}$$

where K is the constant of integration whose value may be found from the initial conditions.

Now when $t = 0$, $i_t = I_0$ (say). Then from Eq. (*iv*) above we get, logh $I_0 = 0 + K$

Putting this value of K in Eq. (*iv*), we have

$$\text{logh}\, i_t-\text{logh } I_0=\frac{R}{L}t \ \text{ or } \ \text{logh } i_t/I_0=-\frac{R}{L}t=\frac{-t}{\lambda} \qquad \therefore \ i_t=I_0\,e^{-t/\lambda} \qquad \text{... (v)}$$

where $\lambda = L/R$ is called the *time-constant* of the circuit. Its reciprocal *R/L* is called the *damping coefficient* of the circuit. The current decreases exponentially as shown in Fig. 22.1 (*b*). From Eq. (*i*) and (*v*), we have

$$i=I_s+I_0e^{-t/\lambda} \qquad \text{... (vi)}$$

If the time is reckoned when the voltage V is applied, so that when $t = 0$, $i = 0$, then from equation (*vi*), we get

$$0=I_s+i_0e^{-0}=I_s+I_0 \quad \therefore \quad I_0=-I_s=-\frac{V}{R}$$

In that case, Eq. (*vi*) becomes

$$i = \frac{V}{R} - \frac{V}{R}e^{-t/\lambda} \qquad \text{... (vii)}$$

$$= \frac{V}{R}(1 - e^{-t/\lambda}) \qquad \text{... (viii)}$$

Curves for I_s and i_t have been plotted in Fig. 22.1 (*b*). The curve for resultant current has been obtained by the superposition of steady-state current I_s (= V/R) and transient current

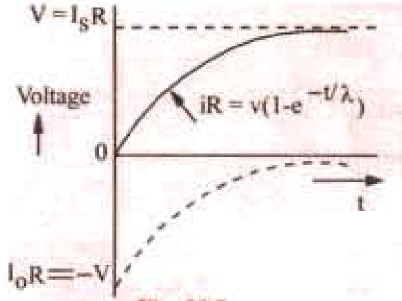


Fig. 22.2

$$i_t = \frac{V}{R}e^{-t/\lambda}$$

Theoretically, the transient current i_t takes infinite time to die off but, in practice, it disappears in a very short time.

The values of resultant, steady-state and transient voltages across the resistor can be found by multiplying Eq. (*vii*) by R and are shown in Fig. 22.2. The e.m.f. of self-induction–$L\,di_t/dt$ is only transient in nature and equals $i_t R$ as seen from Eq. (*ii*) above.

22.5. Short Circuit Current

After some time, the transient current would disappear and the only current flowing in the circuit would be the steady-state current $I_s = V/R$. Let the R-L circuit be closed upon itself *i.e.* be short-circuited by shifting the switch [Fig. 22.1 (*a*)] to position 2. Since the voltage V has been excluded from the circuit, the trapped current I_s will immediately cease to be a steady-state current, but on the other hand, will become the initial value I_0 of a new subsidence transient current i_t. If time is measured at the instant of short-circuit, so that when $t = 0$, the current is $I_s = VR$, then Eq. (v) becomes

$$i_t = \frac{V}{R}e^{-t/\lambda} \qquad \text{... (ix)}$$

This equation has been plotted in Fig. 22.3. The only voltage acting in the circuit is that due to self-induction *i.e.* $-L\,di_t/dt$ which equals $i_t R$.

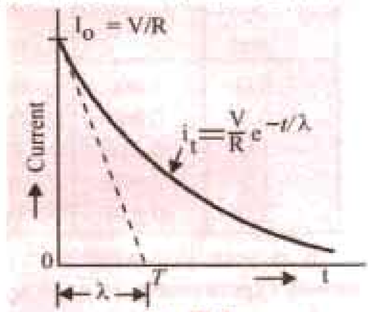


Fig. 22.3

22.6. Time Constant

The time constant of a circuit is defined as the time it would take for the transient current to decrease to zero, if the decrease were linear instead of being exponential.

In other words, it is the time during which the transient current would have decreased to zero, had it maintained its initial rate of decrease.

The initial rate of decrease can be found by differentiating Eq. (*vi*) and putting $t = 0$

$$\therefore \quad \frac{di_t}{dt} = \frac{I_0}{\lambda}e^{-t/\lambda} \qquad \therefore \quad \left(\frac{di_t}{dt}\right)_{t=0} = -\frac{I_0}{\lambda}$$

If the rate of decrease were constant throughout and equal to $-I_0/\lambda$, then the straight line showing the relation between i_t and t would be given by

$$i_t = -\frac{I_0}{\lambda}t$$

The time-period would be equal to the sub-tangent OT drawn to the exponential curve of Fig. 22.3 at $i_t = I_0$ *i.e.* at the beginning of the curve.

If we put $t = \lambda$ in Eq. (v), then $i_t = I_0\,e^{-1} = I_0/e = I_0/2.718 = 0.37\,I_0$

Hence, time period of a circuit is the time during which the transient current decreases to 0.37 of its initial value.

Example 22.1. *A coil having a resistance of 30 ω and an inductance of 0.09 H is connected across a battery of 20 V. Plot the current and its two components. Assume that t = 0 when the circuit is completed.* **(Electromechanic Allahabad Univ. 1992)**

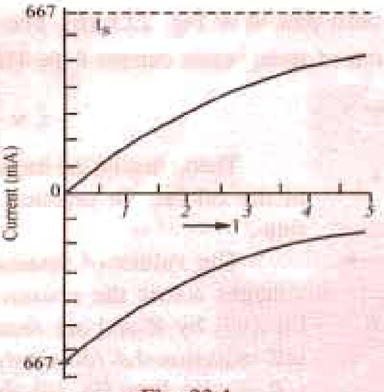


Fig. 22.4

Solution. The two components of the circuit current are (*i*) steady current $I_s = V/R = 20/30 = 2/3$ A = 667 mA and (*ii*) Transient current $i_t = -(V/R)e^{-t/\lambda}$.

Total current is $i = I_s + i_t$. Let us find the value of transient current after various time intervals. In the present case, $\lambda = L/R = 0.09/30 = 0.003$ second = 3 millisecond.

The values of i_t and i at various times are tabulated below. Value of $i = I_s + i_t$

t (ms)	$e^{-t/\lambda}$	i_t (mA)	i	t (ms)	$e^{-t/\lambda}$	i_t (mA)	i
0.0	1.000	−667	0	2.5	0.435	−290	377
0.5	0.847	−565	102	3.0	0.368	−244	423
1.0	0.716	−477	190	3.5	0.311	−208	459
1.5	0.606	−405	262	4.0	0.264	−176	491
2.0	0.514	-344	323	4.5	0.223	−148	519

It is seen that whereas transient current decreases exponentially, total circuit current increases exponentially as expected (Fig. 22.4).

Example 22.2. *A circuit of resistance 10 Ω and inductance 0.1 H in series has a direct voltage of 200 V suddenly applied to it. Find the voltage drop across the inductance at the instant of switching on and at 0.01 second. Find also the flux-linkages at these instants.*

(Basic Electricity, Bombay Univ., 1985)

Solution. (i) Switching instant

At the instant of switching on, $i = 0$, so that iR = 0 hence all applied voltage must drop across the inductance only. Therefore, voltage drop across inductance = **200 V**.

Since at this instant $i = 0$, there are no flux-linkages of the coil.

(ii) When t = 0.01 second

As time passes, current grows so that the applied voltage is partly dropped across the resistance and partly across the coil. Let us first find iR drop for which purpose, we need the value of i at $t = 0.01$ second.

Now, time period of the circuit is $\lambda = L/R = 0.1/10 = 0.01$ second. Since the given time happens to be equal to time constant,

$\therefore \quad i = (200/10) \times 0.632 = 12.64$ A; $iR = 152.64 \times 10 =$ **126.4 V**

Drop across inductance = $\sqrt{200^2 - 126.4^2}$ = **155 V**

Now, $\qquad L = N\,\Phi / i \qquad$ or $\qquad N\Phi = Li$

$\therefore$ Flux-linkages $Li = 0.1 \times 12.64 =$ **1.264 Wb-turns.**

Example 22.3. *A coil of 10 H inductance and 5 Ω resistance is connected in parallel with a 20 Ω resistor across a 100-V d.c. supply which is suddenly disconnected. Find*

(a) the initial rate of change of current after switching.

(b) the voltage across the 20 Ω resistor initially and after 0.3 s.

(c) the voltage across the switch contacts at the instant of separation and

(d) the rate at which the coil is losing stored energy 0.3 second after switching.

Solution. (*a*) Since the steady-state current is zero, $i = I_0\, e^{-t/\lambda}$

Now, when $t = 0$, current is = 100/5 = 20 A. It means the current flowing through the coil immediately before opening the switch is 20 A.

$$\therefore \qquad I_0 = 20 \text{ A}$$

Hence, the above equation becomes $i = 20e^{-t/\lambda}$

Now $\qquad \lambda = L/R = 10/25 = 1/2.5 \qquad \therefore \quad i = 20e^{-2.5t}$

$$\left(\frac{di}{dt}\right)_{t=0} = (-20 \times 2.5e^{-2.5t})_{t=0} = -\mathbf{50\ A/s}$$

The negative sign merely shows that the current is decreasing.

(*b*) After the supply has been disconnected, the current through the 20-Ω resistor is i since it is in series with the coil.

Initial p.d. across the 20 Ω resistor = (current at $t = 0$) × 20 = 20 + 20 = **400 V**

Current through the resistor after 0.3 second = $20e^{-2.5 \times 0.3} = 9.45$ A

$\therefore$ Voltage across the resistor after 0.3 second

= (current at $t = 0.3$ second) × 20 = 9.45 × 20 = **189 V**

(*c*) The e.m.f. induced in the coil at break tends to maintain the current through it in the original direction. Hence, the direction of the current through 20 Ω resistor is upwards so that the p.d. across the switch contacts will be the *sum* of supply voltage and the voltage across 20 Ω resistor.

$\therefore$ Initial voltage across switch contacts = 400 + 100 = **500 V**

(*d*) The rate of loss of energy = power = induced e.m.f. in coil × current (after 0.3 s)

$$= L\left(\frac{di}{dt}\right)_{t=0.3} \times \text{ (after 0.3 second)}$$

Now, after 0.3 second, $i = 9.45$ A

Value of di/dt after 0.3 second = $-20 \times 2.5 \times e^{-0.75} = -23.6$ A/second

$\therefore$ Rate of loss of energy = $-10 \times 23.6 \times 9.45 = -2{,}230$ joule/second

22.7. Transients in R-L Circuits (A.C.)

Let a voltage given by $v = V_m \sin(\omega t + \Psi)$ be *suddenly* applied across an *R-L* circuit [Fig. 22.5 (*a*)] at a time when $t = 0$. It means that the voltage is applied when it is passing through the value $V_m \sin \Psi$. Since the contact may be closed at any point of the cycle, angle Ψ may have any value lying between zero and 2π radians. The resultant current, as before, is given by

$$i = i_s + i_t$$

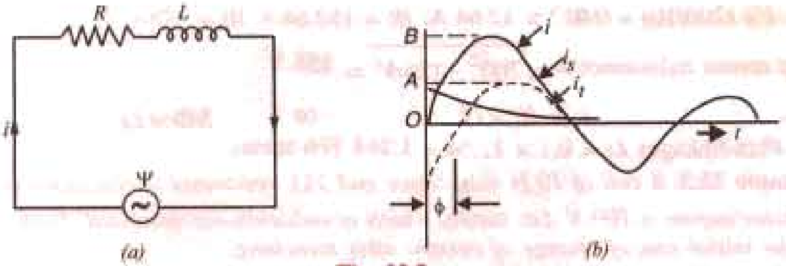


Fig. 22.5

The value of steady-state current is found by the normal circuit theory. The peak steady-state current is given by

$$I_m = \frac{V_m}{\sqrt{R^2 + X_L^{\,2}}} = \frac{V_m}{Z}$$

where $\sqrt{R^2 + X_L^{\,2}}$ is the impedance of the circuit. This current lags behind the applied voltage by an angle ϕ such that $\tan\ \phi = X_L/R =$ or $\phi = \tan^{-1}(X_L/R)$

Hence, the equation for the instantaneous value of the steady-state current becomes

$$i_s = I_m \sin(\omega t + \Psi - \phi)$$

As before, the transient current is given by

$$i_t = I_0\, e^{-t/\lambda} \qquad \therefore \qquad i = I_m \sin(\omega t + \Psi - \phi) + I_0\, e^{-t/\lambda} \qquad \text{... (i)}$$

Now, when $t = 0$, $i = 0$, hence putting these values in Eq. (*i*) above, we get

$$0 = I_m \sin(\Psi - \phi) + I_0 \qquad \therefore \qquad I_0 = -I_m \sin(\Psi - \phi)$$

Hence Eq. (*i*) can be written as

$$i = I_m \sin(\omega t + \Psi - \phi) - I_m \sin(\Psi - \phi) e^{-t/\lambda} \qquad \text{... (ii)}$$

From the above, it is seen that the value of I_0 and hence the size of the transient current depends on angle Ψ *i.e.* it depends on the instant in the cycle at which the circuit is closed. We will consider the following three cases :

Case 1

When $t = 0$, let the voltage pass through its zero value and become positive *i.e.* let $\Psi = 0$. In that case, putting this value of Ψ in Eq. (*ii*), we get

$$i = I_m \sin(\omega t - \phi) - I_m \sin(-\phi) e^{-t/\lambda} = I_m[\sin(\omega t - \phi) + \sin\phi e^{-t/\lambda}]$$

This is shown in Fig. 22.5 (*b*). It is seen that maximum instantaneous peak current *OB* is larger than the normal peak current *OA*.

Case 2

Let $t = 0$ when voltage is passing through its value $V_m \sin \mid\phi$ so that $\Psi = \phi$ or $\Psi - \phi = 0$

In that, $I_0 = 0$, there is no transient current at the time of switching on (*i.e.* $i_t = 0$). It corresponds to the contacts closing at the instant when the steady state current itseelf is zero.

Case 3

When $t = 0$, let the voltage be passing through

$$V_m \sin\left(\phi \pm \frac{\pi}{2}\right) \text{ i.e. } \Psi = \phi \pm \frac{\pi}{2} \text{ and } \Psi - \phi = \pm\pi/2$$

In this case, the transient [as found from Eq. (*ii*)] would be given by

$$i_t = -I_m \sin\left(\pm\frac{\pi}{2}\right).\, e^{-t/\lambda} = \mp I_m e^{-t/\lambda}$$

Under these conditions, the transient would have its maximum possible initial value.

Example 22.4. *A 1.0 H choke has a resistance of 50 Ω. This choke is supplied with an a.c. voltage given by e = 141 sin 314 t. Find the expression for the transient component of the current flowing through the choke after the voltage is suddently switched on.*

(Principles of Elect. Engg-II, Jadavpur Univ. 1986)

Solution. The equation of the transient component of the current is (Art. 22.7 Case 1)

$$i_t = I_m \sin\phi e^{-t/\lambda}$$

Here, $\lambda = L/R = 1/50 = 0.02$ second ; $\mathbf{Z} = 50 + j\,314 = 318\ \angle 80.95°$

$I_m = V_m/Z = 141/318 = 0.443$ A; $\sin 80.95° = 0.9875$

$\therefore$ $i_t = 0.443 \times 0.9875\ e^{-t/0.02} = 0.4376^{-t/0.02}$

Example 22.2. *A 50-Hz sinusoidal voltage of maximum value of 400 V is applied to a series circuit of resistance 10 Ω and inductance 0.1 H. Find tan expression for the value of the current at any instant after the voltage is applied, assuming that voltage is zero at the instant of application. Calculate its value 0.02 second after switching on* **(Electric Circuit, Punjab Univ. 1990)**

Solution. In such cases, as seen from Art. 22.7 (Case 1), the current consists of a steady-state component and a transient component. The equation of the resultant current is

$$i = \underbrace{I_m \sin(\omega t - \phi)}_{\text{steady-state current}} + \underbrace{I_m \sin\phi\, e^{t/\lambda}}_{\text{transient current}}$$

where $I_m = V_m / Z$; $\phi = \tan^{-1}(X_L / R)$; $\lambda = L/R$ second

$R = 10\Omega; X_L = 314 \times 0.1 = 31.4\Omega; \mathbf{Z} = 10 + j31.4 = 33\angle 72.3°$

$I_m = 400/33 = 12.1$ A ; $\phi = 72.3° = 1.26$ rad.

$\sin\phi = \sin 72.3° = 0.9527$; $\lambda = 0.1/10 = 1/100$ second

$i = 121 \quad \{\sin(314\,t - 1.262) + 0.9527\ e^{100t}\}$

Substituting $t = 0.02$ second, we get

$i = 12.1\ (\sin(314 \times 0.02 - 1.262) + 0.9527\ e^{-2})$

$= 12.1\ (\sin 5.02 + 0.9527\ e^{-2}) = 12.1\ (\sin 288° + 0.9527\ e^{-2})$

$= 12.1\ (-\sin 72° + 0.9527 \times 0.1353) = 12.1\ (-0.9511 + 0.1289) =$ **−9.95 A**

Example 22.6. *An alternating voltage $v = 400 \sin(314\,t + \Psi)$ is suddently applied across a coil of resistance 0.2 Ω and inductance 6.36 mH. Determine the first peak value of the resultant current when the transient current has maximum value.*

Solution. Obviously, $\omega = 314$ rad/s

$X_L = \omega L = 314 \times 6.36 \times 10^{-3} = 2\ \Omega$

Coil impedance $\mathbf{Z} = 0.2 + j2 - 2\ \angle 84.3°$

Max. value of steady-state current = 400/2 = 200 A

As seen from Art. 22.7, the maximum value of transient current will occur when

$\Psi = \phi \pm \pi/2$ where $\phi = 84.3°$ *i.e.* the phase angle of the current w.r.t. voltage

$\therefore\ \Psi = 84.3° - 90° = -5.7°$

$\therefore$ resultant current, $i = 400 \sin(314\,t - 90°) + I_0\, e^{-31.4t}$

Now, at $t = 0$, $i = 0$ $\quad\therefore\quad 0 = 400 \sin(-90°) + I_0 \quad \therefore I_0 = 400$ A

Hence, the above equation becomes

$$i = 400 \sin(\omega t - 90°) + 400\ e^{-31.4t}$$

The procedure for determining an exact solution for the first peak of the resultant current is first to differentiate the above expression, next to equate the result to zero and then to solve the resulting expression graphically for t. However, sufficiently accurate result can be obtained by determining the instant at which steady-state current reaches its first positive peak value and then to add to it the value of the transient current at this instant. The first peak value of steady-state current occurs when

$(314\,t - 90°) = \pi/2$ rad ; *i.e.* when $t = \pi/314 = 0.01$ second

At this time, $i_t = 400\ e^{-0.314} = 292$ A

$\therefore$ resultant current i at this time = 200 + 292 = **492 A**

22.8. Transients in R-C Weeries Circuits (D.C.)

When a d.c. voltage V is *suddenly* applied to an R-C series circuit (Fig. 22.6), the voltage v_c across the capacitor rises from zero value to the steady-state value V. If v_c is the voltage across capacitor, v_{tc} the transient voltage, then

$$v_c = V + v_{ct} \qquad \ldots (i)$$

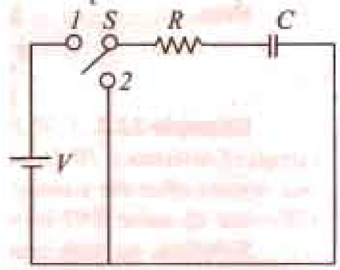


Fig. 22.6

The charging current is maximum at the beginning but then is reduced to zero so that there is no steady-state current but a transient one.

Since the transient current is not associated with any applied voltage, hence

$$i_t R + v_{ct} = 0 \qquad \ldots (ii)$$

Now, capacitor voltage $v_{ct} = q_t/C$

Hence, Eq. (*ii*) becomes

$$i_t R + \frac{q_t}{C} = 0$$

or $$R \cdot \frac{di_t}{dt} + \frac{1}{C} \cdot \frac{dq_t}{dt} = 0 \quad \text{or} \quad \frac{di_t}{dt} = -\frac{1}{CR}\frac{dq_t}{dt} = -\frac{1}{CR} i_t \quad (\because dq_t/dt = i_t)$$

$$\therefore \frac{di_t}{i_t} = -\frac{d_t}{CR}; \qquad \text{As before } i_t = I_0 e^{-t/CR} = I_0 e^{-t/\lambda}$$

where $CR = \lambda$ = time constant. The reciprocal I/CR is known as damping coefficient.

(*i*) Charging Current

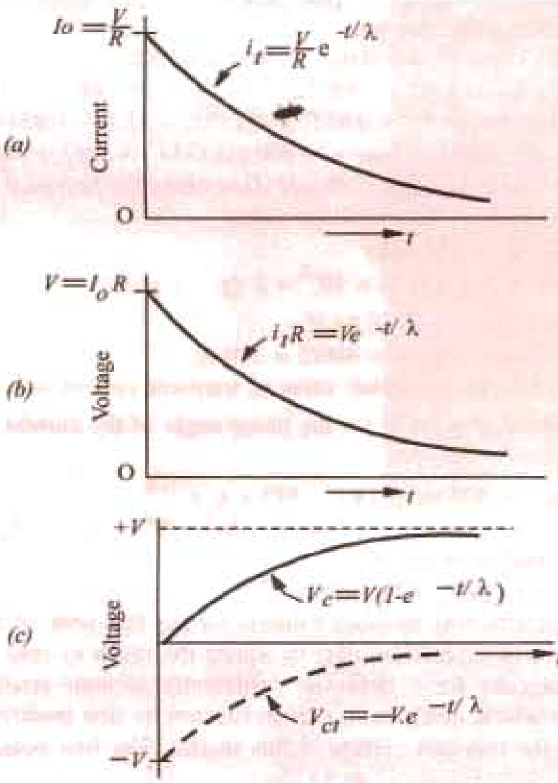


Fig. 22.7

When $t = 0$, transient current $i_t = I_0$, so that from Eq. (*ii*) $v_{ct} = -I_0 R$. Moreover, when $t = 0$, $v_c = 0$, hence from Eq. (*i*), $v_{ct} = -V$

Combining these results, we get

$$I_0 = V/R$$

$$\therefore \qquad i_t = I_0 e^{-t/\lambda} = \frac{V}{R} e^{-t/\lambda}$$

This is plotted in Fig. 22.7 (*a*)

The transient voltage across the resistor R is given by

$$i_t R = \frac{V}{R} e^{-t/\lambda} \times R$$

$$= Ve^{-t/\lambda} \text{ ... Fig. 19.7 } (b)$$

From Eq. (*ii*) the value of transient voltage across the capacitor is $v_a = -i_t R$

Hence, Eq. (*i*) becomes

$$v_c = V - i_t R = V - Ve^{-t/\lambda}$$

or $$v_c = V(1 - e^{-t/\lambda}) \qquad \text{... } (iii)$$

The voltage across the capacitor v_c which is the sum of the transient voltage v_{ct} and steady-state V has been plotted in Fig. 22.7 (*c*).

The charge across the capacitor is given by

$$q = v_c = CV(1 - e^{-t/\lambda}) \text{ or } q = Q(1 - e^{-t/\lambda}) \quad (\therefore Q = CV)$$

(ii) Discharge Current

When the capacitor has become fully charged so that charging current has ceased, then the R-C circuit is short-circuited by shifting the switch S from position 1 to position 2 (Fig. 22.6). On doing so, a transient discharge current will start flowing immediately. If time is reckoned from the instant of short-circuit, then when $t = 0$, $i_t = I_0$, hence from Eq. (*ii*) above $v_{ct} = -I_0 R$. Moreover, when $t = 0$, $v_c = V$. However, since there is no steady-state voltage across the capacitor, from Eq. (*i*), we get $v_c = v_{ct}$. Combining these results, we get

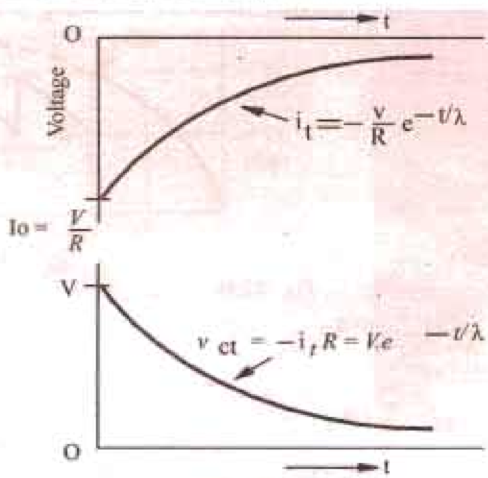


Fig. 22.8

$$I_0 = -V/R$$

$$\therefore \qquad i_t = -\frac{V}{R} e^{-t/\lambda}$$

It is plotted in Fig. 22.8 (*a*). The negative sign shows that discharge current flows in a direction opposite to that in which the chargining current flows. That is why the curve has been drawn below the X-axis. It may be noted that the only voltage in the circuit is v_{ct} which equals $-i_t R$.

Example 22.7. *In a simple saw-tooth generator circuit with the thyrdtron switches on at 150 V and switches off at 10 V. If this circuit is supplied with 250 V d.c. source; find the time period of saw-tooth wave. The resistance and capacitance have the values of 10 kΩ and 1 μF respectively.*

(Principles of Elect. Engg-II, Jadavpur Univ. 1987)

Solution. With reference to Fig. 22.9, let

V = applied voltage

v_{c1} = switching-off voltage of the thyratron = 10 V

v_{c2} = switching-on voltage of the thyratron = 150 V

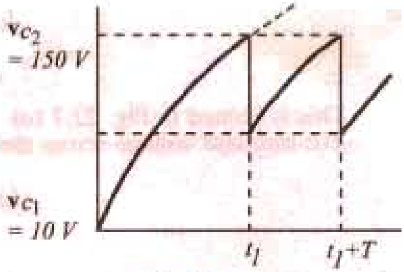


Fig. 22.9

Now, $\quad v = V(1-e^{-t/\lambda}$

$\therefore \quad v_{c1} = V(1-e^{-t/\lambda}) \qquad \text{... (i)}$

$\qquad V(1-e^{(-t_1+T)}/\lambda \qquad \text{... (ii)}$

where T is the time-period of the saw-tooth wave. From Eq. (*i*) and (*ii*), we get

$$T = \lambda \log_e (V - v_{c1}) / (V - v_{c2})$$

Now $\quad \lambda = CR = 10^4 \times 10^{-6} = 10^{-2}$ second

$$V - v_{c1} = 250 - 10 = 240\text{ V}; V - v_{c2} = 250 - 150 = 100\text{ V}$$

$$\therefore \quad \frac{V - v_{c1}}{V - v_{c2}} = \frac{240}{100} = 2.4 \quad \therefore \quad T = 10^{-2} \log_e{}^{2.4} = \mathbf{0.00875 \text{ second}}$$

Example 22.8. *A simple neon-tube time base for a cathode-ray oscillorgraphy employs a 300 kΩ and a 0.016 μF capacitor. The striking and extinction voltages of the neon-tube are 170 V and 140 V respectively. Calculate the frequency of the time base if the supply voltage is 200 V.*

Solution. The voltage across the capactitor increases according to the equation

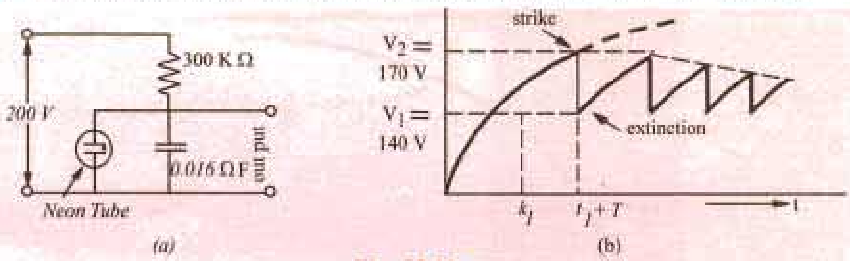


Fig. 22.10

$$v_c = V(1-e^{-t/CR})$$

It is shown in Fig. 22.10 (*b*)

$$\therefore \quad v_{c1} = V(1-e^{-t_1/\lambda}) \qquad \text{... (i)}$$

$$v_{c2} = V(1-e^{(-t_1+T)/\lambda}) \qquad \text{... (ii)}$$

From eq. (*i*) and (*ii*), we get

$$T = \lambda \text{logh} (V - v_{c1})/(V - v_{c2})$$

Now $\quad \lambda = CR = 0.016 \times 10^{-6} \times 300 \times 10^3 = 4.8 \times 10^{-3}$ second

$$V - v_{c1} = 200 - 140 = 60 \text{ V and } V - v_{c2} = 200 - 170 = 30 \text{ V}$$

$$\therefore \quad T = 4.8 \times 10^{-3} \log 60/30 = 1/300 \text{ second}$$

$\therefore \quad$ Frequency of time base = $1/T$ = **300 Hz**

22.9. Transients in R-C Series Circuits (A.C.)

In this case, the resultant currents can be determined in the same way as for an R-L circuit (Art. 22.7). It is given by

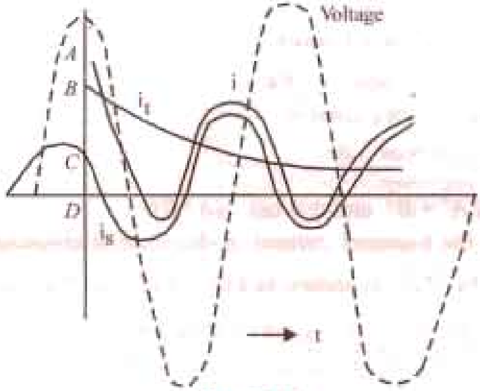


Fig. 22.11

$$i = i_s + i_t = I_m \sin(\omega t + \psi + \phi) + I_0 e^{-t/\lambda} \qquad \text{where} \quad I_m = V_m / \sqrt{R^2 + X_C^{\,2}}$$

$$\text{and} \qquad v = V_m \sin(\omega t + \psi)$$

The value of I_0 as found from initial known conditions ($t = 0, i = 0$) is given by $I_0 = -I_m \sin\ (\psi + \phi)$

Hence, the resultant current becomes

$$i = I_m \sin(\omega t + \psi + \phi)$$
$$- I_m \sin(\psi + \phi) e^{-t/\lambda}$$

As shown in Fig. 22.11, the resultant current at the moment of switch closing is OA and is made up of steady-state current OC and transient current OB.

22.10. Double Energy Transients

In an R-L-C circuit, both electromagnetic and electrostatic energies are involved, hence any sudden change in the conditions of the circuit involves the redistribution of these two forms of energy. The transient currents produced due to this redistribution are known as double-energy transients. The transient current produced may be unidirectional or a decaying oscillatory current.

In an R-L-C circuit, the transient voltages across the three circuit parameters are $i_t, R, (L di_t / dt)$ and q_t / C. Hence, the equation of the transient voltage is

$$i_t R + L\frac{di_t}{dt} + \frac{q_t}{C} = 0 \qquad \ldots (i)$$

Differentiating the above equation and putting i_t for dq_t/dt, we get

$$\frac{d^2 i_t}{dt^2} + \frac{R}{L}\frac{di_t}{dt} + \frac{1}{LC} i_t = 0 \qquad \ldots (ii)$$

This is a linear differential equation of the second order with constant coefficient like 2 (i) given in Art. No. 22.3. Its solution is given by

$$i_t = k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} \qquad \ldots (iii)$$

where k_1 and k_2 are constants whose values are found from the boundary conditions. The values of λ_1 and λ_2 are given by

$$\lambda_1 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \quad \text{and} \quad \lambda_2 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} = -\frac{1}{LC}}$$

Depending on the value of λ_1 and λ_2, four different conditions of the circuit are distinguishable. We will now examine these four conditions in the case of an R-L-C circuit.

Case 1. Loss-free Circuit, $R = 0$ *i.e.* Undamped

In this case, $\lambda_1 = \sqrt{-\frac{1}{LC}} = j - 1\sqrt{LC} = -j\omega$ and $\lambda_2 = -\sqrt{-\frac{1}{LC}} = -j - 1\sqrt{LC} = -j\omega$

Hence, Eq. (iii) given above becomes

$$i_t = k_1 e^{j\omega t} + k_2 e^{-j\omega t} = k_1(\cos\omega t + j\sin\omega t) + k_2(\cos\omega t - j\sin\omega t)$$
$$= (k_1 + k_2)\cos\omega t + j(k_1 - k_2)\sin\omega t$$

or $$i_t = A\cos\omega t + B\sin\omega t \qquad \ldots (iv)$$

where $A = k_1 + k_2$ and $B = j(k_1 - k_2)$

Eq. (*iv*) can be still further simplified to

$$i_t = I_m \sin(\omega t + \phi) \qquad \ldots (v)$$

where $I_m = \sqrt{A^2 + B^2}$ and $\phi = \tan^{-1}(A/B)$

As seen from Eq. (*v*), the transient current in this case is sinusoidal wave of constant peak value and frequency $f = 1/2\pi\sqrt{LC}$ as shown in Fig. 22.12 (*a*). The values of two constant terms I_m and ϕ can be determined from any *two* known initial circuit conditions which are (*i*) the initial current in the inductance and (*ii*) the initial voltage across the capacitor.

Case 2. Low-loss Circuit: $\dfrac{R^2}{4_2} < \dfrac{I}{LC}$ *i.e.* Under-damped

In this case, λ_1 and λ_2 would be conjugate complex numbers because the term under the square root sign in each case would be negative.

$$\therefore \qquad \lambda_1 = \frac{R}{2L} + j\sqrt{\frac{1}{LC}/2 - \frac{R^2}{4L^2}}$$

If $a = \dfrac{R}{2L}$ and $\omega = \sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}$ then $\lambda_1 = -a + j\omega$ and $\lambda_2 = -a - j\omega$

Putting these values in equation (*v*), we get

$$i_t = k_1 e^{(-a+j\omega)t} + k_2 e^{(-a-j\omega)t} = e^{-at}(k_1 e^{j\omega t} + k_2 e^{-j\omega t})$$

This equation can be reduced, as before, to the form

$$i_t = I_m e^{-at}\sin(\omega t + \phi) \qquad \ldots (iv)$$

where I_m and ϕ are constants as before. Equation (*vi*) represents damped transient oscillatory current as shown in Fig. 22.12 (*b*).

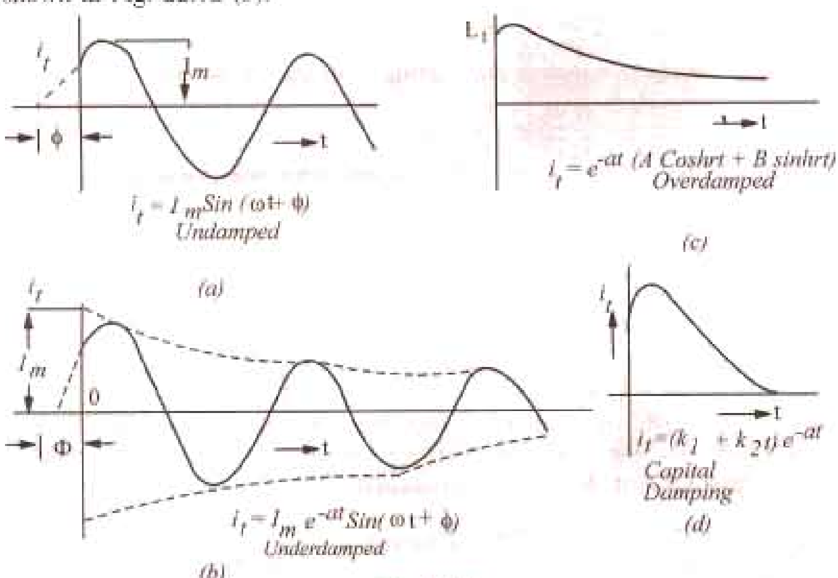


Fig. 22.12

The exponential term e^{-at} which accounts for the decay of oscillations, is called the decay or damping factor or merely *decrement*. It makes each current peak a definite fraction less than that preceding it. The logarithm to the Naperian base '*e*' of the ratio of peaks one cycle apart in time is $a/f = R/2fL$ and is referred to as *logarithmic decrement*. The frequency of damped oscillations is given by

$$f = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$ and is called the natural frequency of the circuit

If $\quad \frac{R^2}{4L^2} < \frac{1}{LC}$, then $f = \frac{1}{2\pi\sqrt{LC}}$

Case 3. High-loss Circuit: $\frac{R^2}{4L^2} > \frac{1}{LC}$ i.e. overdamped

In this case, λ_1 and λ_2 will be pure numbers.

$$\lambda_1 - \frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = -a + \gamma \text{ and } \gamma_2 = -a - \gamma$$

$$\therefore \qquad i_t = k_1 e^{(-a+\gamma)t} + k_2 e^{(-a-\gamma)t} = e^{-at}(k_1 e^{\gamma t} + k_2 e^{-\gamma t})$$

Now $\qquad e^{\gamma t} = \sinh \gamma t + \cosh \gamma t$

and $\qquad e^{-\gamma t} = \cosh \gamma t - \sinh \gamma t$

$$\therefore \qquad i_t = e^{-at}\{(k_1 + k_2)\cosh \gamma t + (k_1 - k_2)\sinh \gamma t\}$$

or $\qquad i_t = e^{-at}(A \cosh \gamma t + B \sinh \gamma t)$

A typical curve of this equation is shown in Fig. 22.12(*c*)

Case 4. $\frac{R^2}{4L^2} = \frac{1}{LC}$ i.e. Critical Damping

In this case, $\lambda_1 = \lambda_2 = -\frac{R}{2L}$

Hence, equation (*iii*) is reduced to

$$i_t = (k_1 + k_2 t)e^{-\frac{R}{2L}t} \text{ or } i_t = (k_1 + k_2 t)e^{-at}$$

It is a case of critical damping because current is reduced to almost zero in the shortest possible time. The above equation has been plotted in Fig. 22.12 (*d*).

Hence, we can summarize as follows:

1. Transient current is an undamped sine wave if $R = 0$
2. Transient current is non-oscillatory if $R < 2\sqrt{L/C}$
3. Transient current is non-oscillatory if $R \geq 2\sqrt{L/C}$
4. Critical damping occurs if $R = 2\sqrt{L/C}$

Example 22.9. *A 5-μF capacitor is discharged suddenly through a coil having an inductance of 2H and a resistance of 200 Ω. The capacitor is initially charged to a voltage of 10 V. Find*

(a) an expression for the current

(b) the additional resistance required to give critical damping.

Solution. Since there is no battery or generator in the circuit (Fig. 22.13), the steady-state current must be zero. It means that resultant current is simply the transient current.

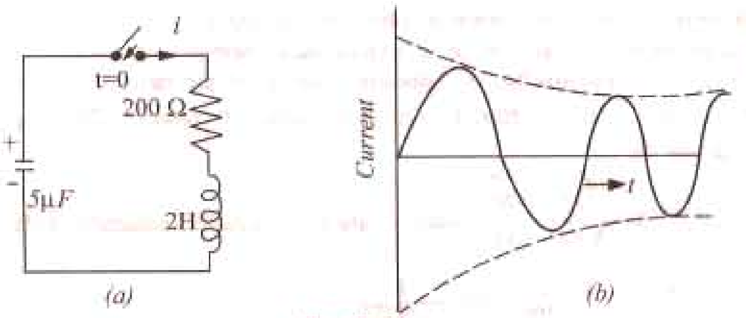


Fig. 22.13

Value of $2\sqrt{L/C} = 2\sqrt{2/5\times10^{-6}}$
$= 1265\ \Omega$

Since $R < 2\sqrt{L/C}$, the circuit is originally oscillatory.
(*a*) the expression for the transient current, therefore, is

$$i_t = I_m e^{-at}\sin(\omega t + \phi)$$

where $a = R/2L = 200/2\times2 = 50$

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \sqrt{100{,}000 - 2500} = 312.3 \text{ rad/s}$$

$$\therefore \quad i_t = I_m e^{-50t}\sin(312.3t + \phi) = i \qquad \text{... } (i)$$

Two initial conditions are known from which I_m and ϕ can be found (*a*) at $t = 0$; $i = 0$ and (*b*) at $t = 0$; $v_L = 10$ V. Applying condition (*a*) to Eq. (*i*), we get

$$0 = I_m \sin\phi, \text{ hence } \phi = 0 \quad \therefore\ i = I_m e^{-50t}\sin 312.3t \qquad \text{... } (ii)$$

Now, at $t = 0$, the voltage across the inductance must be 10 V because the current in the resistance is zero.

i.e. $(L\,di/dt)_{t=0} = 10$ V $\quad \therefore\ (di/dt)_{t=0} = 10/L = 5$ A/s ... (*iii*)

Now, from equation (*ii*), we have

$$\frac{di}{dt} = -50I_m e^{-50t}\sin 312.3t + 312.3 I_m e^{-50t}\cos 312.3t \qquad \text{... } (iv)$$

Putting $t = 0$, it becomes $(di/dt)_{t=0} = 312.3 I_m$

From equation (*iii*), we have

$312.3 I_m = 5 \qquad \therefore I_m = 5/312.3 = 0.016$ A

Hence, the general expression for the current becomes

$$\boldsymbol{i = 0.016\ e^{-50t}\sin 312.3t}$$

It is roughly plotted (the first few cycles only) in Fig. 22.13 (*b*).

(*b*) Critical damping is achieved when $R = 2\sqrt{L/C}$

$$\therefore \quad R = 2\sqrt{2/5\times10^{-6}} = 1265\,\Omega$$

$\therefore$ Additional resistance reqd. = 1265 − 200 = **1065 Ω**

Example 22.10. *A damped oscillation has the equation $i = 50e^{-10t}$ sine 628t. Find the number of oscillations which occurs before the amplitude of the oscillations decays to 1/10th of its undamped value.*

Solution. Undamped amplitude = 50 A
1/10th amplitude = (1/10) × 50 = 5A

Let the time required for this decay be t. Now, the decay of the peak of the oscillations is given by the term $50e^{-10t}$

$\therefore \quad 5 = 50e_1^{-10t} \quad \therefore \quad e_1^{10t_1} = 10$ or $10t_1 = \log h^{10} = 2.3 \log_{10}^{10} = 2.3$

$\therefore \quad t_1 = 0.23$ second

Frequency of oscillations $= 628/2\pi = 100$ Hz.

Hence, the number of oscillations which occur before the amplitude falls to 1/10th of its undamped value is $= 0.23 \times 100 =$ **23**

Example 22.11. *If, in Fig. 22.14, a break occurs at a point marked X, what would be the voltage across the break? It may be assumed that prior to the break, steady conditions existed in the circuit.*

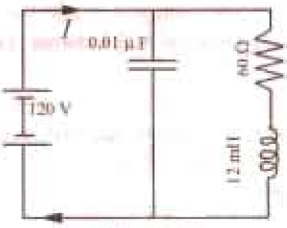


Fig. 22.14

Solution. Steady-state current through the inductance = 120/60 = 2A

Energy stored in the inductor prior to the break

$$= \frac{1}{2}LI^2 = \frac{1}{2} \times 12 \times 10^{-3} \times 4 = 24 \times 10^{-3} \text{ J}$$

Energy initially stored in the capacitor

$$= \frac{1}{2}CV^2 = \frac{1}{2} \times 10^{-8} \times 120^2 = 72 \times 10^{-6} \text{ J} = 0\text{-practically}$$

When the break occurs, the energy stored in the inductor is transferred to the capacitor. If loss of energy during first transfer is neglected, then maximum energy stored in the capacitor is

$$= 20 \times 10^{-3} \text{ J} \quad \therefore \quad \frac{1}{2}CV_m^2 = 24 \times 10^{-3}$$

$$\therefore \quad V_m = \sqrt{2 \times 24 \times 10^{-3} \times 10^8} = 2{,}190 \text{ V}$$

Maximum voltage across the break is = 2190 + 120 = 2310 V

The voltage would be oscillatory because the energy alternates between the inductor and capacitor.

Frequency of voltage oscillation is

$$f = 1/2\pi\sqrt{LC} = 10^5/2\pi \times \sqrt{1.2} = 14{,}530 \text{ Hz}$$

Decay or damping factor $= e^{-at}$ Art. 22.10, Case 2

Here, $a = R/2L = 60/2 \times 12 \times 10^{-3} = 2500$ $\therefore$ damping factor $= e^{-2500t}$

Hence, voltage across the break is

$$= 120 + 2190e^{-2500t}\sin 2\pi \times 14{,}500t = 120 + 2190e^{-2500t}\sin 91{,}290t.$$

Tutorial Problem No. 22.1

1. Deduce an expression for the growth of current in an inductive circuit.

A 15-H inductance coil of 10 Ω resistance is suddenly connected to a 20 V d.c. supply. Calculate:

(*a*) the initial rate of change of current (*b*) the current after 2 second

(*c*) the rate of change of current after 2 second (*d*) the energy stored in the magnetic field in this time

(*e*) the energy lost as heat in this time (*f*) the time constant.

[(*a*) 1.33 A/s (*b*) 1.47 A (*c*) 0.352 A/s (*d*) 16.3 joules (*e*) 19.5 joules (*f*) 1.5 s]

2. A circuit consisting of a 20 Ω resistor in series with a 0.2 H inductor is supplied from 200 V (r.m.s.) 50 Hz a.c. mains. Deduce equations showing how the current varies with time if the supply is suddenly switched on (*a*) at the instant when the voltage is zero (*b*) at the instant when the voltage is a maximum.

[(*a*) 4.11 e^{-100t} + 4.32 sin (314 t − 70°16′) A (*b*) − 1.245 e^{-100t} + 4.32 cos (314t − 72°16′) A]

3. A circuit consisting of a 20 Ω resistor, 20 mH inductor and a 100 μF capacitor in series is connected to a 200 V, d.c. supply. The capacitor is initially uncharged. Determine the equation relating the instantaneous current to the time and find the maximum instantaneous current.

[(20 e^{-500t} sin 500 t) A; 6.44 A]

4. Find an expression for the value of current at any instant after a sinusoidal voltage of amplitude 600 V at 50 Hz is applied to a series circuit of resistance 10 Ω and inductance 0.1 H, assuming that voltage is zero at the instant of switching. Also, find the value of transient current at $t = 0.02$ second.

[– 15.14 A; 2.17 A] (*Electric Circuits and Fields, Gujarat Univ. 1978*)

5. A 40 Ω resistor and a 50 μF capacitor are connected in series and supplied with an alternating voltage $v = 283 \sin 314\, t$. The supply is switched on at the instant when the voltage is zero. Determine the expression for the instantaneous current at time t. **[$-3.18e^{-500t} + 3.76 \sin (314\, t + 57°50')$]**

6. A d.c. voltage of 100 V is suddenly applied to a circuit consisting of a 100 Ω resistor, a 0.1 H inductor and a 100 μF capacitor in series. The capacitor is initially uncharged. Obtain the equation which shows how the capacitor voltage varies with time. **[$100 - 115.3\, e^{-500t} \sin (866\, t + \pi/3)$ V]**

7. The voltage $v = 200 \sin 314t$ is suddenly applied at $t = 0$ to a circuit consisting of a 10 Ω resistor in series with a 0.1 H inductor. Deduce an equation showing how the current varies with time.

[$5.78\, e^{-100t} + 6.06 \sin (314\, t - 72°20')$]

8. A 20 Ω resistor, a 0.01 H inductor and a 100 μF capacitor are connected in series. A d.c. voltage of 100 V is suddenly applied to the circuit. Obtain the equation showing how the current through the circuit varies with time. Find the maximum current and the time at which it occurs. **[$10^4\, e^{-1000t}$; 3.67 A; 0.001 second]**

9. A 4 – μ F capacitor is initially charged to 300 V. It is discharged through a 100 mH inductance and a resistor in series:

(*a*) find the frequency of the discharge if the resistance is zero.

(*b*) how many cycles at the above frequency will occur before the discharge oscillation decays to 1/10 of its initially value if the resistance is 1 Ω.

(*c*) find the value of the resistance which would just prevent oscillations.

[(*a*) 796 Hz (*b*) 36.6 (*c*) 100 Ω]

OBJECTIVE TESTS-22

1. Transient disturbance is produced in a circuit whenever
   (*a*) it is suddenly connected or disconnected from the supply
   (*b*) it is shorted
   (*c*) its applied voltage is changed suddenly
   (*d*) all of the above.
2. There are no transients in pure resistive circuits because they
   (*a*) offer high resistance
   (*b*) obey Ohm's law
   (*c*) have no stored energy
   (*d*) are linear circuits.
3. Transient currents in electrical circuit are associated with
   (*a*) inductors
   (*b*) capacitors
   (*c*) resisters
   (*d*) both (*a*) and (*b*).
4. The transients which are produced due to sudden but energetic changes from one steady state of a circuit to another are called ...... transients.
   (*a*) initiation (*b*) transition
   (*c*) relaxation (*d*) subsidence
5. In an *R-L* circuit connected to an alternating sinusoidal voltage, size of transient current primarily depends on
   (*a*) the instant in the voltage cycle at which circuit is closed
   (*b*) the peak value of steady-state current
   (*c*) the circuit impedance
   (*d*) the voltage frequency.
6. Double-energy transients are produced in circuits consisting of
   (*a*) two or more resistors
   (*b*) resistance and inductance
   (*c*) resistance and capacitance
   (*d*) resistance, inductance and capacitance.
7. The transient current in a loss-free *L-C* circuit when excited from an ac source is a/an ..... sine wave.
   (*a*) over damped (*b*) undamped
   (*c*) under damped
   (*d*) critically damped.
8. Transient current in an *R-L-C* circuit is oscillatory when
   (*a*) $R = 0$ (*b*) $R > 2\sqrt{L/C}$
   (*c*) $R < 2\sqrt{L/C}$ (*d*) $R = 2\sqrt{L/C}$

ANSWERS

1. (*d*) 2. (*c*) 3. (*d*) 4. (*b*) 5. (*a*) 6. (*d*) 7. (*b*) 8. (*c*)

23 SYMMETRICAL COMPONENTS

23.1. Introduction

The method of symmetrical components was first proposed by C.L. Fortescue and has been found very useful in solving unbalanced polyphase circuits, for analytical determination of the

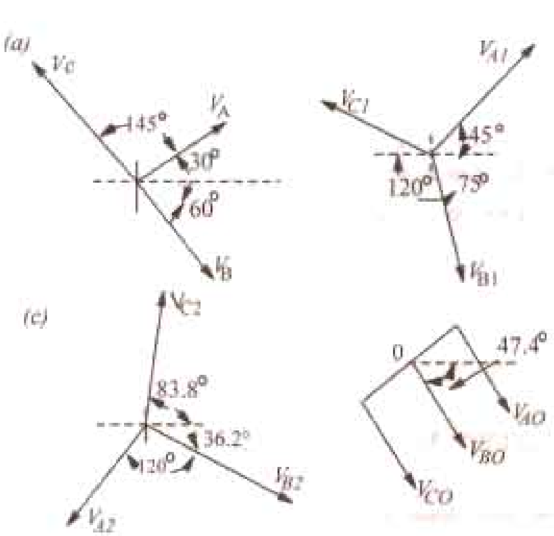


Fig. 23.1

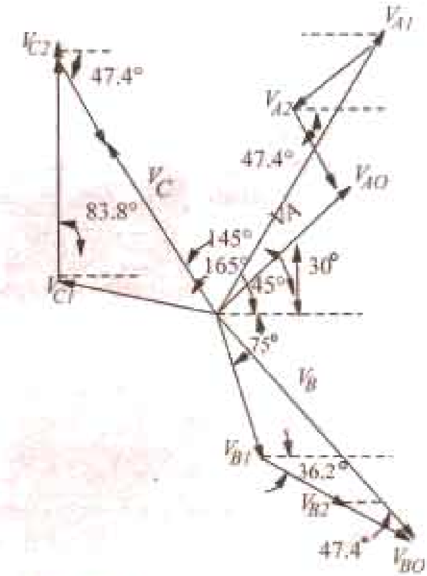


Fig. 23.2

performance of polyphase electrical machinery when operated from a system of unbalanced voltages and for calculation of currents resulting from unbalanced faults. According to Fortescue's theorem, any unbalanced 3-phase system of vectors (whether representing voltages or currents) can be resolved into three *balanced* systems of vectors which are called its '*symmetrical components*'. In Fig. 23.1 (*a*) is shown a set of three unbalanced voltage vectors V_A, V_B and V_C having phase sequence $A \rightarrow B \rightarrow C$. These can be regarded as made up of the following symmetrical components :-

(*i*) A balanced system of 3-phase vectors V_{A1}, V_{B1} and V_{C1} having the phase sequence $A \rightarrow B \rightarrow C$ as the original set of three unbalanced vectors. These vectors constitute the positive-sequence components [Fig. 23.1 (*b*)].

(*ii*) A balanced system of 3-phase vectors V_{A2}, V_{B2} and V_{C2} having phase sequence $A \rightarrow C \rightarrow B$ which is opposite to that of the original unbalanced vectors. These vectors constitute the negative-sequence components [Fig. 23.1 (*c*)].

(*iii*) A system of three vectors V_{A0}, V_{B0} and V_{C0} which are equal in magnitude and are in phase with each other *i.e.* $V_{A0} = V_{B0} = V_{C0}$. These three co-phasal vectors form a uniphase system and are known as zero-sequence components [Fig. 23.1(*d*).]

Hence, it means that an unbalanced 3-phase system of voltages or current can be regarded as due to the superposition of two symmetrical 3-phase systems having opposite phase sequences and a system of zero phase sequence *i.e.* ordinary single-phase current or voltage system. In Fig. 23.2, each of the original vectors has been reconstructed by the vector addition of its positive-sequence, negative - sequence and zero-sequence components. It is seen that

$$V_A = \overset{-ve}{V_{A1}} + \overset{-ve}{V_{A2}} + \overset{zero}{V_{A0}} \qquad \text{... (i)}$$

$$V_B = V_{B1} + V_{B2} + V_{Bo} \qquad \text{... (ii)}$$

$$V_C = V_{C1} + V_{C2} + V_C 0 \qquad \text{... (iii)}$$

23.2 The Positive - sequence Components

As seen from above, the positive-sequence components have been designated as V_{A1}, V_{B1} and V_{C1}. The subscript 1 is meant to indicate that the vector belongs to the positive-sequence system. The letter refers to the original vector of which the positive-sequence vector is a component part.

These positive-sequence vectors are completely determined when the magnitude and phase of any one of these is known. Usually, these vectors are related to each other with the help of the operator a (for details, please refer to Art. 12.11). As seen from Fig. 23. 1 (*b*).

$$\mathbf{V}_{A1} = V_{A1}; V_{B1} - a^2\mathrm{V}_{A1} = V_{A1} \angle -120°; \mathbf{V}_{C1} = a\mathbf{V}_{A1} == V_{A1} \angle 120°$$

23.3. The Negative - sequence Components

This system has a phase sequence of $A \to C \to B$. Since this system is also balanced, it is completely determined when the magnitude and phase of one of the vectors becomes known. The suffix 2 indicates that the vector belongs to the negative-sequence system. Obviously, as seen from Fig. From 23.1 (*c*).

$$\mathbf{V}_{A2} = V_{A2}; \mathbf{V}_{B2} = a\, V_{A2} = \mathbf{V}_{A2} = \angle 120° : \mathbf{V}_{C2} = a^2\mathbf{V}_{A2} = V_{A2} \angle -120°$$

23.4. The Zero - sequence Components

These three vectors are equal in magnitude and phase and hence form what is known as uniphase system. They are designated as V_{A0}, V_{B0} and V_{C0}. Since these are indentical in magnitude

$$\therefore \quad V_{A0} = V_{B0} = V_{C0}$$

23.5. Graphical Composition of Sequence Vectors

Fig. 23.2 Shows how the original vector V_A has been obtained by the addition of V_{A1}, V_{A2} and V_{A0}. The same applies to other vectors V_B and V_C.

For simplicity, let us write V_{A1} as V_1, V_{A2} as V_2 and V_{A0} as V_0. Then

$$\mathbf{V}_A = V_1 + \mathbf{V}_2 + \mathbf{V}_0 \qquad \text{... (iv)}$$

$$\mathbf{V}_B = a^2\mathbf{V}_1 + a\mathbf{V}_2 + \mathbf{V}_0 \qquad \text{... (v)}$$

$$\mathbf{V}_C = a\mathbf{V}_1 + a^2\mathbf{V}_2 + \mathbf{V}_0 \qquad \text{... (vi)}$$

23.6. Evaluation of $\mathbf{V}_{A1}$ or $\mathbf{V}_1$

The procedure for evaluating $\mathbf{V}_1$ is as follows :

Multiplying (*v*) by a and (*vi*) by a^2, we get

$$a\mathbf{V}_B = a^3\mathbf{V}_1 + a^2\mathbf{V}_2 + a\mathbf{V}_0; \qquad a^2\mathbf{V}_C = a^3\mathbf{V}_1 + a^4\mathbf{V}_2 + a^2\mathbf{V}_0$$

Now $\quad a^3 = 1$ and $a^4 = a$, hence

$$a\mathbf{V}_B = \mathbf{V}_1 + a^2\mathbf{V}_2 + a\mathbf{V}_0 \qquad \text{... (vii)}$$

$$a^2\mathbf{V}_C = \mathbf{V}_1 + a\mathbf{V}_2 + a^2\mathbf{V}_0 \qquad \text{.... (viii)}$$

Adding (*iv*), (*vii*) and (*viii*), we get

$$\mathbf{V}_A + a\mathbf{V}_B + a^2\mathbf{V}_C = 3\mathbf{V}_1 + \mathbf{V}_2(1 + a + a^2) + \mathbf{V}_0(1 + a + a^2) = 3\mathbf{V}_1$$

$$\therefore \quad \mathbf{V}_1 = \frac{1}{3}(\mathbf{V}_A + a\mathbf{V}_B + a^2\mathbf{V}_C) = \frac{1}{3}(\mathbf{V}_A + \mathbf{V}_B\angle 120° + \mathbf{V}_C\angle -210°)$$

$$= \frac{1}{3}\left[V_A + V_B\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) + V_C\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\right]$$

This shows that, geometrically speaking, V_1 is a vector one-third as large as the vector obtained by the vector addition of the three original vectors V_A, $V_B\angle 120°$ and $V_C\angle -120°$.

23.7 Evaluation of $\mathbf{V}_{A2}$ or $\mathbf{V}_2$

Multiplying (*vi*) by a and (*v*) by a^2 and adding them to (*iv*) we get

$$a\mathbf{V}_C = a^2\mathbf{V}_1 + a^3\mathbf{V}_2 + a\mathbf{V}_0; \qquad a^2\mathbf{V}_B = a^4\mathbf{V}_1 + a^3\mathbf{V}_2 + a^2\mathbf{V}_0$$

$$\mathbf{V}_A + a^2\mathbf{V}_B + a\mathbf{V}_C = \mathbf{V}_1(1 + a + a^2) + 3\mathbf{V}_2 + \mathbf{V}_0(1 + a + a^2) = 3\mathbf{V}_2 \text{ Now , } 1 + a + a^2 = 0$$

$$\therefore \quad \mathbf{V}_2 = \frac{1}{3}(\mathbf{V}_A + a^2\mathbf{V}_B + a\mathbf{V}_C) = \frac{1}{3}(\mathbf{V}_A + \mathbf{V}_B\angle -120° + \mathbf{V}_C\angle 120°)$$

$$= \frac{1}{3}\left[\mathbf{V}_A + \mathbf{V}_B\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) + \mathbf{V}_C\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\right]$$

23.8. Evaluation of $\mathbf{V}_{A0}$ or $\mathbf{V}_0$

Adding (*iv*), (*v*) and (*vi*), we get $\mathbf{V}_A + \mathbf{V}_B + \mathbf{V}_C = \mathbf{V}_1(1 + a + a^2) + \mathbf{V}_2(1 + a + a^2) + 3\mathbf{V}_0 = 3\mathbf{V}_0$

$$\therefore \quad \mathbf{V}_0 = \frac{1}{3}(\mathbf{V}_A + \mathbf{V}_B + \mathbf{V}_C)$$

It shows that V_0 is simply a vector one third as large as the vector obtained by adding the original vectors V_A, V_B and V_C.

To summarize the above results, we have

(*i*) $\mathbf{V}_1 = \frac{1}{3}(\mathbf{V}_A + a\mathbf{V}_B + a^2\mathbf{V}_C)$ (*ii*) $\mathbf{V}_2 = \frac{1}{3}(\mathbf{V}_A + a^2\mathbf{V}_B + a\mathbf{V}_C)$ (*iii*) $\mathbf{V}_0 = \frac{1}{3}(\mathbf{V}_A + \mathbf{V}_B + \mathbf{V}_C)$

Note. An unbalanced system of 3-phase currents can also be likewise resolved into its symmetrical components. Hence

$$\mathbf{I}_A = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_0; \mathbf{I}_B = a^2\mathbf{I}_1 + a\mathbf{I}_2 + \mathbf{I}_0; \mathbf{I}_C = a\mathbf{I}_1 + a^2\mathbf{I}_2 + \mathbf{I}_0$$

Also, as before $\mathbf{I}_1 = \frac{1}{3}(\mathbf{I}_A + a\mathbf{I}_B + a^2\mathbf{I}_C); \mathbf{I}_2 = \frac{1}{3}(\mathbf{I}_A + a^2\mathbf{I}_B + a\mathbf{I}_C); \mathbf{I}_0 = \frac{1}{3}(\mathbf{I}_A + \mathbf{I}_B + \mathbf{I}_C)$ ... (*ix*)

It shows that I_0 is one-third of the neutral or earth-return current and is zero for an unearthed 3-wire system. It is seen from (*ix*) above that I_0 is zero if the vector sum of the original current vectors is zero. This fact can be used with advantage in making numerical calculations because the original system of vectors can then be reduced to two balanced 3-phase systems having opposite phase sequences.

Example 23.1. *Find out the positive, negative and zero-phase sequence components of the following set of three unbalanced voltage vectors:*

$$V_A = 10\angle 30°; V_B = 30\angle -60°; V_C = 15\angle 145°$$

Indicate on an approximate diagram how the original vectors and their different sequence components are located. (**Principles of Elect. Engg. – 1, Jadavpur Univ. 1987**)

Solution. (*i*) **Positive-sequence vectors**

As seen from Art. 23.6

$$\mathbf{V}_1 = \frac{1}{3}(\mathbf{V}_A + a\mathbf{V}_B + a^2\mathbf{V}_C) = \frac{1}{3}(10\angle 30° + a.30\angle -60° + a^2.15\angle 145°)$$

$$= \frac{1}{3}(10\angle 30° + 30\angle 60° + 15\angle 25°) = 12.42 + j12.43 = 17.6\angle 45°$$

$$\therefore \; \mathbf{V}_{A1} = 17.6\angle 45°; \; \mathbf{V}_{B1} = 17.6\angle 45° \times \angle -120° = 17.6\angle -75°$$

$\mathbf{V}_{C1}' = 17.6\angle 45° \times \angle 120° = 17.6\angle 165°$

These are shown in Fig. 23.1 (*b*)

(*ii*) Negative–sequence vectors

As seen from Art. 23.7,

$$\mathbf{V}_2 = \frac{1}{3}(\mathbf{V}_A + a^2\mathbf{V}_B + a\mathbf{V}_C) = \frac{1}{3}(10\angle 30° + a^2\,30\angle -60° + a.15\angle 145°)$$

$$= \frac{1}{3}(10\angle 30° + 30\angle -180° + 15\angle 265°) = -7.55 - j3.32 = 8.24\angle -156.2°$$

$$\mathbf{V}_{A2} = 8.24\angle -156.2°; \mathbf{V}_{B2} = 8.24\angle -156.2° \times \angle 120° = 8.24\angle -36.2°$$

$$\mathbf{V}_{C2} = 8.24\angle -156.2° \times \angle -120° = 8.24\angle -276.2°$$

These vectors are shown inFig. 23.1 (*c*)

(*iii*) Zero sequence vectors

$$\mathbf{V}_0 = \frac{1}{3}(\mathbf{V}_A + \mathbf{V}_B + \mathbf{V}_C)$$

$$= \frac{1}{3}(10\angle 30° + 30\angle -60° + 15\angle 145°) = 3.8 - j4.12 = 5.6\angle -47.4°$$

These vectors are shown in Fig. 23.1 (*d*).

Example 23.2. *Explain how an unsymmetrical system of 3-phase currents can be resolved into 3 symmetrical component systems.*

Determine the values of the symmetrical components of a system of currents

$I_R = 0 + j120A; I_Y = 50 - j100A; I_B = -100 - j50A$

Phase sequence is RYB. **(Elect. Engg.-I Bombay, Univ. 1986)**

Solution. $\mathbf{I}_R = 0 + j120 = 120\angle 90°$

$$\mathbf{I}_Y = 50 - j100 = 111.8\angle -63.5°; \mathbf{I}_B = -100 - j50 = 111.8\angle -153.5°$$

(*i*) Positive–sequence Components

$$\mathbf{I}_1 = \frac{1}{3}(\mathbf{I}_R + a\mathbf{I}_Y + a^2\mathbf{I}_B) = \frac{1}{3}\left[(0 + j120) + \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)(50 - j100) + \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)(-100 - j50)\right]$$

$$= 22.8 + j108.3 = 110.7\angle 78.1° \quad \therefore \mathbf{I}_{R1} = 110.7\angle 78.1°; \mathbf{I}_{Y1} = 110.7\angle -41.9°; \mathbf{I}_{B1} = 110.7\angle 198.1°$$

(*ii*) Negative–sequence components

$$\mathbf{I}_2 = \frac{1}{3}(\mathbf{I}_R + a^2\mathbf{I}_Y + a\mathbf{I}_B) = \frac{1}{3}(-18.3 + j65.1) = -6.1 + j21.7 = 22.5\angle 105.7°$$

$$\therefore \quad \mathbf{I}_{R2} = 22.5\angle 105.7°; \mathbf{I}_{Y2} = 22.5\angle 225°; \mathbf{I}_{B2} = 22.5\angle -14.3°$$

(*iii*) Zero-sequence component

$$\mathbf{I}_0 = \frac{1}{3}(\mathbf{I}_R + \mathbf{I}_Y + \mathbf{I}_B) = \frac{1}{3}[(0 + j120) + (50 - j100) + (-100 - j50)] = -16.7 - j10$$

As a check, it may be found that

$$\mathbf{I}_R = \mathbf{I}_{R1} + \mathbf{I}_{R2} + \mathbf{I}_0; \mathbf{I}_Y = \mathbf{I}_{Y1} + \mathbf{I}_{Y2} + \mathbf{I}_0; \mathbf{I}_{B1} + \mathbf{I}_{B2} + \mathbf{I}_0$$

Example 23.3. *In a 3-phase, 4-wire system, the currents in the R, Y and B lines under abnormal conditions of loading were as follows:*

$I_R = 100\angle 30°; I_Y = 50\angle 300°; I_B = 30\angle 180°$

Calculate the positive, negative and zero-phase sequence currents in the R-line and the return current in the neutral conductor.

Solution. (*i*) The positive-sequence components of current in the *R*-line is

$$\mathbf{I}_1 = \frac{1}{3}(\mathbf{I}_R + a\mathbf{I}_Y + a^2\mathbf{I}_B)$$

Now $\mathbf{I}_R = 100\angle 30° = 50(\sqrt{3} + j)$

$$\mathbf{I}_Y = 50\angle 300° = 50\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) = 25(1 - j\sqrt{3})$$

$$\mathbf{I}_B = 30\angle 180° = (-30 + j0)$$

$$\mathbf{I}_1 = \frac{1}{3}\left[50(\sqrt{3}+j) + 25(1 - j\sqrt{3}+)\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) + (-30)\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\right] = 58\angle 48.4°$$

(*ii*) The negative–sequence components of the current in the *R*-line is

$$\mathbf{I}_2 = \frac{1}{3}(\mathbf{I}_R + a^2\mathbf{I}_Y + a\mathbf{I}_B)$$

$$= \frac{1}{3}\left[50(\sqrt{3}+j) + 25(1 - j\sqrt{3})\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) + (-30)\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\right] = 18.9\angle 24.9°$$

(*iii*) The zero–sequence component of current in the *R*-line is

$$\mathbf{I}_0 = \frac{1}{3}(\mathbf{I}_R + \mathbf{I}_Y + \mathbf{I}_B) = \frac{1}{3}[50(\sqrt{3}+j) + 25(1 - j\sqrt{3}) - 30] = 27.2\angle 4.7°$$

The neutral current is

$$\mathbf{I}_N = \mathbf{I}_R + \mathbf{I}_Y + \mathbf{I}_B = 3\times\mathbf{I}_0 = 3\times 27.2\angle 4.7° = 81.6\angle 4.7°$$

Example 23.4. *A 3-phase, 4-wire system supplies loads which are unequally distributed on the three phases. An analysis of the currents flowing in the direction of the loads in the R, Y and B lines shows that in the R-line, the positive phase sequence current is 200 $\angle 0°$ A and the negative phase sequence current is 100 $\angle 60°$. The total observed current flowing back to the supply in the neutral conductor is 300 $\angle 300°$ A. Calculate the currents in phase and magnitude in the three lines.*

Assuming that the 3-phase supply voltages are symmetrical and that the power factor of the load on the R-phase is $\sqrt{3}/2$ leading, determine the power factor of the loads on the two other phases.

Solution. It is given that in *R*-phase [Fig. 23.3 (*a*)]

$$\mathbf{I}_{R1} = 200\angle 0° = (200 + j0)\text{ A}; \mathbf{I}_{R2} = 100\angle 60° = (50 + j86.6)\ A$$

$$\mathbf{I}_{R0} = \frac{1}{3}\mathbf{I}_N = (300/3)\angle 300° = (50 - j86.6)\text{ A}$$

$$\mathbf{I}_R = \mathbf{I}_{R1} + \mathbf{I}_{R2} + \mathbf{I}_{R0} = (200 + j0) + (50 + j86.6) + (50 - j86.6) = (300 + j0) = 300\angle 0°$$

Similarly, as seen from Fig. 23.3 (*b*) for the *Y*-phase

$$\mathbf{I}_Y = \mathbf{I}_{Y1} + \mathbf{I}_{Y2} + \mathbf{I}_{Y0} = a^2\mathbf{I}_{R1} + a\mathbf{I}_{R2} + \mathbf{I}_{R0}$$

$$= 200\angle 0° - 120° + 100\angle 60° + 120°$$

$$+ 100\angle 300° = -100 - j173.2 - 100 + 50 - j86.6 = -150 - j259.8 = 300\angle 240°\text{ A}$$

Similarly, as seen from Fig. 23-3 (*c*) for the *B*-phase

$$\mathbf{I}_B = \mathbf{I}_{B1} + \mathbf{I}_{B2} + \mathbf{I}_{B0} = a\mathbf{I}_{R1} + a^2\mathbf{I}_{R2} + \mathbf{I}_{R0} = 200\angle 0° + 120° + 100\angle 60° - 120° + 100\angle 300° = 0$$

Since the power factor of the *R*-phase is $\sqrt{3}/2$ leading, the current I_R leads the voltage V_R by 30° [Fig. 23.3(*d*)]

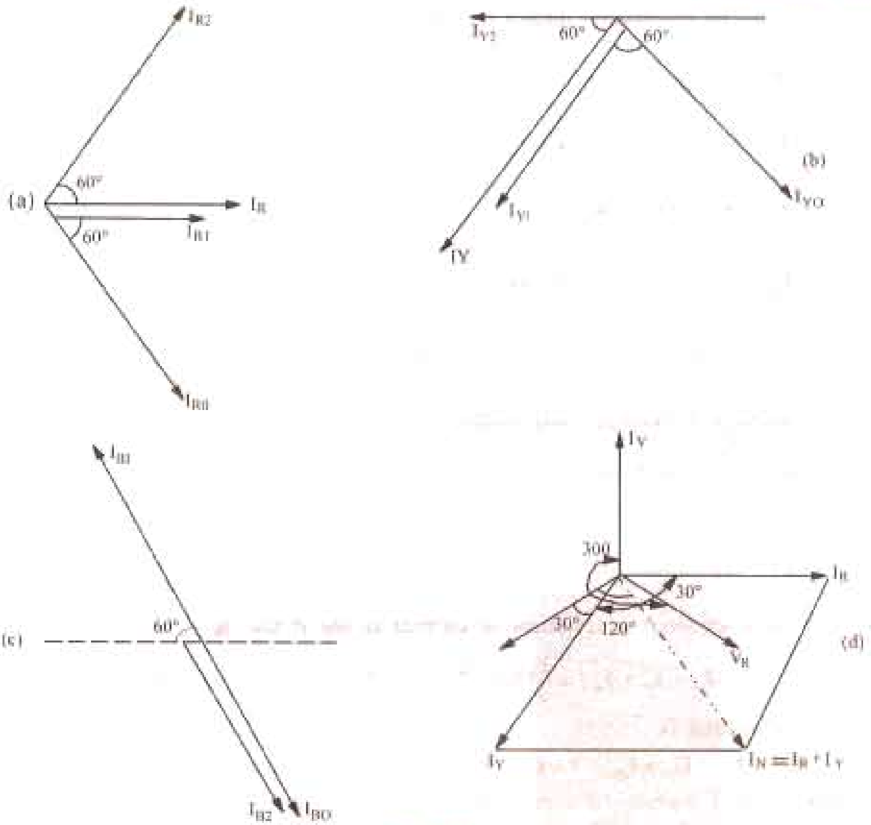


Fig. 23.3

(*d*). Now, phase angle of I_Y is 240° relative to I_R so that I_Y leads its voltage V_Y by 30°. Hence, power factor of *Y* phase is also $\sqrt{3}/2$ leading. The power factor of *B* line is indeterminate because the current in this line is zero.

Example 23.5. *Prove that in a 3-phase system if V_1, V_2 and V_3 are the three balanced voltages whose phasor sum is zero, the positive and negative sequence components can be expressed as*

$$V_{1P} = \left\{\frac{1}{\sqrt{3}}(V_1 + V_2\angle 60°)\right\}\angle 30°; V_{1N} = \left\{\frac{1}{\sqrt{3}}(V_1 + V_2\angle -60°)\right\}\angle -30°$$

Phase sequence is 1-2-3.

A system of 3-phase currents is given as $I_1 = 10 \angle 180°$, $I_2 = 14.14 \angle -45°$ and $I_3 = 10 \angle 90°$. Determine phasor expression for the sequence components of these currents. Phase sequence is 1-2-3. **(Elect. Engg-I, Bombay Univ. 1980)**

Solution. As seen from Art. 23.6

$$\mathbf{V}_{1P} = \frac{1}{3}(\mathbf{V}_1 + a\mathbf{V}_2 + a^2\mathbf{V}_3); \text{Now, } \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 = 0 \quad \therefore \quad \mathbf{V}_3 = -(\mathbf{V}_1 + \mathbf{V}_2)$$

$$\mathbf{V}_{1P} = \frac{1}{3}[\mathbf{V}_1 + a\mathbf{V}_2 - a^2(\mathbf{V}_1 + \mathbf{V}_2)] = \frac{1}{3}[\mathbf{V}_1(1 - a^2) + \mathbf{V}_2(a - a^2)]$$

Now, $1 - a^2 = \frac{2}{3} + j\frac{\sqrt{3}}{2}$ and $a - a^2 = j\sqrt{3}$

$$\therefore\ \mathbf{V}_{1P} = \frac{1}{3}\left[\mathbf{V}_1\left(\frac{3}{2} + j\frac{\sqrt{3}}{2}\right) + j\sqrt{3}\mathbf{V}^2\right] = \frac{1}{\sqrt{3}}\left[\mathbf{V}_1\left(\frac{\sqrt{3}}{2} + j\frac{1}{2}\right) + j\mathbf{V}_2\right]$$

$$= \frac{1}{\sqrt{3}}[\mathbf{V}_1\angle 30^\circ + \mathbf{V}_2\angle 90^\circ] = \left[\frac{1}{\sqrt{3}}(\mathbf{V}_1 + \mathbf{V}_2\angle 60^\circ)\right]\angle 30^\circ$$

Similarly, the negative-sequence component is given by

$$\mathbf{V}_{1N} = \frac{1}{3}(\mathbf{V}_1 + a^2\mathbf{V}_2 + a\mathbf{V}_3) = \frac{1}{3}[\mathbf{V}_1 + a^2\mathbf{V}_2 - a(\mathbf{V}_1 + \mathbf{V}_2)] = \frac{1}{3}[\mathbf{V}_1(1-a) + \mathbf{V}_2(a^2 - a)]$$

Now, $a^2 - a = -j\sqrt{3}$

$$\therefore\ \mathbf{V}_{1N} = \frac{1}{3}\left[\mathbf{V}_1\left(\frac{3}{2} - j\frac{\sqrt{3}}{2}\right) - j\sqrt{3}\mathbf{V}_2\right] = \frac{1}{\sqrt{3}}\left[\mathbf{V}_1\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) - j\mathbf{V}_2\right]$$

$$= \frac{1}{\sqrt{3}}[\mathbf{V}_1\angle -30^\circ + \mathbf{V}_2\angle -90^\circ] = \left\{\frac{1}{\sqrt{3}}(\mathbf{V}_1 + \mathbf{V}_2\angle -60^\circ)\right\}\angle -30^\circ$$

Now $\mathbf{I}_1 = 10\angle 180^\circ = -10 + j0; \mathbf{I}_2 = 14.14\angle -45^\circ = 10 - j10$

$\mathbf{I}_3 = 10\angle 90^\circ = j10$

$$a\mathbf{I}_2 = 14.14\angle 75^\circ = 3.66 + j13.66; a^2\mathbf{I}_2 = 14.14\angle -165^\circ = -13.66 - j3.66$$

$$a\mathbf{I}_3 = 14.14\angle 21-^\circ = -12.25 - j707; a^2\mathbf{I}_3 = 14.14\angle -30^\circ = 12.25 - j7.07$$

$$\mathbf{I}_{1P} = \frac{1}{3}(\mathbf{I}_1 + a\mathbf{I}_2 + a^2\mathbf{I}_3) = \frac{1}{3}(5.91 + j6.59) = 1.97 + j2.2$$

$$\mathbf{I}_{1N} = \frac{1}{3}(\mathbf{I}_1 + a^2\mathbf{I}_2 + a\mathbf{I}_3) = \frac{1}{3}(-35.91 - j10.73) = -11.97 - j3.58$$

$$\mathbf{I}_{10} = \frac{1}{3}(\mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3) = 0$$

Tutorial Problems No. 23.1

1. The following currents were recorded in the *R, Y*, and *B* lines of a 3-phase system under abnormal conditions: **[$\mathbf{I}_R = 300\angle 300^\circ$ A; $\mathbf{I}_Y = 500\angle 240^\circ$ A; $\mathbf{I}_B = 1{,}000\angle 60^\circ$ A]**

Calculate the values of the positive, negative and zero phase-sequence components.

[$\mathbf{I}_1 = 536\angle -44^\circ 20'$ A; $\mathbf{I}_2 = 372\angle 171^\circ$ A; $\mathbf{I}_0 = 145\angle 23^\circ 20'$]

2. Determine the symmetrical components of the three currents $I_0 = 10\angle 0^\circ$; $\mathbf{I}_B = 100\angle 250^\circ$ and $\mathbf{I}_C = 10\angle 110^\circ$ A **[$\mathbf{I}_1 = (39.45 + j522)$ A; $\mathbf{I}_2 = (-20.24 + j22.98)$ A; $\mathbf{I}_0 = (-9.21 - j28.19)$ A]**

(Elect. Meas & Measuring Instru., Madras Univ. June 1976)

3. The three current vectors of a 3-phase, four-wire system have the following values: $\mathbf{I}_A = 7 + j_0$, $\mathbf{I}_B = -12 - j13$ and $\mathbf{I}_C = -2 + j3$. Find the symmetrical components. The phase sequence is *A, B, C*

[$\mathbf{I}_0 = -2.33 - j3.33$; $\mathbf{I}_{A1} = (27.75 - j3.67)$; $\mathbf{I}_{B1} = (-17.05 - j22.22)$; $\mathbf{I}_{C1} = (-10.7 + j25.9)$; $\mathbf{I}_{A2} = (0.25 + j13.67)$ A; $\mathbf{I}_{B2} = (-11.97 - j6.62)$ A; $\mathbf{I}_{C2} = (11.73 - j7.05)$ A]

23.9. Zero-Sequence Components of Current and Voltage

Any circuit which allows the flow of positive-sequence currents will also allow the flow of negative-sequence currents because the two are similar. However, a fourth wire is necessary if zero-sequence components are to flow in the lines of the 3-phase system. It follows that the line currents of 3-phase 3-wire system can contain no zero sequence components whether it is delta - or star-connected. The zero sequence components of line-to-line voltages are non-existent regardless of the degree of imbalance in these voltages. It means that a set of unbalanced 3-phase, line-to-line voltages may be represented by a positive system and a negative system of balanced voltages. This fact is of considerable importance in the analysis of 3-phase rotating machinery. For example, the operation of an induction motor when supplied from an unbalanced system of 3-phase voltages, may be analysed on the basis of two balanced systems of voltages of opposite phase sequence.

Let us consider some typical 3-phase connections with reference to zero-sequence components of current and voltage.

(a) Four-wire Star Connection. Due to the presence of the fourth wire, the zero sequence currents may flow. The neutral wire carries only the zero-sequence current which is the sum of the zero-sequence currents in the three lines. Since the sum of line voltages is zero, there can be no zero sequence component of line voltages.

(b) Three-wire Star Connection. Since there is no fourth or return wire, zero-sequence components of current cannot flow. The absence of zero-sequence currents may be explained by considering that the impedance offered to these currents is infinite and that this impedance is situated between the star points of the generator and the load. If the two star points were joined by a neutral, only zero-sequence currents will flow through it so that only zero-sequence voltage can exist between the load and generator star points. Obviously, no zero-sequence component of voltage appears across the phase load.

(c) Three-wire Delta Connection. Due to the absence of fourth wire, zero-sequence components of currents cannot be fed into the delta-connected load. However, though line currents have to sum up to zero (whereas phase currents need not do so) it is possible to have a zero-sequence current circulating in the delta-connected load.

Similarly, individual phase voltages will generally possess zero-sequence components though components are absent in the line-to line voltage.*

23.10. Unbalanced Star Load Supplied from Unbalanced Three-phase Three-wire System

In this case, line voltages and load currents will consist of only positive and negative-sequence components (but no zero-sequence component). But load voltages will consist of positive, negative and zero-sequence components.

Let the line voltages be denoted by $\mathbf{V}_{RY}$, $\mathbf{V}_{YB}$ and $\mathbf{V}_{BR}$, line (and load) currents by $\mathbf{I}_R$, $\mathbf{I}_Y$ and $\mathbf{I}_B$, the load voltages by $\mathbf{V}_{RN}$, $\mathbf{V}_{YN}$ and $\mathbf{V}_{BN}$ and load impedances by $\mathbf{Z}_R$, $\mathbf{Z}_Y$, and $\mathbf{Z}_B$ (their values being the same for currents of any sequence).

Obviously, $\mathbf{V}_{RN} = \mathbf{I}_R\mathbf{Z}_R$; $\mathbf{V}_{YN} = \mathbf{I}_Y\mathbf{Z}_Y$ and $\mathbf{V}_{BN} = \mathbf{I}_B\mathbf{Z}_B$

If $\mathbf{V}_{RN1}$, $\mathbf{V}_{RN2}$ and $\mathbf{V}_0$ are the symmetrical components of $\mathbf{V}_{RN}$, then we have

$$\mathbf{V}_0 = \frac{1}{3}(\mathbf{V}_{RN} + \mathbf{V}_{YN} + \mathbf{V}_{BN}) = \frac{1}{3}(\mathbf{I}_R\mathbf{Z}_R + \mathbf{I}_Y\mathbf{Z}_Y + \mathbf{I}_B\mathbf{Z}_B)$$

$$= \frac{1}{3}[\mathbf{Z}_R(\mathbf{I}_{R1} + \mathbf{I}_{R2}) + \mathbf{Z}_Y(\mathbf{I}_{Y1} + \mathbf{I}_{Y2}) + \mathbf{Z}_B(\mathbf{I}_{B1} + \mathbf{I}_{B2})]$$

$$= \frac{1}{3}[\mathbf{Z}_R(\mathbf{I}_{R1} + \mathbf{I}_{R2}) + \mathbf{Z}_Y(a^2\mathbf{I}_{R1} + a\mathbf{I}_{P2}) + \mathbf{Z}_B(a\mathbf{I}_{R1} + a^2\mathbf{I}_{R2})]$$

$$= \mathbf{I}_{R1}.\frac{1}{3}(\mathbf{Z}_R + a^2\mathbf{Z}_Y + a\mathbf{Z}_B) + \mathbf{I}_{R2}.\frac{1}{3}(\mathbf{Z}_R + a\mathbf{Z}_Y + a^2\mathbf{Z}_B) = \mathbf{I}_{R1}\mathbf{Z}_{R2} + \mathbf{I}_{R2}\mathbf{Z}_{R1} \quad \text{...}(i)$$

$$\mathbf{V}_{RN1} = \frac{1}{3}(\mathbf{V}_{RN} + a\mathbf{V}_{YN} + a^2\mathbf{V}_{BN}) = \frac{1}{3}(\mathbf{I}_R\mathbf{Z}_R + a\mathbf{I}_Y\mathbf{Z}_Y + a^2\mathbf{I}_B\mathbf{Z}_B)$$

$$= \frac{1}{3}[\mathbf{Z}_R(\mathbf{I}_{R1} + \mathbf{I}_{R2})] + a\mathbf{Z}_Y(\mathbf{I}_{Y1} + \mathbf{I}_{Y2}) + a^2\mathbf{Z}_B(\mathbf{I}_{B1} + \mathbf{I}_{B2})$$

$$= \mathbf{I}_{R1}.\frac{1}{3}(\mathbf{Z}_R + \mathbf{Z}_Y + \mathbf{Z}_B) + \mathbf{I}_{R2}.\frac{1}{3}(\mathbf{Z}_R + a^2\mathbf{Z}_Y + a\mathbf{Z}_B) = \mathbf{I}_{R1}\mathbf{Z}_0 + \mathbf{I}_{R2}\mathbf{Z}_{R2} \quad \text{...}(ii)$$

$$\mathbf{V}_{RN2} = \mathbf{I}_{R2}\mathbf{Z}_0 + \mathbf{I}_{R1}\mathbf{Z}_{R1}$$

---

* However, under balanced conditions, the phase voltages will possess no zero-sequence components.

Similarly, $\mathbf{V}_{YN1} = \mathbf{I}_{Y1}\mathbf{Z}_0 + \mathbf{I}_{Y2}\mathbf{Z}_{Y2} = a^2(\mathbf{I}_{R1}\mathbf{Z}_0 + \mathbf{I}_{R2}\mathbf{Z}_{R2}) = a^2\mathbf{V}_{RN1}$

$$\mathbf{V}_{BN1} = \mathbf{I}_{B1}\mathbf{Z}_0 + \mathbf{I}_{B2}\mathbf{Z}_{B2} = a(\mathbf{I}_{R1}\mathbf{Z}_0 + \mathbf{I}_{R2}\mathbf{Z}_{R2}) = a\mathbf{V}_{RN1}$$

$$\mathbf{V}_{YN2} = \mathbf{I}_{Y2}\mathbf{Z}_0 + \mathbf{I}_{Y1}\mathbf{Z}_{Y1} = a(\mathbf{I}_{R2}\mathbf{Z}_0 + \mathbf{I}_{R1}\mathbf{Z}_{R1}) = a\mathbf{V}_{RN2}$$

$$\mathbf{V}_{BN2} = \mathbf{I}_{B2}\mathbf{Z}_0 + \mathbf{I}_{B1}\mathbf{Z}_{B1} = a^2(\mathbf{I}_{R2} + \mathbf{Z}_0 + \mathbf{I}_{R1}\mathbf{Z}_{R1}) = a^2\mathbf{V}_{RN2}$$

Now, $\mathbf{V}_{RN1}$ and $\mathbf{V}_{RN2}$ may be determined from the relation between the line and phase voltages as given below:

$$\mathbf{V}_{RY} = \mathbf{V}_{RN} + \mathbf{V}_{NY} = \mathbf{V}_{RN} - \mathbf{V}_{YN} = \mathbf{V}_0 + \mathbf{V}_{RN1} + \mathbf{V}_{RN2} - (\mathbf{V}_0 + \mathbf{V}_{RN1} + \mathbf{V}_{YN2}]$$

$$= \mathbf{V}_{RN1}(1-a^2) + \mathbf{V}_{RN2}(1-a) = \frac{1}{2}\sqrt{3}[\mathbf{V}_{RN1}(\sqrt{3}+j1) + \mathbf{V}_{RN2}(\sqrt{3}-j1)] \quad \text{... (iii)}$$

$$\mathbf{V}_{YB} = \mathbf{V}_{YN} + \mathbf{V}_{NB} = \mathbf{V}_{YN} - \mathbf{V}_{BN} = \mathbf{V}_0 + \mathbf{V}_{YN1} + \mathbf{V}_{YN2} - (\mathbf{V}_0 + \mathbf{V}_{BN1} + \mathbf{V}_{BN2})$$

$$= \mathbf{V}_{RN1}(a^2 - a) - \mathbf{V}_{RN2}(a - a^2) = -j\sqrt{3}.\mathbf{V}_{RN1} + j\sqrt{3}\mathbf{V}_{RN2}$$

$$\therefore \quad \mathbf{V}_{RN2} = \mathbf{V}_{RN1} - j\mathbf{V}_{YB} / \sqrt{3}$$

Substituting this value of $\mathbf{V}_{RN2}$ in Eq. (*iii*) above and simplifying, we have

$$\mathbf{V}_{RN1} = \frac{1}{3}[\mathbf{V}_{RY} + \frac{1}{2}\mathbf{V}_{YB}(1 + j\sqrt{3})] \quad \text{... (iv)}$$

$$\mathbf{V}_{RN2} = \frac{1}{3}[\mathbf{V}_{RY} + \frac{1}{2}\mathbf{V}_{YB}(1 - j\sqrt{3})] \quad \text{... (v)}$$

Having known $\mathbf{V}_{RN1}$ and $\mathbf{V}_{RN2}$, the currents $\mathbf{I}_{R1}$ and $\mathbf{I}_{R2}$ can be determined from the following equations:

$$\mathbf{I}_{R1} = \frac{(\mathbf{V}_{RN1}\mathbf{Z}_0 - \mathbf{V}_{RN2}\mathbf{Z}_{R2})}{\mathbf{Z}_0^2 - \mathbf{Z}_{R1}\mathbf{Z}_{R2}} \quad \text{... (vi)}$$

$$\mathbf{I}_{R2} = \frac{(\mathbf{Z}_{RN2}\mathbf{Z}_0 - \mathbf{V}_{RN1}\mathbf{Z}_{R1})}{\mathbf{Z}_0^2 - \mathbf{Z}_{R1}\mathbf{Z}_{R2}} \quad \text{... (vii)}$$

Alternatively, when $\mathbf{I}_{R1}$ becomes known, $\mathbf{I}_{R2}$ may be found from the relation

$$\mathbf{I}_{R2} = \frac{\mathbf{V}_{RN2} - \mathbf{I}_{R1}\mathbf{Z}_{R1}}{\mathbf{Z}_0} \quad \text{... (viii)}$$

The symmetrical components of the currents in other phases can be calculated from $\mathbf{I}_{R1}$ and $\mathbf{I}_{R2}$ by using the relations given in Art. 23.8.

The phase voltages may be calculated by any one of the two methods given below:

(*i*) directly by calculating the products $\mathbf{I}_R\mathbf{Z}_R$, $\mathbf{I}_Y\mathbf{Z}_Y$ and $\mathbf{I}_B\mathbf{Z}_B$.

(*ii*) by first calculating the zero-sequence component $\mathbf{V}_0$ of the phase or load voltages and then adding this to the appropriate positive and negative sequence components.

For example, $\mathbf{V}_{RN} = \mathbf{V}_0 + \mathbf{V}_{RN1} + \mathbf{V}_{RN2}$

23.11. Unbalanced Star Load Supplied from Balanced Three-phase, Three-wire System

It is a special case of the general case considered in Art. 23.10 above. In this case, the symmetrical components of the load voltages consist only of positive and zero-sequence components. This fact may be verified by substituting the value of $\mathbf{V}_{YB}$ in Eq. (*v*) of Art. 23.10. Now, in a balanced or symmetrical system of positive-phase sequence

$$\mathbf{V}_{YB} = \mathbf{V}_{RY}\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

Substituting this value in Eq. (*v*) above, we have

$$\mathbf{V}_{RN2} = \frac{1}{3}[\mathbf{V}_{RY} + \frac{1}{2}\mathbf{V}_{YB}(1 - j\sqrt{3})]$$

$$= \frac{1}{3}\left[\mathbf{V}_{RY} + \frac{1}{2}.\mathbf{V}_{RY}\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)(1 - j\sqrt{3})\right] = \frac{1}{3}-(\mathbf{V}_{RY} - \mathbf{V}_{RY}) = 0$$

Hence, substituting this value of $\mathbf{V}_{RN2}$ in Eq. (*iv*) and (*vi*) of Art. 23.10, we get

$$\mathbf{V}_{RN1} = j\mathbf{V}_{YB}\sqrt{3} = \frac{1}{2}\mathbf{V}_{RY} - j\frac{1}{2}\mathbf{V}_{RY}/\sqrt{3} \qquad ...(i)$$

$$\mathbf{I}_{R1} = \mathbf{V}_{RN1}\mathbf{Z}_0/(\mathbf{Z}_0^2 - \mathbf{Z}_{R1}\mathbf{Z}_{R2}) = \mathbf{V}_{RN1}\frac{\mathbf{Z}_R + \mathbf{Z}_Y + \mathbf{Z}_B}{\mathbf{Z}_R\mathbf{Z}_Y + \mathbf{Z}_Y\mathbf{Z}_B + \mathbf{Z}_B\mathbf{Z}_R} \qquad ...(ii)$$

$$\mathbf{I}_{R2} = -\mathbf{I}_{R1}.\mathbf{Z}_{R1}/\mathbf{Z}_0 = -\mathbf{R}\mathbf{I}_0 = -\mathbf{I}_{R1}\frac{\mathbf{Z}_R + a\mathbf{Z}_Y + a^2\mathbf{Z}_B}{\mathbf{Z}_R + \mathbf{Z}_Y + \mathbf{Z}_B} \qquad ...(iii)$$

Example 23.6 illustrates the procedure for calculating the current in an unbalanced star-connected load when supplied from a symmetrical three-wire system.

Example 23.6. *A symmetrical 3-phase, 3-wire, 440-V system supplies an unbalanced Y-connected load of impedances $Z_R = 5 \angle 30° \ \Omega$; $Z_Y = 10 \angle 45° \ \Omega$, $Z_B = 10 \angle 60° \ \Omega$. Phase sequence is $R \to Y \to B$. Calculate in the rectangular complex form the symmetrical components of the currents in the R-line.* **(Elect. Engg.-I, Bombay Univ. 1985)**

Solution. As shown in Fig. 23.4, the branch impedances are

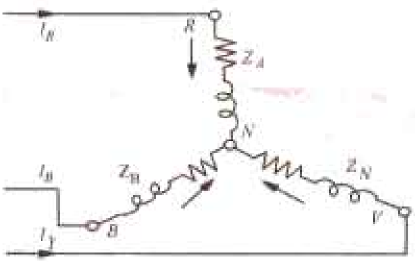


Fig. 23.4

$$\mathbf{Z}_R = 5\angle 30° = (4.33 + j2.5)\Omega$$

$$\mathbf{Z}_Y = 10\angle 45° = (7.07 + j7.07)\Omega$$

$$\mathbf{Z}_B = 10\angle 60° = (5 + j8.66)\Omega$$

$$a\mathbf{Z}_Y = 10\angle 165° = 10(-0.966 + j0.259) = -9.66 + j2.59)\Omega$$

$$a^2\mathbf{Z}_Y = 10\angle -75° = 10(0.259 - j0.966) = 2.59 - j9.66$$

$$aZ_B = 10\angle 180° = (-10 + j0)$$

$$a^2\mathbf{Z}_B = 10\angle -60° = 10(0.5 - j0.866) = (5 - j8.66)$$

The symmetrical component impedances required for calculation purposes are as follows:

$$\mathbf{Z}_0 = \frac{1}{3}(\mathbf{Z}_R + \mathbf{Z}_Y + \mathbf{Z}_B) = \frac{1}{3}(16.4 + j18.23) = (5.47 + j6.08)\Omega$$

$$\mathbf{Z}_{R1} = \frac{1}{3}(\mathbf{Z}_R + \mathbf{Z}_Y + a^2\mathbf{Z}_B)$$

$$= \frac{1}{3}(4.33 + j2.5 - 9.66 + j2.59 + 5 - j8.66)$$

$$= \frac{1}{3}(-0.33 - j3.57) = -0.11 - j1.19$$

$$\mathbf{Z}_{R2} = \frac{1}{3}(\mathbf{Z}_R + a^2\mathbf{Z}_Y + a\mathbf{Z}_B)$$

$$= \frac{1}{3}(4.33 + j2.5 + 2.59 - j9.66 - 10 + j0) = \frac{1}{3}(-3.08 - j7.16) = -1.03 - j2.55$$

Now

$$\mathbf{Z}_0{}^2\mathbf{Z}_{R1}\mathbf{Z}_{R2} = \frac{1}{3}(\mathbf{Z}_R\mathbf{Z}_Y + \mathbf{Z}_Y\mathbf{Z}_B + \mathbf{Z}_B\mathbf{Z}_R)$$

$$= \frac{1}{3}[50\angle 75° + 100\angle 105° + 50\angle 90°] = (-4.31 + j65)$$

Let $\mathbf{V}_{RY}$ be taken as the reference vectors so that $\mathbf{V}_{RY} = (440 + j0)$

Then, $\mathbf{V}_{RN1} = \frac{1}{2}\mathbf{V}_{RY} - j.\frac{1}{2}\mathbf{V}_{RY}/\sqrt{3} = \frac{1}{2}(440+j0) - j\frac{1}{2}(440+j0)/\sqrt{3} = (220 - j127)$

Now, $\mathbf{I}_{R_1} = \dfrac{\mathbf{V}_{RN1}.\mathbf{Z}_0}{(\mathbf{Z}_0^{\;2} - \mathbf{Z}_{R_1}\mathbf{Z}_{R_2})} = \dfrac{(220-j127)(5.47+j6.08)}{(-4.31+j65)} = \dfrac{1986+j643}{(-4.31+j65)}$

$$= \frac{2088\angle 18^\circ}{66.4\angle 93.8^\circ} = 31.6\angle -75.8^\circ \neq (7.75 - j30.6)\text{A}$$

$$\mathbf{I}_{R_2} = -\mathbf{I}_{R_2}\mathbf{Z}_{R_1}/\mathbf{Z}_0$$

$$= \frac{-(7.75-j30.6)(-0.11-j1.19)}{(5.47+j6.08)} = \frac{(37.25+j5.88)}{(5.47+j6.08)}$$

$$= \frac{37.75\angle 9^\circ}{8.18\angle 48^\circ} = 4.61\angle -39^\circ = (3.28 - j2.66)\text{ A}$$

Hence, the symmetrical components of $\mathbf{I}_R$ are:-

Positive-sequence component = $(7.75 - j30.6)$ A

Negative-sequence component = $(3.28 - j2.66)$ A

Note. (*i*) $\mathbf{I}_R = \mathbf{I}_{R1} + \mathbf{I}_{R2}$

(*ii*) Symmetrical components of other currents are

$$\mathbf{I}_Y = a^2\mathbf{I}_{B1} \text{ and } \mathbf{I}_{Y2} = a\mathbf{I}_{R2}\;:\;\mathbf{I}_{B1} = a\mathbf{I}_{R1} \text{ and } \mathbf{I}_{B2} = a^2\mathbf{I}_{R2}$$

Example 23.7. *A balanced star-connected load takes 75 A from a balanced 3-phase, 4-wire supply. If the fuses in two of the supply lines are removed, find the symmetrical components of the line currents before and after the fuses are removed.*

Solution. The circuit is shown in Fig. 23.5.

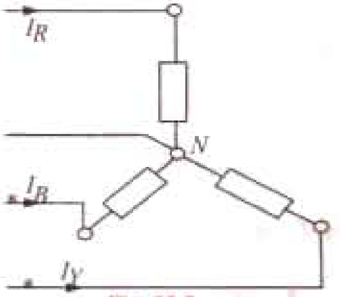


Fig. 23.5

Before uses are removed

$$\mathbf{I}_R = 75\angle 0^\circ; \mathbf{I}_Y = 75\angle -120^\circ; \mathbf{I}_B = 75\angle 120^\circ$$

$$\mathbf{I}_1 = \frac{1}{3}(\mathbf{I}_R + a\mathbf{I}_Y + a^2\mathbf{I}_B) = \frac{1}{3}(75\angle 0^\circ + 75\angle 0^\circ + 75\angle 360^\circ) = 75\angle 0^\circ \text{ A}$$

$$\mathbf{I}_2 = \frac{1}{3}(\mathbf{I}_R + a^2\mathbf{I}_Y + a\mathbf{I}_B) = \frac{1}{3}(75\angle 0^\circ + 75\angle 120^\circ + 75\angle 240^\circ) = 0$$

$$\mathbf{I}_0 = \frac{1}{3}(\mathbf{I}_R + \mathbf{I}_Y + \mathbf{I}_B) = \frac{1}{3}(75\angle 0^\circ + 75\angle -120^\circ + 75\angle 120^\circ) = 0.$$

After fuses are removed

$$\mathbf{I}_R = 75\angle 0^\circ; \mathbf{I}_Y = \mathbf{I}_B = 0$$

$$\mathbf{I}_1 = \frac{1}{3}(75\angle 0^\circ + 0 + 0) = 25\angle 0^\circ$$

$$\mathbf{I}_2 = 25\angle 0^\circ, \mathbf{I}_0 = 25\angle 0^\circ$$

Example 23.8. *Explain the terms: positive-, negative- and zero-sequence components of a 3-phase voltage system. A star-connected load consists of three equal resistors, each of 1 Ω resistance. When the load is connected to an unsymmetrical 3-phase supply, the line voltages are 200 V, 346 V and 400 V.*

Find the magnitude of the current in any one phase by the method of symmetrical components. **(Power Systems-II, A.M.I.E. 1989)**

Solution. This question could be solved by using the following two methods:

(*a*) As shown in Fig. 23.6(*a*), the line voltages form a closed right-angled triangle with an angle = $\tan^{-1}(346/200) = \tan^{-1}(1.732) = 60^\circ$

Hence, if $\mathbf{V}_{AB} = 200\angle 0^\circ$, then $\mathbf{V}_{BC} = 346\angle -90^\circ = -j346$ and $\mathbf{V}_{CA} = 400\angle 120^\circ$.

As seen from Eq. (*iv*) and (*v*) of Art. 23.10

$$\mathbf{V}_{AN1} = \frac{1}{3}\left[\mathbf{V}_{AB} + \frac{1}{2}\mathbf{V}_{BC}(1 + j\sqrt{3})\right]$$

$$= \frac{1}{3}\left[200 + \left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)(-j346)\right] = 166.7 - j57.7$$

$$\mathbf{V}_{AN2} = \frac{1}{3}\left[\mathbf{V}_{AB} + \mathbf{V}_{BC}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right)\right]$$

$$= \frac{1}{3}\left[200 + \left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)(-j346)\right] = -33.3 - j57.7$$

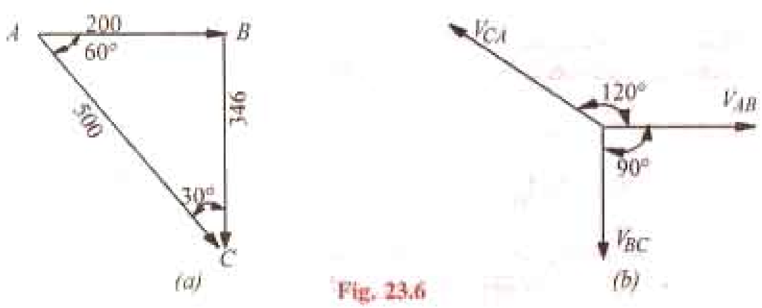


Fig. 23.6

Now, $\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C = 3$

$$\mathbf{Z}_{A1} = \mathbf{Z}_A + a\mathbf{Z}_B + a^2\mathbf{Z}_C = 0 \text{ (because } \mathbf{Z}_A = \mathbf{Z}_B = \mathbf{Z}_C \text{ and } 1 + a + a^2 = 0\text{)}$$

Similarly, $\mathbf{Z}_{A2} = \mathbf{Z}_A + a^2\mathbf{Z}_B + a\mathbf{Z}_C = 0$

Hence, from Eq. (*ii*) of Art. 23.10 we have $\mathbf{V}_{AN1} = \mathbf{I}_{A1}\mathbf{Z}_0 \quad \therefore \quad \mathbf{I}_{A1} = \mathbf{A}_{AN1}/\mathbf{Z}_0$

Now, $\mathbf{Z}_0 = \frac{1}{3}(\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C) = 3/3 = 1\,\Omega \quad \therefore \quad \mathbf{I}_{A1} = (166.7 - j57.7)\text{ A}$

$$\mathbf{I}_{AS} = \mathbf{V}_{AN2}/\mathbf{Z}_0 = -33.3 - j57.7 \quad \therefore \quad \mathbf{I}_A = \mathbf{I}_{A1} + \mathbf{I}_{A2} \quad (\because \mathbf{I}_0 = 0)$$

$$\therefore \qquad \mathbf{I}_A = (166.7 - j57.7) + (-33.3 - j57.7) = 133.4 - j115.4 = 176.4\angle -40.7°\text{ A}$$

(*b*) Using Millman's theorem and taking the line terminal *A* as reference point, the voltage between *A* and the neutral point *N* is

$$\mathbf{V}_{NA} = \frac{\mathbf{V}_{BA}\mathbf{Y}_B + \mathbf{V}_{CA}\mathbf{Y}_C}{\mathbf{Y}_A + \mathbf{Y}_B + \mathbf{Y}_C} = \frac{200\angle 180° \times 1 + 400\angle 120° \times 1}{3} = -133.3 + j115.3 = 176.4\angle 139.3°$$

$$\mathbf{V}_{AN} = 176.4\angle 139.3° - 180° = 176.4\angle -40.7° \quad \therefore \quad \mathbf{I}_A = \mathbf{I}_{AN}/\mathbf{Z} = 176.4\angle -40.7° \text{ ... as before}$$

Example 23.9. *Three equal impedances of (8 + j6) are connected in star across a 3-phase, 3-wire supply. The phase voltages are V_A = (220 + j0), V_B = (– j220) and V_C = (– 100 + j220) V. If there is no connection between the load neutral and the supply neutral, calculate the symmetrical components of A-phase current and the three line currents.*

Solution. Since there is no fourth wire, there is no zero-component current. Moreover, $\mathbf{I}_A + \mathbf{I}_B + \mathbf{I}_C = 0$. The symmetrical components of the A-phase voltages are

$$\mathbf{V}_0 = \frac{1}{3}(220 - j220 - 100 + j220) = 40\ \mathbf{V}\,*$$

$$\mathbf{V}_{A1} = \frac{1}{3}[220 + (-0.5 + j0.866)(-j220) + (-0.5 - j0.866)(-100 + j220)]$$

$$= \frac{1}{3}[(660 + j86.6)] = (220 + j28.9)\ \mathrm{V}$$

$$\mathbf{V}_{A2} = \frac{1}{3}[220 + (-0.5 - j0.866)(-j220) + (-0.5 + j0.866)(-100 + j220)]$$

$$= \frac{1}{3}(-120 - j86.6) = (-40 - j28.9)\ \mathbf{V}$$

The component currents in phase A are

$$\mathbf{I}_{A1} = \frac{\mathbf{V}_{A1}}{(8 + j6)} = \frac{220 + j28.9}{(8 + j6)} = (19.33 - j10.89)\ \mathrm{A}$$

$$\mathbf{I}_{A2} = \frac{\mathbf{V}_{A2}}{(8 + j6)} = \frac{(-40 - j28.9)}{(8 + j6)} = (-4.93 - j0.09)\ \mathrm{A}$$

$$\mathbf{I}_A = \mathbf{I}_{A1} + \mathbf{I}_{A2} = (14.4 - j10.8)\mathrm{A}; \mathbf{I}_B = \mathbf{I}_{B1} + \mathbf{B}_{B2} = a^2\mathbf{I}_{A1} + a\mathbf{I}_{A2}$$

$$\mathbf{I}_B = \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)(19.93 - j10.89) + \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)(-4.93 - j0.09)$$

Example 23.10. *Two equal impedance arms AB and BC are connected to the terminals A, B, C of a 3-phase supply as shown in Fig. 23.7. Each capacitor has a reactance of $X = \sqrt{3}R$. A high impedance voltmeter V is connected to the circuit at points P and Q as shown. If the supply line voltages V_{AB}, V_{BC}, V_{CA} are balanced, determine the reading of the voltmeter (a) when the phase sequence of the supply voltages is $A \to B \to C$ and (b) when the phase sequence is reversed. Hence, explain how this network could be employed to measure, respectively, the positive and negative phase sequence voltage components of an unbalanced 3-phase supply.*

Solution. The various currents and voltages are shown in Fig. 23.7.

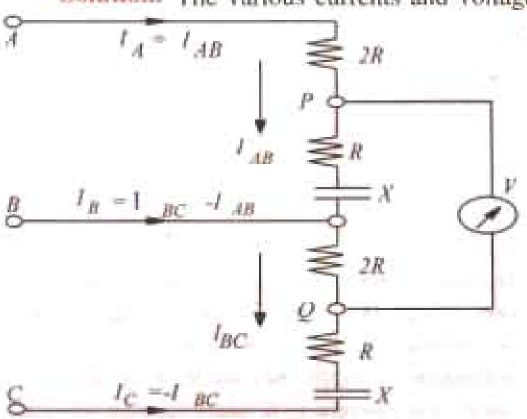


Fig. 23.7

Phase Sequence ABC

Taking $\mathbf{V}_{AB}$ as the reference vector, we have

$\mathbf{V}_{AB} = V(1 + j0); \mathbf{V}_{BC} = a^2\mathrm{V}; \mathbf{V}_{CA} = a\mathrm{V}$

As seen from the diagram,

$$\mathbf{I}_{AB} = \frac{\mathbf{V}_{AB}}{3R - jX} = \frac{V}{3R - j\sqrt{3}R} = \frac{V}{\sqrt{3}R(\sqrt{3} - j)}$$

$$\mathbf{I}_{BC} = \frac{\mathbf{V}_{BC}}{3R - jX} = \frac{a^2V}{\sqrt{3}R(\sqrt{3} - j)}$$

Now, $\mathbf{V}_{PO} + \mathbf{V}_{OQ} = \mathbf{V}_{PQ}$

Also, $\mathbf{V}_{PQ} = \mathbf{I}_{AB}(R - jX) + \mathbf{I}_{BC}.2R$

*However, it is not required in the problem.

$$= \frac{V}{\sqrt{3}R(\sqrt{3}-j)}.R(1-j\sqrt{3}) + \frac{a^2V}{\sqrt{3}R(\sqrt{3}-j)}.2R$$

$$= \frac{V}{\sqrt{3}(\sqrt{3}-j)}.(1-j\sqrt{3}+2a^2) \quad \left(\because \; a^2 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

$$= \frac{V}{\sqrt{3}(\sqrt{3}-j)}.(-j2\sqrt{3}) \qquad \therefore \; V_{PQ} = \frac{V}{\sqrt{3}\times 2}\times 2\sqrt{2} = V$$

Hence, the voltmeter which is not phase sensitive will read the line voltage when phase sequence is $A \to B \to C$.

Phase Sequence ACB

In this case, $\mathbf{V}_{AB} = V(1+j0)$; $\mathbf{V}_{BC} = aV$; $\mathbf{V}_{CA} = a^2V$

Also, $\mathbf{V}_{PQ} = \mathbf{V}_{PO} + \mathbf{V}_{OQ}$

$$= \mathbf{I}_{AB}(R-jX) + \mathbf{I}_{BC}.2R. = \frac{V(R-jX)}{\sqrt{3}.R(\sqrt{3}-j)} + \frac{aV.2R}{\sqrt{3}.R(\sqrt{3}-j)}$$

$$= \frac{V}{\sqrt{3}(\sqrt{3}-j)}(1-j\sqrt{3}+2a) = \frac{V}{\sqrt{3}(\sqrt{3}-j)}(1-j\sqrt{3}-1+j\sqrt{3}) = 0$$

Hence, when the phase sequence is reversed, the voltmeter reads zero.

It can be proved that with sequence ABC, the voltmeter reads the positive sequence component ($\mathbf{V}_1$) and with phase sequence ACB, it reads the negative sequence component ($\mathbf{V}_2$) with phase sequence ABC

$$\mathbf{V}_{PQ} = \frac{\mathbf{V}_{AB}}{3R-j\sqrt{3}.R}(R-j\sqrt{3}R) + \frac{\mathbf{V}_{BC}}{3R-j\sqrt{3}R}2R = \frac{1}{3-j\sqrt{3}}[(\mathbf{V}_1+\mathbf{V}_2)(1-j\sqrt{3}) + 2(a^2\mathbf{V}_1 + a\mathbf{V}_2)]$$

$$= \frac{1}{3-j\sqrt{3}}[\mathbf{V}_1 1-j\sqrt{3}+2a^2) + \mathbf{V}_2(1-j\sqrt{3}+2a)] = \frac{1}{3-j\sqrt{3}}\mathbf{V}_1(-j2\sqrt{3}) \qquad \therefore \; \mathbf{V}_{PQ} = \mathbf{V}_1$$

With phase sequence ACB

$$\mathbf{V}_{PQ} = \frac{1}{3-j\sqrt{3}}[(\mathbf{V}_1+\mathbf{V}_2)(1-j\sqrt{3}) + 2(a\mathbf{V}_1 + a^2\mathbf{V}_2)]$$

$$= \frac{1}{3-j\sqrt{3}}[\mathbf{V}_1(1-j\sqrt{3}+2a) + \mathbf{V}_2(1-j\sqrt{3}+2a^2)]$$

$$= \frac{1}{3-j\sqrt{3}}[\mathbf{V}_2(-j2\sqrt{3})] \quad \therefore \; V_{PQ} = V_2$$

23.12. Measurement of Symmetrical Components of Circuits

The apparatus consists of two identical current transformers, two impedances of the same ohmic value, one being more inductive than the other to the extent that its phase angle is 60° greater and two identical ammeters A_1 greater and two identical ammeters A_1 and A_2 as shown in Fig. 23.8 (*a*). It can be shown that A_1 reads positive-sequence current only while A_2 reads negative-sequence current only. If the turn ratio of the current transformer is K, then keeping in mind that zero-sequence component is zero, we have

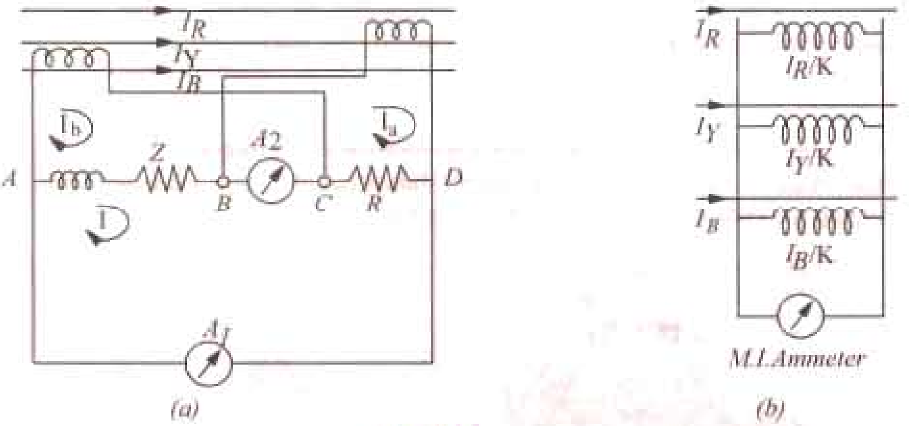


Fig. 23.8

$$\mathbf{I}_a = \mathbf{I}_R / K = (\mathbf{I}_1 + \mathbf{I}_2)/K$$

where $\mathbf{I}_1$ and $\mathbf{I}_2$ are the positive and negative-sequence components of the line current respectively.

Similarly,

$$\mathbf{I}_b = \mathbf{I}_Y / K = (a^2\mathbf{I}_1 + a\mathbf{I}_2)/K$$

If the impedance of each ammeter is $R_A + jK_A$, then impedance between points B and D is

$$\mathbf{Z}_{BD} = (R + R_A + jX_A)$$

The value of $\mathbf{Z}$ is so chosen that

$$\mathbf{Z}_{AC} = \mathbf{Z} + R_A + jX_A = (R + R_A + jX_A)\angle 60° = \mathbf{Z}_{BD}\angle 60°$$

For finding the current read by A_1, imagine a break at point X. Thevenin voltage across the break X is

$$\mathbf{V}_{th} = \mathbf{I}_b\mathbf{Z}_{AC} + \mathbf{I}_a\mathbf{Z}_{BD} = \mathbf{I}_b\mathbf{Z}_{BD} = \mathbf{I}_b\mathbf{Z}_{BD}\angle 60° + \mathbf{I}a\mathbf{Z}_{BD} = (\mathbf{I}_b\angle 60° + \mathbf{I}_a)\mathbf{Z}_{BD}$$

Total impedance is series with this Thevenin voltage is

$$\mathbf{Z}_T = \mathbf{Z}_{AC} + \mathbf{Z}_{BD} = \mathbf{Z}_{BD}\angle 60° + \mathbf{Z}_{BD} = \left(\frac{3}{2} + j\frac{\sqrt{3}}{2}\right)\mathbf{Z}_{BD}$$

The current flowing normally through the wire in which a break has been imagined is

$$\mathbf{I} = \frac{\mathbf{V}_{th}}{\mathbf{Z}_T} = \frac{\mathbf{I}_b\angle 60° + \mathbf{I}_a}{(3/2) + j(\sqrt{3}/2)} = \frac{1}{K}\frac{(a^2\mathbf{I}_1 + a\mathbf{I}_2)\angle 60° + \mathbf{I}_1 + \mathbf{I}_2}{\sqrt{3}\angle 30°}$$

$$= \frac{1}{K}\frac{I_1(1\angle 300° + 1) + I_2(1\angle 180° + 1)}{\sqrt{3}\angle 30°} = \frac{I_1}{K}\angle -60°$$

It means that A_1 reads positive-sequence current only. The ammeter A_2 reads current which is

$$= \mathbf{I}_a + \mathbf{I}_b - \mathbf{I} = \frac{1}{K}(I_1 + I_2 + I_1\angle 240° + I_2\angle 120° - I_2\angle -60°)$$

$$= \frac{I_1}{K}\left(1 - \frac{1}{2} - j\frac{\sqrt{3}}{2} - \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) + \frac{I_2}{K}(1 + 1\angle 120°) = \frac{I_2}{K}\angle 60°$$

In other words, A_2 reads negative-sequence current only.

Now, it will be shown that the reading of the moving-iron ammeter of Fig. 23-8(b) is proportional to the zero-sequence component.

$$\mathbf{I}_0 = \frac{1}{3}(\mathbf{I}_R + \mathbf{I}_Y + \mathbf{I}_B) = \frac{K}{3}\left(\frac{\mathbf{I}_R}{K} + \frac{\mathbf{I}_Y}{K} + \frac{\mathbf{I}_B}{K}\right)$$

$$= \frac{K}{3} \times \text{(current through the ammeter)} = \frac{K}{3} \times \text{ammeter reading.}$$

It is obvious that for I_0 to be present, the system must be 3-phase, 4-wire. However, when the fourth wire is available, then I_0 may be found directly by finding the neutral current I_N. In that case $\mathbf{I}_0 = \frac{1}{3} I_N$.

23.13. Measurement of Positive and Negative-sequence *Voltages

With reference to Fig. 23.9 (*a*), it can be shown that the three voltmeters indicate only the positive-sequence component of the 3-phase system.

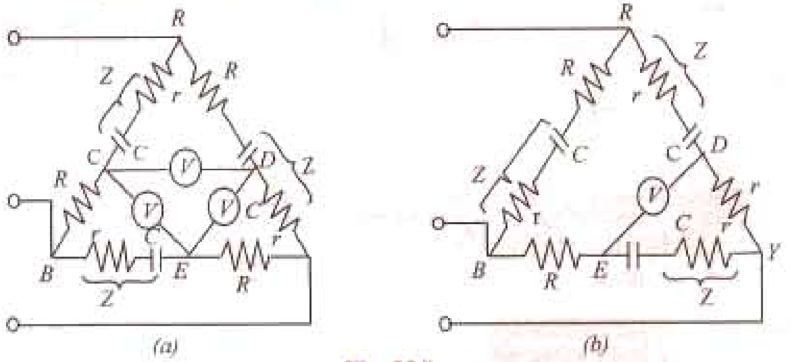


Fig. 23.9

The line-to-neutral voltage can be written (with reference to the red phase) as

$$\mathbf{V}_{RN} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_0; \mathbf{V}_{YN} = a^2\mathbf{V}_1 + a\mathbf{V}_2 + \mathbf{V}_0; \mathbf{V}_{BN} = a\mathbf{V}_1 + a^2\mathbf{V}_2 + \mathbf{V}_0$$

$$\mathbf{V}_{RY} = \mathbf{V}_{RN} + \mathbf{V}_{NY} = \mathbf{V}_{RN} - \mathbf{V}_{YN} = (1-a^2)\mathbf{V}_1 + (1-a)\mathbf{V}_2$$

$$\mathbf{V}_{YB} = \mathbf{V}_{YN} + \mathbf{V}_{NB} = \mathbf{V}_{YN} - \mathbf{V}_{BN} = (a^2 - a)\mathbf{V}_1 + (a - a^2)\mathbf{V}_2$$

$$\therefore \quad \mathbf{V}_{DY} = \frac{\mathbf{V}_{RY}(r + 1/j\omega C)}{(R + r + 1/j\omega C)} \qquad \mathbf{V}_{YE} = \frac{\mathbf{V}_{YB}.R}{(R + r + 1/j\omega C)}$$

$$\therefore \quad \mathbf{V}_{DE} = \frac{(r + 1/j\omega C)}{(R + r + 1/j\omega C)}\left[\mathbf{V}_{RY} + \frac{R}{(r + 1/j\omega C)}\mathbf{V}_{YB}\right]$$

The different equating of the bridge circuit are so chosen that

$$\frac{R}{r + j1/\omega C} = -a^2 = \frac{1}{2} + j\frac{\sqrt{3}}{2} \quad \text{or} \quad R = \frac{r}{2} + \frac{\sqrt{3}}{2\omega C} + \frac{1}{j2\omega C} + j\frac{\sqrt{3}r}{2}$$

Equating the *j*-terms or quadrature terms on both sides, we have

$$0 = \frac{1}{j2\omega C} + j\frac{\sqrt{3}}{2} \qquad \therefore \quad \frac{1}{\omega C} = \sqrt{3}r \qquad \text{... (i)}$$

Similarly, equating the reference or real terms, we have

$$R = \frac{r}{2} + \frac{\sqrt{3}}{2\omega C} = \frac{r}{2} + \frac{3r}{2} = 2r \qquad \text{... (ii)}$$

$$\therefore \quad \frac{r + 1/j\omega C}{R + r + 1/\omega C} = \frac{r - j\sqrt{3}r}{3r - j\sqrt{3}r} = \frac{r(1 - j\sqrt{3})}{r(3 - j\sqrt{3})} = \frac{1}{\sqrt{3}}\angle -30°$$

*It is supposed to possess infinite impedance and $$\mathbf{V}_{DE} = \frac{1\angle -30^\circ}{\sqrt{3}}[(1-a^2)\mathbf{V}_1 + (1-a)\mathbf{V}_2 - a^2(a^2-a)\mathbf{V}_1 - a^2(a-a^2)\mathbf{V}_2]$$

$$= \frac{1\angle -30^\circ}{\sqrt{3}}(1-a^2-a^4+a^3)\mathbf{V}_1 \qquad [\because\ 1-a = a^2(a-a^2)] = \sqrt{3}\mathbf{V}_1\angle -30^\circ$$

Hence, the voltmeter connected between points D and E measures $\sqrt{3}$ times the positive sequence component of the phase voltage. So do the other two voltmeters.

In Fig. 23.9 (b), the elements have been reversed. It can be shown that provided the same relation is maintained between the elements, the high impedance voltmeter measures $\sqrt{3}$ times the negative-sequence component of phase voltage.

23.14. Measurement of Zero-sequence Component of Voltage

The zero-sequence voltage is given by

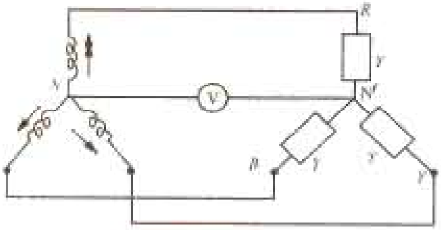


Fig. 23.10

$$\mathbf{V}_0 = \frac{1}{3}(\mathbf{V}_{RN} + \mathbf{V}_{YN} + \mathbf{V}_{BN})$$

Fig. 23.10 indicates one method of measuring $\mathbf{V}_0$. As seen

$$\mathbf{V}_{N'N} = \frac{\mathbf{V}_{RN}\mathbf{Y} + \mathbf{V}_{YN}\mathbf{Y} + \mathbf{V}_{BN}\mathbf{Y}}{3\mathbf{Y}}$$

$$= \frac{1}{3}(\mathbf{V}_{RN} + \mathbf{V}_{YN} + \mathbf{V}_{BN}) = \mathbf{V}_0$$

Hence, voltmeter connected between the neutral points measures the zero-sequence component of the voltage.

Tutorial Problem No. 23.2

1. Explain the essential features in the representation of an unsymmetrical three-phase system of voltages or currents by symmetrical components.

In a three-phase system, the three line currents are; $\mathbf{I}_R = (30 + j50)$ A, $\mathbf{I}_Y = (15 - j45)$ A, $\mathbf{I}_B = (-40 + j70)$ A. Determine the values of the positive, negative and zero sequence components.

[52.2 A, 19.3 A, 25.1 A] (*London Univ.*)

2. The phase voltages of a three-phase, four-wire system are; $\mathbf{V}_{RN} = (200 + j0)$ V, $\mathbf{V}_{YN} = (0 - j200)$ V, $\mathbf{V}_{BN} = (-100 + j\,100)$ V. Show that these voltages can be replaced by symmetrical components of positive, negative and zero sequence and calculate the magnitude of each component.

Sketch vector diagrams representing the positive, negative and zero-sequence components for each of the three phases.

[176 V', 38 V, 47.1 V] (*London Univ.*)

3. Explain, with the aid of a diagram of connections, a method of measuring the symmetrical components of the currents in an unbalanced 3-phase, 3-wire system.

If in such a system, the line currents, in amperes, are

$$\mathbf{I}_R = (10 - j2), \mathbf{I}_Y = (-2 - j4), \mathbf{I}_B = (-8 + j6)$$

Calculate their symmetrical components.

[$\mathbf{I}_{R0} = \mathbf{I}_{Y0} = \mathbf{I}_{B0} = 0$; $\mathbf{I}_{R1} = 7.89 + j0.732$; $\mathbf{I}_{Y1} = a^2\,\mathbf{I}_{R1}$; $\mathbf{I}_{B1} = a\mathbf{I}_{R1}$, $\mathbf{I}_{R2} = 2.113 - j2.732$; $\mathbf{I}_{Y2} = a\mathbf{I}_{R2}$; $\mathbf{I}_{B2} = a^2\mathbf{I}_{R2}$]

OBJECTIVE TESTS – 23

1. The method of symmetrical components is very useful for
   (*a*) solving unbalanced polyphase circuits
   (*b*) analysing the performance of 3-phase electrical machinery
   (*c*) calculating currents resulting from unbalanced faults
   (*d*) all of the above
2. An unbalanced system of 3-phase voltages having *RYB* sequence actually consists of
   (*a*) a positive-sequence component
   (*b*) a negative-sequence component
   (*c*) a zero-sequence component
   (*d*) all of the above.
3. The zero-sequence component of the unbalanced 3-phase system of vectors V_A, V_B and V_C is .... of their vector sum.
   (*a*) one-third (*b*) one-half
   (*c*) two-third (*d*) one-fourth
4. In the case of an unbalanced star-connected load supplied from an unbalanced 3-ϕ, 3 wire system, load currents will consist of
   (*a*) positive-sequence components
   (*b*) negative-sequence components
   (*c*) zero-sequence components
   (*d*) only (*a*) and (*b*).

ANSWERS

1. (*d*) 2. (*d*) 3. (*a*) 4. (*d*)

* It is supposed to possess infinite impedance

24 INTRODUCTION TO ELECTRICAL ENERGY GENERATION

24.1. Preference for Electricity

Energy is vital for all living-beings on earth. Modern life-style has further increased its importance, since a faster life means faster transport, faster communication, and faster manufacturing processes. All these lead to an increase in energy required for all those modern systems.

Arising out of comparison of status of nations, the progress is related in terms of per capita consumption of electrical energy (i.e. kWH consumed per person per year). At present, this parameter for India is about 300, for UK it is 12 to 15 times more, and for USA, it is about 30 times more.

It simply means that Electrical energy is the most popular form of energy, whether we require it in the usable thermal form (= heating applications), in mechanical form (= electrical motor-applications in Industries), for lighting purposes (= illumination systems), or for transportation systems.

Following are the main reasons for its popularity.

1. Cleaner environments for user
2. Higher efficiency
3. Better controllability
4. Easier bulk-power, long-distance transportation of power using overhead transmission or underground cables
5. Most versatile devices of energy conversions from Electrical to other forms are available for different purposes, such as thermal, illumination, mechanical, sound, chemical, etc.

24.2. Comparison of Sources of Power

While selecting a method of generating electricity, following factors are taken into account for purposes of comparison:

(*a*) Initial cost: For a given rating of a unit (in the minds of planners), investment must be known. Naturally, lower the initial cost, better it is.

(*b*) Running Cost:- To produce a given amount of electrical energy, the cost of conversion process (including proportional cost of maintenance/repairs of the system) has to be known.

(*c*) Limitations:- Whether a particular resource is available, whether a unit size of required rating is available from a single unit or from an array of large number of units, and whether a particular method of generation is techno-economically viable and is time-proven, are typical queries related to the limitations of the concerned method.

(*d*) 1) perpetuity, 2) efficiency, 3) reliability, 4) cleanliness & 5) simplicity.
It is naturally desirable that the source must have perpetuity (= be of endless duration), high conversion efficiency, and reliability (in terms of availability in appropriate quantity). The energy conversion must be through a cleaner process (specially from the view- points of toxicity, pollution or any other hazardous side effects). Further, a simpler overall system is always preferred with regards to maintenance/repairs problems and is supposed to be more reliable.

24.3. Sources for Generation of Electricity

Following types of resources are available for generating electrical energy (No doubt, this list can be extended to include some more up-coming resources. The following list, however, gives the popular and potential resources).

(a) Conventional methods

(*a*) **Thermal:** Thermal energy (from fossil fuels) or Nuclear Energy used for producing steam for turbines which drive the alternators (= rotating a.c. generators).

(*b*) **Hydro-electric:** Potential of water stored at higher altitudes is utilized as it is passed through water-turbines which drive the alternators.

(b) Non-conventional methods

(*c*) **Wind power:** High velocities of wind (in some areas) are utilized in driving wind turbines coupled to alternators. Wind power has a main advantage of having zero production cost. The cost of the equipment and the limit of generating-unit-rating is suitable for a particular location (= geographically) are the important constraints. This method has exclusive advantages of being pollution free and renewable. It is available in plentiful quantity, at certain places. It suffers from the disadvantages of its availability being uncertain (since dependent on nature) and the control being complex (since wind-velocity has wide range of variation, as an input, and the output required is at constant voltage and constant frequency). Single large-power units cannot be planned due to techno-economic considerations.

(*d*) **Fuel cells:** These are devices which enable direct conversion of energy, chemically, into electrical form. This is an up-coming technology and has a special merit of being pollution-free and noise-free. It is yet to become popular for bulk-power generation.

(*e*) **Photo voltaic cells:** These directly convert solar energy into electrical energy through a chemical action taking place in solar cells. These operate based on the photo-voltaic effect, which develops an emf on absorption of ionizing radiation from Sun.

Power Scenario In India:-

Following approximate statistical data give an idea about some aspects in this regard.

Total installed capacity	:	150,000 MW
Hydropower	:	50,000 MW
Nuclear	:	10,000 MW
Thermal (fossil fuels)	:	80,000 MW
Other methods	:	10,000 MW

Other methods include partly exploited Potential such as Wind, Solar, Co-generation, Methods using Bio-fuels, etc.

24.4. Brief Aspects of Electrical Energy Systems

24.4.1. Utility and Consumers

At generating stations, power is generated at the best locations. Load-centres are generally away from these. Generation-units and Loads are connected by transmission systems. Thus, the energy system is divided into two main parts.

(*a*) Utility (including sources and transmission network) and,

(*b*) Consumers (who utilize the electrical energy)

24.4.2. Why is the three- phase a. c. system most popular?

(*a*) It is well known that a.c. generation is simpler (than d.c. generation through electrical machines because of absence of commutators in a.c. machines). Further, mechanical commutation system in d.c. machines sets an upper limit of their size, while the rating of the individual generators in *modern power stations* is too large, say about 1500 times the rating of a single largest feasible d.c. machine. A.C. further facilitates in stepping down or stepping-up of a voltage to suit a particular requirement, with the help of a simple device, the well-known transformer.

(*b*) Changing over from a.c. to d.c. is very easy these days due to the rectifiers of sufficiently high power ratings, so that a wide range of d.c. – applications can now be catered to.

(*c*) As the number of phases goes on increasing the power-output (from a device using a given quantity of active material, namely, that used for core and for winding) increases, but the number of circuits (*i.e.* connecting lines/ wires, switches, etc) also increases.

These two points are contradictory. A choice will be in favour of such a number (of phases) which will be high enough from power-output point of view but low enough from viewpoints of complexity of connecting/controlling large number of phases. There is a Golden compromise. It has resulted into popularity of the three-phase a. c. systems over the entire time period. Now it has almost become a standard practice for all purposes.

24.4.3. Cost of Generation

Cost of generation for one unit of electrical energy depends on the method of generation, formulae worked to assess its running cost under the specified conditions, and the cost of transmission line loss to transport power upto the load. These days, a modern utility (= electricity board) has a large number of generators sharing the responsibility of supplying power to all the customers connected to the Grid (= common supply-network). Then, for an increase in load-demand, at a known location, the most economical generating unit is to be identified and that unit should be monitored to meet the increased demand.

24.4.4. Staggering of Loads during Peak-demand Hours

Incentives to consumers (by way of supplying at reduced rates during light load hours e.g. Late night 'hours, after noon hours) help in even demand throughout the day.

24.4.5. Classifications of Power Transmission

(*a*) Using underground cables or using overhead transmission lines.
(*b*) Extra High Voltage A.C.-versus-Extra High Voltage D.C. transmission systems.

24.4.6. Selecting A.C. Transmission Voltage for a Particular Case

In general, for transporting a given power of V I watts, either V can be high or V can be low. Accordingly, the current can be either low or high respectively. Higher voltage means higher cost of insulation, and larger clearances. Higher current means larger cross section of conductors. Considering these together, the most economical voltage has to be found out for a particular requirement.

Kelvin's Laws give a guideline for this. (These are discussed in Art. 47.21 in Vol. III, of this book).

24.5. Conventional Sources of Electrical Energy

Thermal (coal, gas, nuclear) and hydro-generations are the main conventional methods of generation of Electrical Energy. These enjoy the advantages of reaching perfections in technologies for these processes. Further, single units rated at large power-outputs can be manufactured along with main components, auxiliaries and switch- gear due to vast experiences during the past century. These are efficient and economical.

These suffer from the disadvantages listed below:

1. The fuels are likely to be depleted in near future, forcing us to conserve them and find alternative resources.
2. Toxic, hazardous fumes and residues pollute the environment.
3. Overall conversion efficiency is poor.
4. Generally, these are located at remote places with respect to main load centres, increasing the transmission costs and reducing the system efficiency.
5. Maintenance costs are high.

   Out of these, only two such types will be dealt here, which have a steam turbine working as the prime mover. While remaining two use Internal Combustion Engines (I.C. engines) or Gas turbine as the prime mover, and these will *not* be dealt with, in this Introductory treatment.

The steam-turbine driven systems are briefly discussed below.

24.5.1. Steam Power Stations (Coal-fired)

India has rich stocks of coal as a natural resource. Chemical energy stored within the coal is finally transformed into Electrical energy through the process of these stations. Heat released by the combustion of coal produces steam in a boiler at elevated temperatures and pressures. It is then passed through steam turbines, which drive the alternator, the output of which is the electrical energy.

Figure 24.1 shows a simple schematic diagram of a modern coal-fired thermal station.

In India, coal is generally of low grade containing ash upto 40 %. It poses two problems. (*i*) Calorific value is low and hence system efficiency can be increased only by additional processes like pulverizing the coal and using oil firing to start with, (*ii*) large volumes of ash have to be handled after ensuring that ash is extracted to the maximum possible extent (upto 99 %) by using electrostatic precipitators, before flue gases from boiler are finally passed on to the atmosphere.

Coal is burnt in the boiler. This heat converts water into steam when passed through the boiler tubes. Modern plants have super heaters to raise the temperature and pressure of steam so that plant efficiency is increased. Condenser and cooling tower deal with steam coming out of turbine. Here, maximum heat is extracted from steam (which then takes the form of water) to pre-heat the incoming water and also to recycle the water for its best utilization.

Steam-turbine receives controlled steam from boiler and converts its energy into mechanical energy which drives the 3-ph a. c. generator (=alternator). The alternator delivers electrical energy, at its rated voltage (which may be between 11 to 30 kV). Through a circuit breaker, the step up transformer is supplied. Considering the bulk-power to be transmitted over long distances, the secondary rating

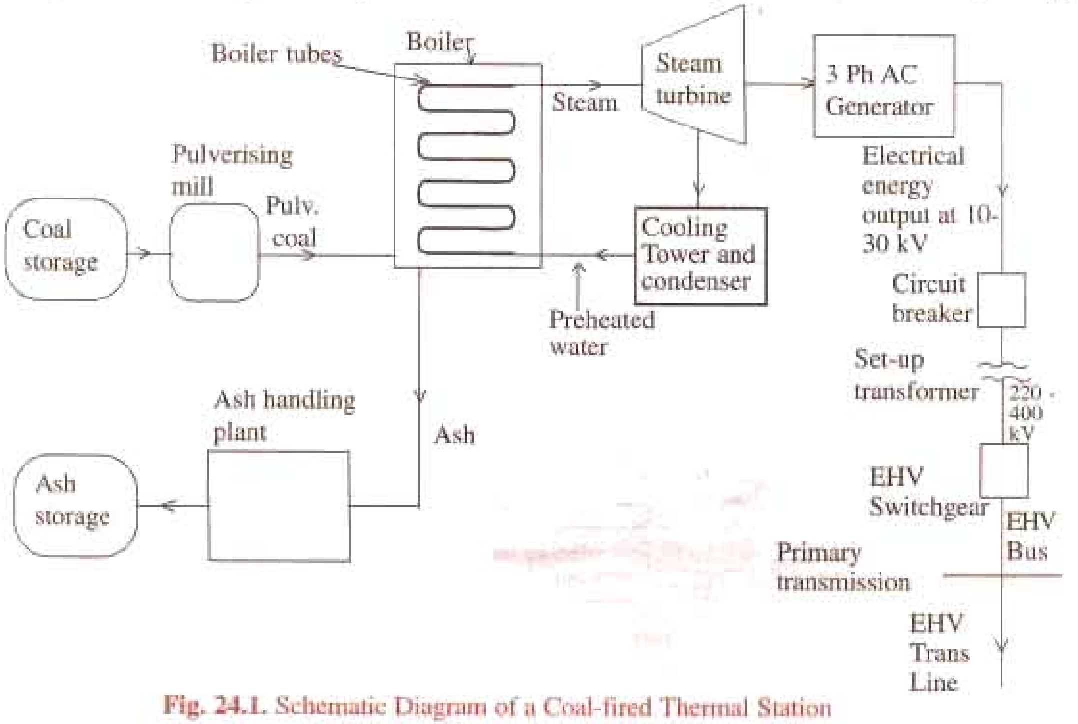


Fig. 24.1. Schematic Diagram of a Coal-fired Thermal Station

of this transformer may be 220 or 400 kV (as per figures for India). Through Extra-High Voltage Switch gear, the Bus is energized and the EHV primary transmission line can transport power to the Load centres connected to it.

A modern coal-fired thermal power station consumes about 10 % of its power for supplying to the Auxiliaries. These are mainly as follows.

(*a*) Main-exciter for alternator. (*b*) Water pumps.
(*c*) Fans. Forced draught and Induced draught fans for Pre-heaters and Chimney.
(*d*) Coal handling plant including pulverising mill.
(*e*) Ash handling plant including Electrolytic Precipitator.

Naturally, whenever such a station is to be brought into operation (either at commissioning or after repairs/maintenance schedule) the power required for the auxiliaries has to be supplied by the grid. Once the system is energized fully, it will look after supplying power to its own auxiliaries.

Merits of Coal-fired thermal stations

1. Fuel (=coal) is cheap. 2. Less initial cost is required. 3. It requires less space.
4. As a combination of all above points, the cost of generating unit of electrical energy is less.

Demerits

1. Atmospheric pollution is considerable.
2. Coal may have to be transported over long distances, in some cases, after some years, and then the energy cost may be quite high.

24.5.2. Nuclear Power Stations

Nuclear energy is available as a result of fission reaction. In a typical system, Uranium 235 is bombarded with neutrons and Heat energy is released. In chain-reaction, these release more neutrons, since more Uranium 235 atoms are fissioned. Speeds of Neutrons must be reduced to critical speeds for the chain reaction to take place. Moderators (= speed-reducing agents like graphite, heavy water, etc). are used for this purpose. Nuclear fuel rods (of Uranium 235) must be embedded in speed-reducing agents. Further, control rods (made of cadmium) are required since they are strong neutron-absorbers and help in finely regulating this reaction so that power control of the generator is precisely obtainable. When control rods are pulled out and are away from fuel rods, intensity of chain reaction increases, which increases the power output of the system. While if they are pushed in and closer to the fuel rods, the power-output decreases. Thus, the electrical load demand on the generator decides (automatically) the control-rod positions through a very sophisticated control system.

Fig. 24.2 shows a basic scheme of such a Nuclear power-station.

Advantages of Nuclear Generation

1. Quantity of fuel required is small for generating a given amount of electrical energy, compared to that with other fuels.
2. It is more reliable, cheaper for running cost, and is efficient when operated at rated capacity.

Disadvantages

1. Fuel is expensive and not abundantly available everywhere. 2. It has high capital cost.
3. Maintenance charges are high. 4. Nuclear waste disposal is a problem.

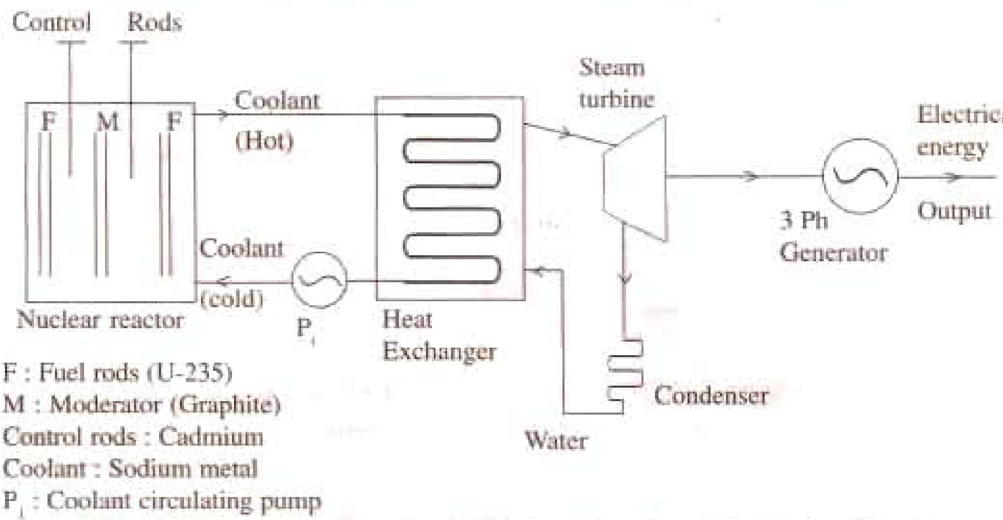


Fig. 24.2. Schematic Diagram of a Nuclear Power Station indicating Main Components

24.5.3. Hydroelectric Generation

Water-reservoir at higher altitudes is a pre-requisite for this purpose. Power-house is located at a lower level. The difference in these two levels is known as "Head."

Based on the "Heads", the Hydroelectric stations are categorized below:

1. Low head up to 60 metres. 2. Medium head between 60 & 300 metres.
3. High heads above 300 metres.

In this method of generation, water from higher height is passed through penstock as controlled in the valve-house, into the water turbine. Thus, potential energy of water stored at higher altitudes is first converted into Kinetic energy. As the water reaches the turbine, it gains speed after losing the Potential energy. Kinetic energy of this speedy water drives the water turbine, which converts this into mechanical output. It drives the coupled generator, which gives Electrical energy output.

A schematic diagram of such a system is shown in fig.

The valve house has a controlling valve (=main sluice valve) and a protecting valve (= an automatic, isolating, "butterfly" type valve). As is obvious, power control is done by the main sluice valve, while "butterfly" valve comes into action if water flows in opposite direction as a result of a sudden drop in load on the generator. Otherwise, the penstock is subjected to extreme strains and it has a tendency to burst due to pressure of water as a result of sudden load reduction.

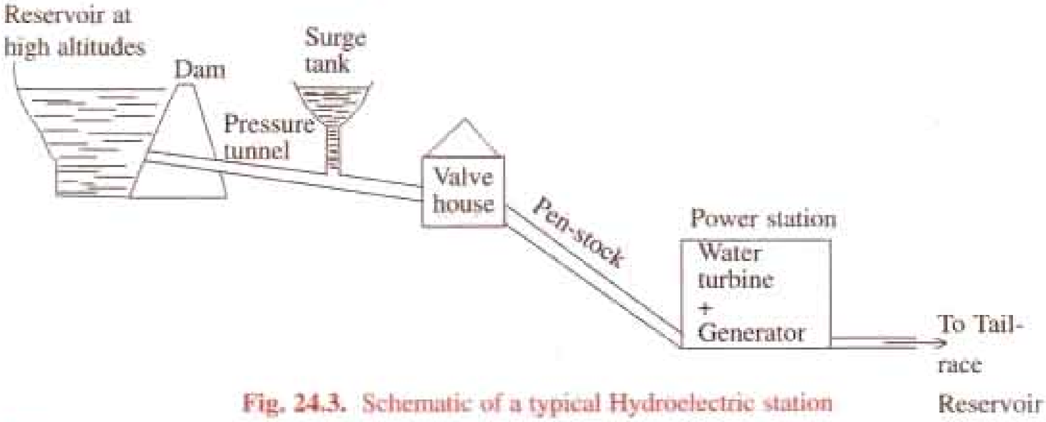


Fig. 24.3. Schematic of a typical Hydroelectric station

Note: Variation of Head (low-, medium-, or high-head) will affect the block-schematic

After doing the work (of imparting its energy to the water turbine), the water is allowed to pass into the tail-race reservoir.

The water turbines are essentially low-speed prime movers. In that, the best operating speed is dependent on the head. Alternators coupled to water turbines thus have large number of poles (since P= 120 f/N). Such alternators have the Salient-Pole type rotor.

There are different types of water turbines suitable for different cases (i.e. Heads, Power rating, Load-variation curve, etc). Since, this is only an elementary treatment, these aspects will not be discussed here.

24.6. Non-Conventional Energy Sources

Considering the previously discussed thermal methods of conventional energy-generations, it is necessary to understand the non-conventional energy sources, since they have two points in their favour.

(a) Non-polluting processes are used.

(b) Perpetuity and renewability of the main source (which is a natural atmospheric resource) generally exists.

The non-conventional energy sources are further advantageous due to virtually zero running cost, since wind energy or solar energy is the input-source of power.

However, they are disadvantageous due to high initial cost (per MW of installed capacity) and due to uncertainty resulting out of weather changes. For example, dense clouds (or night hours) lead to non-availability of solar energy. Similarly, "still-air" condition means no possibility of wind power generation, and during stormy weathers, wind turbines cannot be kept in operation (due to dangerously high speeds they would attain if kept in operation).

24.6.1. Photo Voltaic Cells (P.V. Cells or SOLAR Cells)

When ionized solar radiation is incident on a semi-conductor diode, energy conversion can take place with a voltage of 0.5 to 1 volt (d.c.) and a current density of 20-40 mA/cm^2, depending on the materials used and the conditions of Sunlight. Area of these solar cells decides the current output. An array of large number of such diodes (i.e. Solar cells) results into higher d.c. output voltage.

Since, the final form of electrical energy required is generally an alternating current, it is realized from d.c. using inverters.

At quite a few locations in India, for realizing few hundred kilowatts of power-rating, huge arrays are accommodated in horizontal as well as vertical stacks, so that land area required is not too vast. Electrically, they are connected in series and in parallel combinations of cells so that rated voltage & current are realized.

Just to understand the principle of operation of solar cells, let a semi-conductor diode receive ionized radiation from Sun, as in fig - 24.4

Typical materials used for these cells are: material doped with boron, cadmium sulphide, gallium-arsenide, etc. Their choice is mainly decided by conversion efficiency. Best material may lead to the efficiency being typically 15%. Since solar energy is available free of cost, this low-efficiency does not matter.

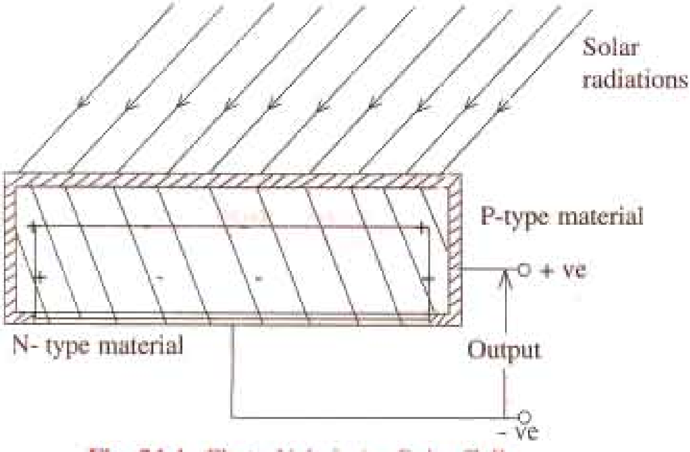


Fig. 24.4. Photo Voltaic (or Solar Cell)

This method suffers from the disadvantages of having high initial cost and uncertainty (since dependent on weather conditions) including non operative night periods. Main advantages are: (*i*) no running cost (however, replacements of components may be a botheration), (*ii*) no pollution, (*iii*) location can be near the load (hence transportation of power is not required over long distances), (*iv*) since natural source is involved, it is perpetual.

Individual stations using solar cells are in operation with ratings of the order of 250-1000 kW. With manufacturing costs of semi-conductor devices going down and with the advent of better and better quality of cells which will be available in future, this method of generation has bright prospects.

24.6.2. Fuel Cells

24.6.2.1. Principle of Operation

In Fuel cells, negative porous electrode is fed by hydrogen and the positive porous electrode is fed by oxygen. Both the electrodes are immersed in an electrolyte. The porous electrodes are made of such a conducting material that both the fuel (oxygen and hydrogen) and the electrolyte can pass through them. Such a material for electrodes is nickel. The electrolyte is a solution of sulphuric acid or potassium hydroxide. The electrodes have a catalyst (= platinum or sintered nickel) which break the fuelcompound into more reactive atoms.

24.6.2.2. Chemical Process (with Acidic Electrolyte)

At Negative Electrode : $2H_2 \longrightarrow 4H^+ + 4e$

These hydrogen ions enter the solution (=electrolyte) leaving behind electrons which pass through external circuit to the positive electrode.

At Positive Electrode: $O_2 + 4H^+ + 4e \rightarrow 2H_2O$

Thus, the combination of Hydrogen and Oxygen results into water at the positive electrode. Water is the waste-product of the cell, which is harmless. The process is, thus, pollution free. There is no source of energy required, since the process is basically **"chemical"** in nature.

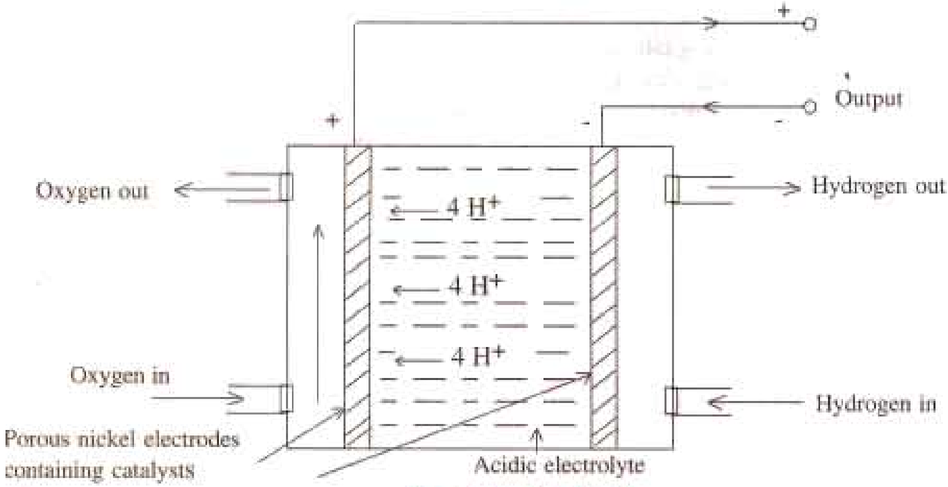


Fig. 24.5. Fuel Cell

24.6.2.3. Schematic Diagram. It is given in fig 24.5

24.6.2.4. Array for Large outputs

A fuel cell has a d. c. output voltage typically of 1.23 volts at normal atmospheric pressure and temperature. Raising pressure and temperature increases this voltage. To realize large output parameters (= voltage, current, and hence power), an array of a large number of fuel cells (connected in series as well as in parallel) is made. Voltage levels of 100 to 1000 V and power levels in kilowatts can be realized.

24.6.2.5. High Lights

1. Pollution-free, noiseless.
2. No outside source of energy is required.
3. Efficient.
4. No restriction on location

    (*a*) High initial cost. (*b*) Working life is short.

Note: Solar energy can also be used for generating electrical energy through an intermediate stage of producing steam, which is used later for driving an alternator. However, this method is not discussed here.

24.6.3. Wind Power

24.6.3.1. Background

Wind power has been in use for serving the mankind, since centuries through what has been popularly known as "Wind-mills." There is no "electrical" stage of energy in old-styled uses where wind-velocity is directly used for performing the jobs such as wheat-grinding, pumping water for irrigation, sailing vessels, etc. It enjoys the advantages of being plentiful, inexhaustible, renewable and non-polluting, over and above being cheap for running costs. It suffers from disadvantages of being unreliable, and being economically un-viable for large power generation. In India, a large number of such units with small and medium power ratings (up to 100 kW) are already in operation mainly in coastal or hilly areas. With the modern system, it is now preferred to have suitable power-control circuits on the output side of wind-generators so that these can pump energy into low voltage lines of the grid over a wide range of variation of wind speeds.

24.6.3.2. Basic Scheme

A large variety of wind-turbines naturally exist arising out of large variation in wind-pattern all over and out of different manufacturers producing systems with different designs. Since the aim here is to understand the basic system, only one type of system is presented here.

In Fig. 24.6 (*a*) an arrangement wherein, a horizontal three-bladed system is shown mounted

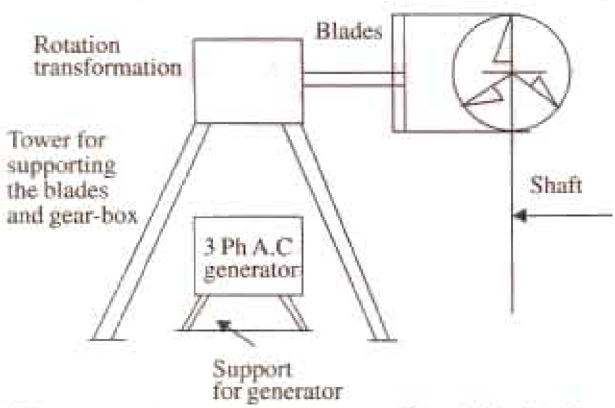


Fig. 24.6 (a). Wind-generation, a Schematic view

Fig. 24.6 (b). Part-side-view to show a typical three bladed wind turbine

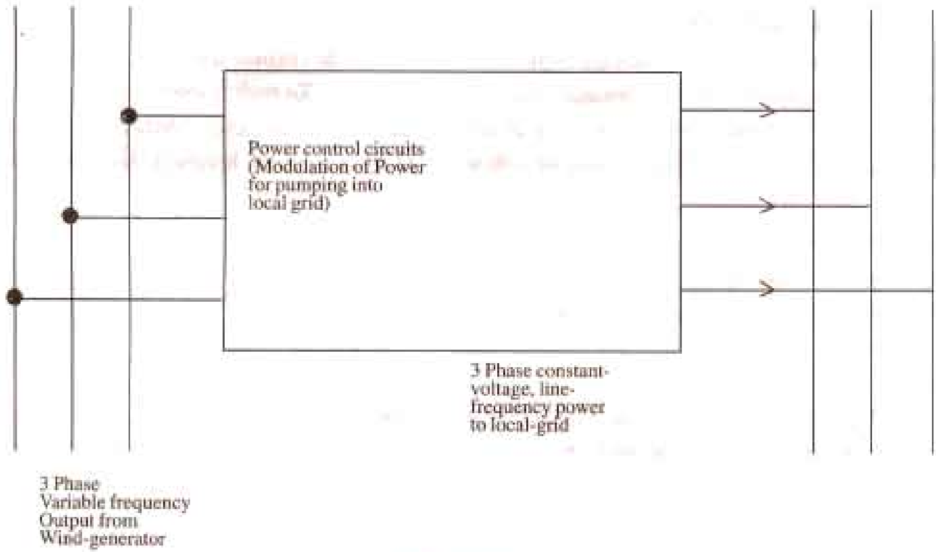


Fig. 24.7

on a tower. Through rotation-transformation using gears to step up the speed and to link the horizontal axis of turbine with vertical axis of generator. The speed of wind varies, as such turbine speed also varies so that output frequency and voltage of three-phase alternator vary over a wide range. Further, its waveform is also a distorted one. To increase its utility, it is necessary to modulate (through proper power-control) to derive line-frequency constant voltage output and hook-up to local grid for pumping the available wind-energy into it. This is schematically represented in fig 24.7.

24.6.3.3. Indian Scenario

Wind farms have been located where a large number of wind generators of ratings of few hundred kilowatts are in operation. For every unit, there is a safe wind - speed zone. If the wind-speed is below this, there is no appreciable power output, hence, it is better that the system is *not* brought into operation. If the wind speeds are too high, it is mechanically unsafe and hence it is *not* to be operated, even if the energy available is higher. This is decided by automated system. Such forms are located in coastal regions and in hilly areas. Because of the metering of energy received by the local grid, the investor can get a good return through payment from the grid-authorities. Hence, this has become a medium-level and attractive industrial investment.